☆ Adding two angular momentum

$|j_1 m_1\rangle \otimes |j_2 m_2\rangle$ can be expanded in terms of

$|j_1 j_2; m_1 m_2\rangle$ or $|j_1 j_2; j m\rangle$ basis vector

☆
$$|j_1 j_2; j m\rangle = \sum_{m_1 m_2} |j_1 j_2; m_1 m_2\rangle \underline{\langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle}$$

$$= \sum_{m_1 m_2} \underline{C^{jm}_{m_1 m_2}} \, |j_1 j_2; m_1 m_2\rangle$$

↑ Clebsh-Gordan coefficient

or

$$|j_1 j_2 m_1 m_2\rangle = \sum_{j} C^{jm}_{m_1 m_2} |j_1 j_2; j m\rangle$$

$\boxed{\vec{j} = \vec{j}_1 + \vec{j}_2}$ Act $\underline{J_z}$ on both sides of the equation.

$$\Rightarrow \quad m\,|j_1 j_2; j m\rangle = \sum_{m_1 m_2} C^{jm}_{m_1 m_2} (m_1 + m_2)\,|j_1 j_2; m_1 m_2\rangle$$

Times $\langle j_1 j_2 m_1 m_2|$ on both side

$$\Rightarrow \quad m\, C^{jm}_{m_1 m_2} = (m_1 + m_2)\, C^{jm}_{m_1 m_2}$$

$$\Rightarrow \quad \boxed{\text{If} \quad m - m_1 - m_2 \neq 0 \qquad C^{jm}_{m_1 m_2} = 0}$$

☆ Apply J_+ on both sides.

$$\sqrt{j(j+1) - m(m+1)}\;|j_1 j_2; j\, m+1\rangle$$

$$= \sum_{m_1 m_2} C^{j}_{m_1 m_2} \left\{ \begin{array}{l} \sqrt{j_1(j_1+1) - m_1(m_1+1)}\;|j_1 j_2; m_1+1, m_2\rangle \\ + \sqrt{j_2(j_2+1) - m_2(m_2+1)}\;|j_1 j_2; m_1\, m_2+1\rangle \end{array} \right\}$$

times $\langle j_1 j_2 m_1 m_2|$ from the left

$$\Rightarrow \sqrt{j(j+1)-m(m+1)}\, C^{j\,m+1}_{m_1, m_2} = \sqrt{j_1(j_1+1)-m_1(m_1-1)}\, C^{j\,m}_{m_1-1, m_2} + \sqrt{j_2(j_2+1)-m_2(m_2-1)}\, C^{j\,m}_{m_1, m_2-1}$$

similarly, by acting J_- on the equation, we have

$$\Rightarrow \sqrt{j(j+1)-m(m-1)}\, C^{j\,m-1}_{m_1 m_2} = \sqrt{j_1(j_1+1)-m_1(m_1+1)}\, C^{j\,m}_{m_1+1, m_2} + \sqrt{j_2(j_2+1)-m_2(m_2+1)}\, C^{j\,m}_{m_1, m_2+1}$$

Proof of the Wigner-Eckart theorem

* From $[J_z, T^q_k] = q\, T^q_k$

$$\langle \alpha'; j', m' | J_z T^q_k | \alpha; j m\rangle - \langle \alpha' j' m' | T^q_k J_z | \alpha\, j m\rangle$$

$$= q\, \langle \alpha' j' m' | T^q_k | \alpha\, j m\rangle$$

$$\Rightarrow (m' - m - q)\, \langle \alpha' j' m' | T^q_k | \alpha\, j m\rangle = 0$$

namely if $m' - m - q \neq 0$

$$\langle \alpha' j' m' | T^q_k | \alpha\, j m\rangle = 0$$

* From $[J_-, T^q_k] = \sqrt{k(k+1) - q(q-1)}\; T^{q-1}_k$

$$\langle \alpha' j' m' | J_- T^q_k | \alpha\, j m\rangle - \langle \alpha' j' m' | T^q_k J_- | \alpha\, j m\rangle$$

$$= \sqrt{k(k+1) - q(q-1)}\; \langle \alpha' j' m' | T^{q-1}_k | \alpha\, j m\rangle$$

$$\Rightarrow \sqrt{k(k+1)-q(q-1)}\,\langle \alpha' j' m' | T_k^{q-1} | \alpha j m \rangle$$

$$+\sqrt{j(j+1)-m(m-1)}\,\langle \alpha' j' m' | T_k^{q} | \alpha j\, m-1 \rangle$$

$$= \sqrt{j'(j'+1)-m'(m'+1)}\,\langle \alpha' j'\, m'+1 | T_k^{q} | \alpha j m \rangle$$

(The recursion relation is same as the CG from J_+)

✱ From $[J_+, T_k^q] = \sqrt{k(k+1)-q(q+1)}\; T_k^{q+1}$

$$\Rightarrow \langle \alpha' j' m' | J_+ T_k^q | \alpha j m \rangle - \langle \alpha' j' m' | T_k^q J_+ | \alpha j m \rangle$$

$$= \sqrt{k(k+1)-q(q+1)}\,\langle \alpha' j' m' | T_k^{q+1} | \alpha j m \rangle$$

$$\Rightarrow \sqrt{j'(j'+1)-m'(m'-1)}\,\langle \alpha' j'\, m'-1 | T_k^{q} | \alpha j m \rangle$$

$$= \sqrt{j(j+1)-m(m+1)}\,\langle \alpha' j' m' | T_k^{q} | \alpha j\, m+1 \rangle$$

$$+ \sqrt{k(k+1)-q(q+1)}\,\langle \alpha' j' m' | T_k^{q+1} | \alpha j m \rangle$$

(Same as the recursion relations for CG from J_-)

This is the same recursion relations for the CG coefficients with $(j_1\ j_2\ m_1\ m_2) \Rightarrow (j, k, m, q)$

$\Rightarrow$ Therefore the m behavior of $\langle \alpha' j' m' | T_q^k | \alpha j m \rangle$ must be proportional to the CG coefficient, We write

$$\langle \alpha' j' m' | T_q^k | \alpha j m \rangle = \langle j\, k; m\, q | j\, k\, j' m' \rangle$$

$$\times \Big(\underbrace{\text{terms independent of } m', m \text{ and } q}_{= \underbrace{\frac{1}{\sqrt{2j+1}}}_{\text{convention}} \times (\text{reduced matrix})} \Big)$$

This part is called

Recap: ~~Wigner-Eckart Theorem~~

$$\langle j'm' | T_L^M | j m \rangle = \langle j L ; m M | j' m' \rangle \frac{\langle j' \| T_L \| j \rangle}{\sqrt{2j'+1}}$$

$\langle \|\cdot\| \rangle$ is called the "reduced matrix element".

It depends on j, j', L. But it is independent of the quantum numbers m, m' and M.

* The WE Theorem: The matrix element can be seperated into a pure geometrical factor (the CG coeff) and a factor containing the dynamics (the reduced matrix element)

⇒ Selection rules:

$\langle j' m' | T_L^M | j m \rangle \neq 0 \Rightarrow \boxed{m' = M + m}$

also the "triangle rule",

$j' = j+L, \quad j+L-1, \quad ---- \quad |j-L|$

* A tensor of zero rank can only connect states of the same total j.

* A tensor of first rank can only connect states $j' = j$ or $j' = j \pm 1$ (However, it can't connect $j=0$ to $j'=0$)

* A tensor of the second rank can only connect states

  $j' = j, \quad j \pm 1, \quad j \pm 2$

  However, it can't connect states with $j' = j = 0$, $j' = j = \frac{1}{2}$, or $j=1$, $j'=0$ (or vice versa)

An example of WE Theorem application.

$$\langle 2,0 |\, Y_{10} \,| 1,0 \rangle$$

$$= \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \quad Y_{20}^{*}(\theta,\phi)\, Y_{10}(\theta,\phi)\, Y_{10}(\theta,\phi)$$

$$= \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)\, \left(\sqrt{\frac{3}{4\pi}}\cos\theta\right)^2$$

$$= \frac{2\pi}{\pi}\,\frac{\sqrt{15}}{8}\,\sqrt{\frac{3}{4\pi}} \int_{-1}^{1} dx\, (3x^2-1)\,x^2 = \frac{1}{\sqrt{5\pi}}$$

By WE Theorem

$$\frac{1}{\sqrt{5\pi}} = \underbrace{\langle 1 1 ; 0 0 | 2, 0 \rangle}_{= \sqrt{\frac{2}{3}}} \frac{\langle 2 \| Y_1 \| 1 \rangle}{\sqrt{2\cdot 1+1}}$$

from the CG coeff table

$$\Rightarrow \quad \langle 2 \| Y_1 \| 1 \rangle = \frac{1}{\sqrt{5\pi}}\,\sqrt{3}\cdot\sqrt{\frac{3}{2}} = \frac{3}{\sqrt{10\pi}}$$

Then by WE theorem and the CG table, we can quickly obtain the following ~~result~~ matrix elements:

$$\langle 2,1 |\, Y_{10} \,| 1,1 \rangle = \underbrace{\langle 1 1 ; 0 1 | 2 1 \rangle}_{= \frac{1}{\sqrt{2}}} \frac{\langle \| 1 \| \rangle}{\sqrt{3}} = \frac{1}{\sqrt{5\pi}}\,\frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{2}{3}}} = \frac{1}{2}\sqrt{\frac{3}{5\pi}}$$

$$\langle 2,2 |\, Y_{11} \,| 1,1 \rangle = \underbrace{\langle 1 1 ; 1 1 | 2 2 \rangle}_{= 1} \frac{\langle \| 1 \| \rangle}{\sqrt{3}} = \sqrt{\frac{3}{10\pi}}$$

$$\langle 2\, {-1} |\, Y_{1-1} \,| 1,0 \rangle = \underbrace{\langle 1 1 ; -1\, 0 | 2\, {-1} \rangle}_{= \frac{1}{\sqrt{2}}} \frac{\langle \| 1 \| \rangle}{\sqrt{3}} = \frac{1}{2}\sqrt{\frac{3}{5\pi}}$$

$$\langle 2,0 |\, Y_{1-1} \,| 1,1 \rangle = \underbrace{\langle 1 1 ; -1, 1 | 2, 0 \rangle}_{= \frac{1}{\sqrt{6}}} \frac{\langle \| 1 \| \rangle}{\sqrt{3}} = \frac{1}{2}\,\frac{1}{\sqrt{5\pi}}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$