Scattering

- Why scattering?

$\Delta x \Delta p \geq \frac{\hbar}{2}$, in order to probe $\Delta x \simeq d$

we need momentum transfer $\Delta p \gtrsim \frac{\hbar}{2d}$

$d \sim 0.2\,fm\ (= 2\times 10^{-16} m)$ $\quad E \sim 1\,GeV$
size of proton

$\sim 2\times 10^{-10} m\ (2\,Å)$ $\quad E \sim 1\,keV$
size of atom

- Rutherford α particle on gold foil $\Longleftrightarrow$ J. J. Thomson
nuclei plum-pudding model

- On the other hand, resonance scattering is used to study the excitation spectrum of a system.

Lippman - Schwinger Eq.

We first study t-independ formalism for scattering

short-ranged potential $V(x)$ around $\vec{x} \sim 0$

$$\left[-\frac{\hbar^2}{2m} \vec{\nabla}^2 + V(x) \right] \psi(x) = E \psi(x)$$

when $|x| \gtrsim a$, "the size" of the scatterer, $V(x) \simeq 0$

for $|x| \gg a$,

$$\underbrace{-\frac{\hbar^2}{2m} \vec{\nabla}^2}_{H_0} \psi = E\psi, \quad \Rightarrow \text{plane wave } \langle \vec{x} | \psi \rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} e^{i\vec{k}\cdot\vec{x}}$$

$E = \frac{\hbar^2 \vec{k}^2}{2m}$, note, it is not normalizable.
we just follow Sakurai's notation

$$(H_0 + V(x)) |\psi\rangle = E |\psi\rangle$$

or $$(E - H_0) |\psi\rangle = V(x) |\psi\rangle$$

Naively, $$|\psi\rangle = \frac{1}{E - H_0} V(x) |\psi\rangle$$

Moreover, the solution is not unique, we can add a plane wave $|\phi\rangle$ to the solution.

$$\boxed{|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0 + i\epsilon} V(x)\,|\psi\rangle}$$

This is called Lippman-Schwinger Eq.

- ☆ Meaning of the eq.
- ☆ To avoid the singularity, $(+i\epsilon)$.

Then apply $(E-H_0)$ on both sides of the eq.

$$(E-H_0)\,|\psi\rangle = 0\,|\phi\rangle + V(x)\,|\psi\rangle$$

$\Rightarrow$ Indeed, this is the solution to the Schrödinger Eq.

In the x-rep.

$$\psi(x) = \langle \vec{x}|\psi\rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} e^{i\vec{k}\cdot\vec{x}} + \langle \vec{x}|\frac{1}{E-H_0+i\epsilon} V(x)\,|\psi\rangle$$

The 2nd term :

$$\int d^3\vec{x}'\ \langle\vec{x}|\frac{1}{E-H_0+i\epsilon}|\vec{x}'\rangle\langle\vec{x}'|V(x)|\psi\rangle$$

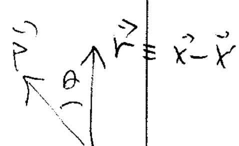
$$= \int d^3\vec{x}'\ \underbrace{\langle\vec{x}|\frac{1}{E-H_0+i\epsilon}|\vec{x}'\rangle}_{G(\vec{x},\vec{x}')} V(\vec{x}')\,\psi(\vec{x}')$$

This is the Green's function as you learnt in classical E&M.

We can work out the Green's function. by inserting the complete set of states.

$$G(\vec{x},\vec{x}') = \int d^3\vec{p}\ \langle\vec{x}|\vec{p}\rangle \frac{1}{E - \frac{\vec{p}^2}{2m} + i\epsilon}\langle\vec{p}|\vec{x}'\rangle$$

$$= \int d^3\vec{p}\ \frac{e^{i\frac{\vec{x}\cdot\vec{p}}{\hbar}}}{(2\pi\hbar)^{\frac{3}{2}}}\ \frac{1}{E - \frac{\vec{p}^2}{2m} + i\epsilon}\ \frac{e^{-i\frac{\vec{x}'\cdot\vec{p}}{\hbar}}}{(2\pi\hbar)^{\frac{3}{2}}}$$

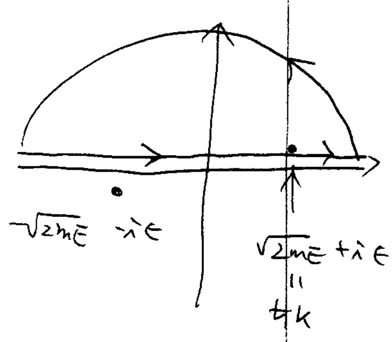


$$= \int_0^\infty 2\pi p^2 dp \int_{-1}^{1} d(\cos\theta)\ \frac{1}{(2\pi\hbar)^3}\ \frac{e^{i\frac{pr}{\hbar}\cos\theta}}{E - \frac{p^2}{2m} + i\epsilon}$$

$$= \int_0^\infty \frac{2\pi}{(2\pi\hbar)^3}\, p^2 dp \left(\frac{\hbar}{ipr}\right) \frac{e^{\frac{ipr}{\hbar}} - e^{-\frac{ipr}{\hbar}}}{E - \frac{p^2}{2m} + i\epsilon} \longrightarrow \text{change variable } x = -p$$

$$= \frac{1}{(2\pi\hbar)^2} \int_{-\infty}^{\infty} \left(\frac{-2m}{ir}\right) \frac{p\,dp\, e^{\frac{ipr}{\hbar}}}{p^2 - 2mE - i\epsilon}$$

$$= -\frac{2m}{4\pi r \hbar^2}\, e^{ikr}$$



Therefore

$$\psi(\vec{x}) = \underbrace{\frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\, e^{i\vec{k}\cdot\vec{x}}}_{\text{incident particle with a fixed 3-momentum}} - \frac{2m}{\hbar^2}\int dx' \underbrace{\frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|}}_{\text{spherical wave originated from the scatterer}} V(\vec{x}')\,\psi(\vec{x}')$$

✱ If we had chosen the opposite B.C. $\frac{1}{E - H_0 - i\epsilon}$

2nd term : coming from infinity and converging at the origin

$\longrightarrow$ practically impossible to arrange.

Typically, the exp is done by placing the detector far away from the scatterer, $|\vec{x}| \gg a$,

The integration over $\vec{x}'$ is limited within the size "a".

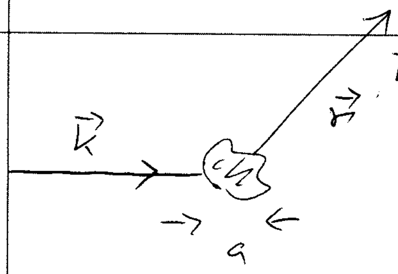
$$\Rightarrow \quad |\vec{x}| \gg |\vec{x}'|$$

and $|\vec{x} - \vec{x}'| \simeq \sqrt{\vec{x}^2 - 2\vec{x}\cdot\vec{x}'} \simeq |\vec{x}| - \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|}$

In this limit, the Lippmann-Schwinger Eq becomes

$$\psi(x) \simeq \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\, e^{i\vec{k}\cdot\vec{x}} - \frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int dx'\, e^{-i\vec{k}'\cdot\vec{x}'}\, V(\vec{x}')\,\psi(\vec{x}')$$

$r = |\vec{x}|$, $\vec{k}' = |k|\,\hat{r}$



note $|\vec{k}'| = |\vec{k}|$

$$\Rightarrow \psi(\vec{x}) \simeq \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\left[e^{i\vec{k}\cdot\vec{x}} + \frac{e^{ikr}}{r} f(\vec{k}', \vec{k})\right]$$

where $$f(\vec{k}', \vec{k}) = -\frac{(2\pi\hbar)^{\frac{3}{2}}}{4\pi}\frac{2m}{\hbar^2}\int dx'\, e^{-i\vec{k}'\cdot\vec{x}'}\, V(x')\,\psi(x')$$

$$\frac{d\sigma}{d\Omega}d\Omega = \frac{\text{\# of particles scattered into } d\Omega \text{ per unit time}}{\text{incident particle flux}}.$$

$$= \frac{r^2 d\Omega\, |\vec{j}_{scatt}|}{|\vec{j}_{incid}|} = |f(\vec{k}', \vec{k})|^2\, d\Omega$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = |f(\vec{k}', \vec{k})|^2} \qquad \sigma_{total} = \int d\Omega \frac{d\sigma}{d\Omega}$$

$\Downarrow$ note: no incident wave included

$\Downarrow$ next topic: optical theorem

The above discussion can be refined by considering wave packet instead of plane wave.

$$\psi \sim \left(\frac{1}{2\pi d^2}\right)^{\frac{3}{4}} e^{i\vec{k}\cdot\vec{x}}\, e^{-i\left(\frac{\hbar\vec{k}^2}{2m}t\right)}\, e^{-\frac{(\vec{x}-\hbar\vec{k}t)^2}{4m^2d^2}}$$

As long as $d \gg a$, no diff between plane-wave and wave packet

Born approximation

replacing $|\psi\rangle$ on the RHS by the incident plane wave $\frac{1}{(\sqrt{2\pi\hbar})^3} e^{i\vec{k}\cdot\vec{x}}$

$$\Rightarrow \psi^{(1)}(x) \simeq \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\left[e^{i\vec{k}\cdot\vec{x}} - \frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d^3x'\, e^{-i\vec{k}'\cdot\vec{x}'} V(\vec{x}')\, e^{i\vec{k}\cdot\vec{x}'}\right]$$

$$= \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\left[e^{i\vec{k}\cdot\vec{x}} - \frac{2m}{\hbar}\frac{e^{ikr}}{4\pi r}\int d^3x'\, e^{i\vec{q}\cdot\vec{x}'} V(\vec{x}')\right] \qquad \vec{q} \equiv \vec{k}-\vec{k}'$$

$$f^{(1)}(\vec{q}) = \frac{-2m}{\hbar^2}\frac{(2\pi\hbar)^{\frac{3}{2}}}{4\pi}\int \frac{d^3x'}{(2\pi\hbar)^{\frac{3}{2}}} e^{i\vec{q}\cdot\vec{x}'} V(\vec{x}')$$

$$= -m\sqrt{\frac{2\pi}{\hbar}}\quad \tilde{V}(\vec{q})$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{2\pi m^2}{\hbar}\,|\tilde{V}(\vec{q})|^2}$$

This is to be compared with the Dyson series ~ and we use $\frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}}$ in box normalization

$$\langle \vec{k}'|U_I|\vec{k}\rangle = \cdots + \left(-\frac{\hbar}{i}\right)\int_0^t dt' \,\langle k'|V_I(t')|\vec{k}\rangle$$

$$W = \frac{2\pi}{\hbar}\sum_{k'}\delta\left(\frac{\hbar^2\vec{k}'^2}{2m} - \frac{\hbar^2\vec{k}^2}{2m}\right)|\langle\vec{k}'|V|\vec{k}\rangle|^2$$

$$= \frac{2\pi}{\hbar}\sum_{k'}\delta\left(\frac{\hbar^2\vec{k}'^2}{2m} - \frac{\hbar^2\vec{k}^2}{2m}\right)\frac{1}{V^2}\left|\int d^3x'\, e^{i\vec{q}\cdot\vec{x}'} V(\vec{x}')\right|^2$$

$$\vec{k} = \frac{2\pi}{L}\vec{n} \quad\Rightarrow\quad d\vec{n}^3 = \frac{V}{(2\pi)^3}dk^3$$

$$\frac{dW}{d\Omega}\Delta\Omega = \frac{2\pi}{\hbar}\frac{V}{(2\pi)^3}\frac{m}{\hbar^2}\Delta\Omega\int \frac{k'^2 dk'}{k'}\,\delta(k'-k)\frac{1}{V^2}\left|\int d^3x'\, e^{i\vec{q}\cdot\vec{x}'} V(\vec{x}')\right|^2$$

$$\Rightarrow \frac{dW}{d\Omega} = \frac{m}{\hbar^2(2\pi)^2}\frac{k'}{V}\left|\int d^3x'\, e^{i\vec{q}\cdot\vec{x}} V(x')\right|^2$$

$$\vec{j} = \frac{1}{V}\cdot\vec{v} = \frac{1}{V}\frac{\hbar\vec{k}}{m} \quad\Rightarrow\quad \boxed{\frac{d\sigma}{d\Omega} = \frac{\frac{dW}{d\Omega}}{|\vec{j}|} = \frac{2\pi m^2}{\hbar}\,|\tilde{V}(\vec{q})|^2}$$

☆ The two treatments have the same result!!