Cartesian Tensors versus Irreducible Tensor

In classical physics, $V_i \to \sum_j R_{ij} V_j$

$T_{ijk\cdots} \to \sum_{i'j'k'\cdots} R_{ii'} R_{jj'} R_{kk'} \cdots T_{i'j'k'\cdots}$

rank (of the tensor) $\equiv$ # of indices

The simplest rank 2 Cartesian tensor is dyadic, formed out of two vectors $\vec{U}$ and $\vec{V}$.

$$T_{ij} = \underbrace{U_i}_{3} \times \underbrace{V_j}_{3} = 9 \text{ d.o.f.}$$

The trouble is that it is reducible, namely, it can be decomposed into objects that transform differently under rotations.

$$U_i V_j = \underbrace{\frac{\vec{U}\cdot\vec{V}}{3}}_{\text{a scalar}} \delta_{ij} + \frac{1}{2}\left[\underbrace{U_i V_j - U_j V_i}\right] + \left[\underbrace{\frac{U_i V_j + U_j V_i}{2} - \frac{\vec{U}\cdot\vec{V}}{3}\delta_{ij}}\right]$$

$= \epsilon_{kij} U_i V_j = (\vec{U}\times\vec{V})_k$ a vector

3×3 symmetric matrix + traceless $= (6-1) = 5$ d.o.f.

$$9 = 1 + 3 \text{ d.o.f.} + 5$$

$\sim \ell = 0$ $\qquad \sim \ell = 1$: $|1 0\rangle, |1 1\rangle, |1, -1\rangle$ $\qquad \sim \ell = 2$: $|2 0\rangle, |2 \pm 1\rangle, |2 \pm 2\rangle$

This is the motivation of introducing <u>irreducible spherical tensors</u>.

m - independence

$|\alpha\ j\ m\rangle$: Subspace of angular momentum j
α: complementary information

$|\beta, j, m'\rangle$: Another subspace of another β of the same j

consider the matrix elements:

$$\langle\beta, j, m-1|\alpha, j, m-1\rangle = \frac{\langle\beta, j, m-1|J_-|\alpha, j, m\rangle}{\sqrt{j(j+1)-m(m-1)}}$$

Hermitian conjugate of J_-

$$= \frac{(|\beta, j, m-1\rangle, J_-|\alpha, j, m\rangle)}{\sqrt{j(j+1)-m(m-1)}} = \frac{(J_+|\beta, j, m-1\rangle, |\alpha, j, m\rangle)}{\sqrt{j(j+1)-(m-1)(\underline{m-1}+1)}}$$

equal

$$= (|\beta, j, m\rangle, |\alpha, j, m\rangle) = \underline{\langle\beta\, j\, m|\alpha\, j\, m\rangle}$$

⇒ The matrix element is invariant under equal rotations of both the bra and the ket.

⇒ Both matrix elements behave identically under rotation

⇒ have the <u>same</u> angular momentum.

☆ Moreover, the matrix element is independent of m because rotations mix values of m.

⇒ We can denote $\langle\beta, j\, m|\alpha\, j\, m\rangle = \langle\beta\, j\|\alpha\, j\rangle$

α, β are additional information to specify the states.

$\boxed{\text{Vector operators}}$ like $\vec{x}, \vec{p}, \vec{L}$ etc..

transform infinitesimally by

$$e^{i\theta \vec{n}\cdot\vec{J}}\, \vec{V}\, e^{-i\theta\vec{n}\cdot\vec{J}} = \vec{V} + \theta\cdot\vec{n}\times\vec{V}$$

$$\Rightarrow i[\vec{n}\cdot\vec{J}, \vec{V}] = \vec{n}\times\vec{V}$$

$$i[\vec{n}_\alpha \vec{J}_\alpha, \vec{V}_j] = \epsilon_{j\alpha\beta}\, \vec{n}_\alpha \vec{V}_\beta$$

or $\quad [J_n, V_i] = i\,\epsilon_{nik} V_k$

It works for $\vec{V} = \vec{J}$, just the ordinary angular momentum.

as $\quad [J_z, J_\pm] = \pm J_\pm \qquad J_\pm = J_x \pm i J_y$

and $\quad [J_+, J_-] = 2J_z$

We also have

$$[J_+, \underbrace{V_x - iV_y}_{\sqrt{2}V_-}] = 2\overset{V_0}{\widetilde{V_z}}, \qquad [J_+, V_z] = -V_x - iV_y$$

$$[J_-, V_x + iV_y] = -2V_z \qquad [J_-, V_z] = V_x - iV_y$$

...

define $\quad V_0 \equiv V_z, \quad V_{+1} \equiv \frac{-1}{\sqrt{2}}(V_x + iV_y), \quad V_{-1} = \frac{1}{\sqrt{2}}(V_x - iV_y)$

then,

$$[J_\pm, V_q] = \sqrt{1(1+1) - q(q\pm 1)}\; V_{q\pm 1}$$

$$[J_z, V_q] = q\, V_q$$

$\Rightarrow$ vector operator acts like a spin-1 object.

Tensor operator T_k^q

Spherical Harmonics $Y_\ell^m \to Y_k^q(\hat{\theta}, \hat{\phi})$

Wave function $\Rightarrow$ operators.

The spherical harmonics (as operators) satisfy the following commutation relations:

$$[J_z, Y_k^q] = q Y_k^q, \quad [J_\pm, Y_k^q] = \sqrt{k(k+1) - q(q\pm1)}\; Y_k^{q\pm1}$$

Similarly, the tensor operators T_k^q are defined in the same manner. (just $\vec{x} \to V_x$,
$\vec{y} \to V_y$
$\vec{z} \to V_z$)

and

$$[J_z, T_k^q] = q T_k^q, \quad [J_\pm, T_k^q] = \sqrt{k(k+1) - q(q\pm1)}\; T_k^{q\pm1}$$

Examples are

① $L_z \sim T_1^0$

② $\mp \frac{L_\pm}{\sqrt{2}} \sim T_1^{\pm1}$

③ $\vec{r}^2 \sim T_0^0$

④ x or $p_x \sim -T_1^1 + T_1^{-1}$

⑤ $x^2 \sim$ linear combination of T_2^m

$\Rightarrow$ Any analytic function of x, y and z can be expanded in terms of irreducible tensor operators.

(under rotations, the operators mix only amongst irreducible subset

Since T_k^q rotates like an angular momentum state, one can define the state

$$|\beta, J, M\rangle \equiv \sum_{q,m} \langle k\, j; q\, m \,|\, J, M\rangle\, T_k^q\, |\beta\, j\, m\rangle$$

rotates as an object with angular momentum J and projection M.

The Clebsch-Gordan coefficients is denoted as

$$\langle k, j; q\, m \,|\, J\, M\rangle$$

$\Rightarrow$ $\boxed{T_k^q \text{ can be treated as an state } |q, k\rangle}$