Quantum Mechanics - 2

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We have encountered the tensor operators many times, but without introducing the proper tools to handle them.

examples:

1. EDM $\vec{J} \propto \vec{S}$

2. Hyperfine structure

$$H_1 = 2 g_N \mu_N \mu_B \frac{\vec{I}\cdot\vec{L}}{r^3} + 2 g_N \mu_N \mu_B \left[\frac{8}{3}\pi \delta^3(\vec{r})\, \vec{I}\cdot\vec{S} + \frac{1}{r^3} \vec{I}_i \left(\frac{3 r_i r_j - r^2 \delta_{ij}}{r^2} \right) \vec{S}_j \right]$$

moreover, to calculate the matrix elements, we have used the projection theorem

$$\vec{V} \rightarrow \frac{(\vec{V}\cdot\vec{J})\cdot\vec{J}}{j(j+1)\hbar^2}$$

3. Multipole

		#	
Electric	monopole	1	$= 2^0$
Magnetic	dipole	2	$= 2^1$
E	Quadrapole	4	$= 2^2$
M	Octupole	8	$= 2^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

but selection rule?

All of them have something to do with the Wigner-Eckart theorem

electric dipole transition $\Rightarrow$ $\Delta \ell = 0, \pm 1$.

forbidden by parity

What does it look like ?

For a rotational invariant system:

$$\langle \alpha', j', m' | T^{(k)}_{(q)} | \alpha, j, m \rangle = C^{j'm'}_{jk;mq} \frac{\langle \alpha' j' \| T^{(k)} \| \alpha, j \rangle}{\sqrt{2j'+1}}$$

☆ Angular dependence of these matrix-elements can be factored out, and it is given by the Clebsch-Gordon coefficient.

☆ To evalute $\langle \alpha' j' m' | T^{(k)}_{(q)} | \alpha, j m \rangle$ with various combinations of m, m' and q' (for a given set of α, α', j' & j)
We only need to know just ONE of them, all others can be related through the C-G coefficient.

☆ An irreducible tensor operator $T^{(k)}_{q}$ can be viewed as angular momentum operator, and the selection rules follow.

(you should ~~read~~ go through this part in Sakurai)

Now, let's slowly moving to the end.

under rotation (R), R: 3×3 rotation matrix

ket $|\alpha\rangle \rightarrow D(R)|\alpha\rangle$

the expectation value of $\vec{V}$ is assumed to change as follow

$$\langle\alpha|V_i|\alpha\rangle \rightarrow \langle\alpha|D^\dagger(R)\, V_i\, D(R)|\alpha\rangle = \sum_j R_{ij}\langle\alpha|V_j|\alpha\rangle$$

For any arbitrary $|\alpha\rangle$ the above to be hold

$$\Rightarrow \boxed{D^\dagger(R)\, V_i\, D(R) = \sum_j R_{ij} V_j}$$

For an infinitesimal rotation

$$D(R) = \mathbb{1} - \frac{i\epsilon\, \vec{J}\cdot\vec{n}}{\hbar}$$

$$\left(\mathbb{1} + \frac{i\epsilon\, \hat{n}\cdot\vec{J}}{\hbar}\right) V_i \left(\mathbb{1} - \frac{i\epsilon\, \hat{n}\cdot\vec{J}}{\hbar}\right) = R_{ij}(\hat{n};\epsilon)\, V_j$$

$$\Rightarrow \quad V_i + \frac{i\epsilon}{\hbar}[\hat{n}\cdot\vec{J}, V_i] = R_{ij}(\hat{n};\epsilon)\, V_j$$

choose $\hat{n} = \hat{z}$. $\Rightarrow R(\hat{z};\epsilon) = \begin{pmatrix} 1 & -\epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So $i=1$ $\quad V_x + \frac{i\epsilon}{\hbar}[J_z, V_x] = V_x - \epsilon V_y$

$i=2$ $\quad V_y + \frac{i\epsilon}{\hbar}[J_z, V_y] = V_y + \epsilon V_x$

$i=3$ $\quad V_z + \frac{i\epsilon}{\hbar}[J_z, V_z] = V_z$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \quad \begin{array}{l} [J_z, V_x] = i\hbar V_y \\ [J_z, V_y] = -i\hbar V_x \\ [J_z, V_z] = 0 \end{array}$$

Or, more generally

$$[\hat{n}\cdot\vec{J}, \vec{V}] = -i\hbar\, \hat{n}\times\vec{V}$$

Taking the j-th component,

$$n_i[J_i, V_j] = -i\hbar\, \epsilon_{jik}\, n_i V_k$$

Since $\hat{n}$ is an arbitrary unit vector,

$$\Rightarrow \boxed{[J_i, V_j] = i\hbar\, \epsilon_{ijk} V_k}$$

from this, you can derive (as HW) the Dirac identity

$$[J^2, [J^2, \vec{V}]] = 2\hbar^2 (J^2 \vec{V} + \vec{V} J^2) - 4\hbar^2 (\vec{V}\cdot\vec{J})\vec{J}$$

By this, we can prove the projection theorem;

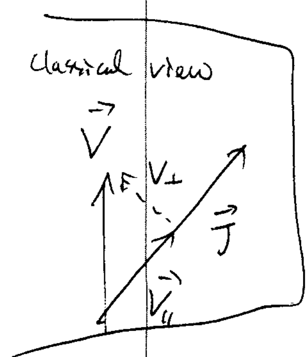
L.H.S. $\langle \gamma' j m' | J^2[J^2, \vec{V}] - [J^2, \vec{V}] J^2 | \gamma j m \rangle$

$$= \hbar^2 j(j+1) \langle \gamma' j m' | 0 | \gamma j m \rangle = 0$$

R.H.S.

$$0 = 2\hbar^2 \langle \gamma' j m' | J^2 \vec{V} + \vec{V} J^2 | \gamma j m \rangle - 4\hbar^2 \langle \gamma' j m' | (\vec{V}\cdot\vec{J})\vec{J} | \gamma j m \rangle$$

$$= 4\hbar^2 \left\{ j(j+1)\hbar^2 \langle \gamma' j m' | \vec{V} | \gamma j m \rangle - \langle \gamma' j m' | (\vec{V}\cdot\vec{J})\vec{J} | \gamma j m \rangle \right\}$$



$$\Rightarrow \boxed{\langle \gamma' j m' | \vec{V} | \gamma j m \rangle = \langle \gamma' j m' | \frac{(\vec{V}\cdot\vec{J})\vec{J}}{j(j+1)\hbar^2} | \gamma j m \rangle}$$

This is the projection theorem

Problem 4 in HW 4, just start with

$$\langle j m | \vec{V} | j m' \rangle = \underbrace{\lambda}_{\text{a const}} \langle j m | \vec{J} | j m' \rangle$$