Time dependent Perturbation

☆ Schrödinger picture $|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\psi(0)\rangle$, H_s

(Interaction picture (Dirac picture)

Heisenberg picture $|\psi(t)\rangle = |\psi(0)\rangle$, $H_H = U^\dagger H_s U$

☆ $H = \underset{\sim}{H_0} + V$,

say, we know how to solve it exactly.

then $i\hbar\frac{d}{dt}|\alpha, t\rangle_s = H|\alpha, t\rangle_s$, $\quad i\hbar\frac{d}{dt}O_s = 0$

☆ Interaction picture:

$$|\alpha, t\rangle_I = e^{+\frac{iH_0t}{\hbar}}|\alpha, t\rangle_s$$

then

$$i\hbar\frac{d}{dt}|\alpha, t\rangle_I = -H_0 e^{\frac{iH_0t}{\hbar}}|\alpha, t\rangle_s + e^{\frac{iH_0t}{\hbar}}\left(i\hbar\frac{d}{dt}|\alpha, t\rangle_s\right)$$

$$= e^{\frac{iH_0t}{\hbar}}(-H_0 + H)|\alpha, t\rangle_s$$

$$= e^{\frac{iH_0t}{\hbar}}(V|\alpha, t\rangle_s) = \left(e^{\frac{iH_0t}{\hbar}} V e^{-\frac{iH_0t}{\hbar}}\right) e^{+\frac{iH_0t}{\hbar}}|\alpha, t\rangle_s$$

$$= V_I|\alpha, t\rangle_I$$

$$\Rightarrow \boxed{i\hbar\frac{d}{dt}|\alpha, t\rangle_I = V_I|\alpha, t\rangle_I}$$

☆ $|\alpha, t\rangle_I$ does not evolve if $V_I = 0$, and it does if $V_I \neq 0$.

☆ $V_I = e^{\frac{iH_0t}{\hbar}} V e^{-\frac{iH_0t}{\hbar}}$ this is to compare with $O_H = e^{\frac{iH_0t}{\hbar}} O_s e^{-\frac{iH_0t}{\hbar}}$

☆ In the interaction picture

$$O_I(t) = e^{\frac{iH_0t}{\hbar}} O e^{-\frac{iH_0t}{\hbar}}, \text{ therefore } \boxed{i\hbar\frac{d}{dt}O_I(t) = [O_I(t), H_0]}$$

☆ The time evolution is solely due to H_0, not H nor V.

	Heisenberg	Interaction	Schrödinger
State ket	no evolution	by V_I	by H
Operators	by H	by H_0	no evolution

Dyson series Introducing the time-evolution operator

$$|\alpha, t\rangle_I = U_I(t)\,|\alpha, t=0\rangle_I$$

which satisfies

$$i\hbar \frac{d}{dt} U_I(t) = V_I\, U_I(t) \quad \text{with B.C.} \quad U_I(0) = \mathbb{1}$$

then

$$U_I(t) = \mathbb{1} - \frac{i}{\hbar}\int_0^t V_I(t')\, U_I(t')\, dt'$$

One can check that it satisfies both the D.E & the B.C.

Now, we can solve this equation iteratively,

$O(V^0)$, $\quad U_I(t) = \mathbb{1}$

$O(V^1)$, $\quad U_I(t) = \mathbb{1} - \frac{i}{\hbar}\int_0^t V_I(t')\, dt'$

$O(V^2)$, $\quad U_I(t) = \mathbb{1} - \frac{i}{\hbar}\int_0^t V_I(t')\left[\mathbb{1} - \frac{i}{\hbar}\int_0^{t'} V_I(t'')\, dt''\right] dt'$

$$= \mathbb{1} - \frac{i}{\hbar}\int_0^t V_I(t')\, dt' + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\; V_I(t')\, V_I(t'')$$

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the $O(V^n)$ term in $U_I(t)$ is

$$\boxed{\left(\frac{-i}{\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\; V_I(t_1)\, V_I(t_2) \cdots V_I(t_n)}$$

The infinite series expansion of the time evolution operator in power of V_I in the <u>interaction picture</u> is called the Dyson series.

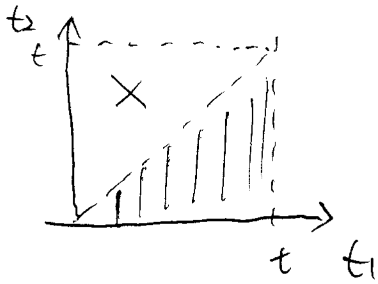
It can be simplified by using the time-order product.

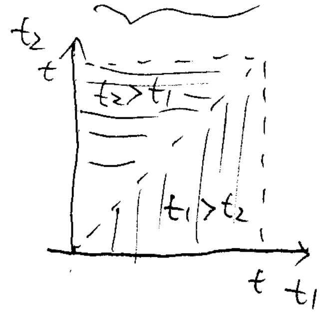
For example, two operators $A(t)$ & $B(t)$,

$$T[A(t), B(t')] = \begin{cases} A(t)B(t') & \text{if } t>t' \\ B(t')A(t) & \text{if } t'>t \end{cases}$$

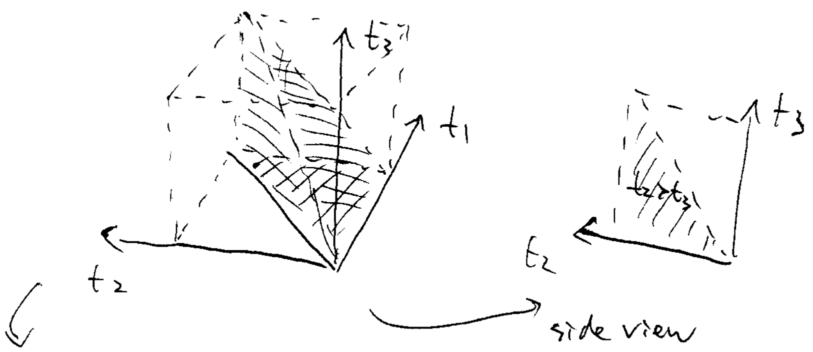
Then the 2nd order piece in $U_I(t)$ can be written as

$$\left(\frac{-i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \, V_I(t_1) V_I(t_2) = \frac{1}{2}\left(\frac{-i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^t dt_2 \, T[V_I(t_1), V_I(t_2)]$$





Similarly, for the 3rd order piece



the volume is

$$\left(\frac{1}{2}\right)\int_0^1 dt_3 \, (t_3)^2 = \frac{1}{6}$$

$\Rightarrow$ $\frac{1}{6}$ volume of the cube

3! ways to order the 3 times.

$$\left(\frac{-i}{\hbar}\right)^3 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \; V_I(t_1) V_I(t_2) V_I(t_3)$$

$$= \left(\frac{1}{3!}\right)\left(\frac{-i}{\hbar}\right)^3 \int_0^t dt_1 dt_2 dt_3 \, T[V_I(t_1), V_I(t_2), V_I(t_3)]$$

$\Rightarrow$ The n-th order piece becomes

$$\boxed{\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n \int_0^t dt_1 dt_2 \cdots dt_n \, T[V_I(t_1), V_I(t_2), \cdots V_I(t_n)]}$$

★ Therefore, with the help of this new notation, the time-evolution operator in the interaction picture can be written simply as

$$\boxed{U_I(t) = T\left[\exp\left(-i\int_0^t V_I(t')\,dt'\right)\right]}$$

(this expression is understood in terms of its Taylor expansion)

• side remark:

Actually, the Path-Integral formulation gives this expression right away

$$\int Dx(t)\, e^{\frac{iS[x(t)]}{\hbar}}\, x(t_n)\cdots x(t_1) = \langle x_f, t_f |\, T[x(t_n)\cdots x(t_1)]\, |\, x_i, t_i\rangle$$

dictated by H ; Heisenberg operators.

$$S[x(t)] = \int_{t_i}^{t_f} dt\,(p\dot{q} - H) = \int_{t_i}^{t_f} dt\,[p\dot{q} - H_0 - V]$$

$$= S_0[x(t)] - \int_{t_i}^{t_f} dt\, V$$

$$\Rightarrow \int Dx(t)\, e^{\frac{iS[x(t)]}{\hbar}} = \int Dx(t)\, e^{\frac{iS_0[x(t)]}{\hbar}}\, e^{-i\int_{t_i}^{t_f} dt\,\frac{V}{\hbar}}$$

$$= \langle x_f, t_f |\, T\left[e^{-i\int_{t_i}^{t_f}\frac{V_I\,dt}{\hbar}}\right] |\, x_i, t_i\rangle$$

dictated by H_0

• Also, the time evolution to the full H is related to

$$U(t) = e^{-\frac{iH_0 t}{\hbar}}\, U_I(t)$$

& $$i\hbar\frac{d}{dt}U(t) = H\,U(t)$$

check:

$$i\hbar\frac{d}{dt}\left(e^{-\frac{iH_0t}{\hbar}}U_I(t)\right) = H_0 e^{-\frac{iH_0t}{\hbar}}U_I(t) + e^{-\frac{iH_0t}{\hbar}}\left(i\hbar\frac{d}{dt}U_I(t)\right)$$

$$= \left[H_0 + e^{-\frac{iH_0t}{\hbar}}V_I\, e^{+\frac{iH_0t}{\hbar}}\right]e^{-\frac{iH_0t}{\hbar}}U_I(t)$$

$$= (H_0 + V)\,U(t) = H\,U(t)$$

★

Transition Probability

$H_0|i\rangle = E_i|i\rangle$ and $H_0|f\rangle = E_f|f\rangle$, stay there forever.

With V, we would like to know at $t=0$, $i \rightarrow t=t_f$, f

In the Schrödinger picture

$$A(i\to f,t) = \langle f|U(t)|i\rangle$$
$$= \langle f|\, e^{\frac{-iH_0t}{\hbar}}\, U_I(t)\,|i\rangle$$
$$= e^{\frac{-iE_ft}{\hbar}}\,\langle f|U_I(t)|i\rangle$$

Then the probability

$$P(i\to f,t) = |A(i\to f,t)|^2 = |\langle f|U_I(t)|i\rangle|^2$$

※ Note that, $|f\rangle$ must be the eigenstate of H_0!

otherwise, the probability calculated in Int pic. may not be the same as that in Schrödinger picture.

And $\langle f|U_I(t)|i\rangle$ can be worked out with the Dyson series.