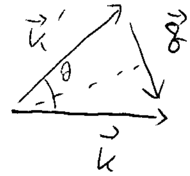
Example of Coulomb Scattering

$$V(r) = \frac{Z}{r}$$



The Born approximation gives

$$f(\vec{q}) = -m\sqrt{\frac{2\pi}{\hbar}}\ \hat{V}(\vec{q}) \qquad \vec{q} = \vec{k} - \vec{k}', \quad |q| = 2|\vec{k}|\sin\frac{\theta}{2}$$

Brute force Fourier transformation.

$$\hat{V}(\vec{q}) = \int \frac{d^3x}{(2\pi\hbar)^{\frac{3}{2}}} e^{i\vec{q}\cdot\vec{x}} V(\vec{x}) = \frac{2\pi}{(2\pi\hbar)^{\frac{3}{2}}} \int_0^\infty r^2 dr \int_{-1}^{1} d(\cos\theta)\, e^{iqr\cos\theta}\, V(r)$$

$$= \frac{2\pi}{(2\pi\hbar)^{\frac{3}{2}}} \int_0^\infty \frac{r\,dr}{iq} \left(e^{iqr} - e^{-iqr}\right) V(r)$$

$$= \frac{4\pi}{(2\pi\hbar)^{\frac{3}{2}}} \frac{1}{q}\, Im\left[\int_0^\infty e^{iqr} V(r)\, r\, dr\right]$$

$V(r) = \frac{Z}{r} \Rightarrow$ div. to tame it $\Rightarrow V(r) = \frac{Z}{r} e^{-\epsilon r}$

and let $\epsilon \to 0$ at the end of calculation.

$$\lim_{\epsilon\to 0} Im \int_0^\infty e^{iqr} e^{-\epsilon r} dr = \lim_{\epsilon\to 0} Im\left(\frac{-1}{iq - \epsilon}\right) = \lim_{\epsilon\to 0} Im\left(\frac{\epsilon + iq}{\epsilon^2 + q^2}\right)$$

$$= \lim_{\epsilon\to 0} \frac{q}{q^2 + \epsilon^2} = \frac{1}{q}$$

$$\Rightarrow \hat{V}(\vec{q}) = \frac{4\pi}{(2\pi\hbar)^{\frac{3}{2}}} \frac{Z}{q^2} \Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{4\pi Z}{(2\pi\hbar)^{\frac{3}{2}} q^2}\right)^2 m^2 \frac{2\pi}{\hbar}$$

$$= \frac{Z^2 m^2}{4\hbar^4 k^4 \sin^4\frac{\theta}{2}} = \boxed{\frac{Z^2}{16 E^2 \sin^4\frac{\theta}{2}}}$$

For Yukawa

$$\hat{V}(\vec{q}) = \frac{4\pi Z}{(2\pi\hbar)^{\frac{3}{2}}(q^2 + \mu^2)}$$

Born Expansion.

Go to higher order by iteratively inserting the R.H.S. of the Eq at a given order in V back into the $|\psi\rangle$. then we have the infinite series.

$$|\psi\rangle = |\phi\rangle + \frac{1}{E-H_0+i\epsilon} V(x)\,|\psi\rangle$$

1st $$|\psi\rangle^{(1)} = |\phi\rangle + \frac{1}{E-H_0+i\epsilon} V(x)\,|\phi\rangle$$

2nd $$|\psi\rangle^{(2)} = |\phi\rangle + \frac{1}{E-H_0+i\epsilon} V(x)\left[\,|\phi\rangle + \frac{1}{E-H_0+i\epsilon} V(x)\,|\phi\rangle\right]$$

$$= |\phi\rangle + \frac{1}{E-H_0+i\epsilon} V(x)\,|\phi\rangle + \frac{1}{E-H_0+i\epsilon} V(x) \frac{1}{E-H_0+i\epsilon} V(x)\,|\phi\rangle$$

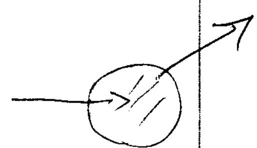
higher order $$|\psi\rangle = \underbrace{|\phi\rangle}_{①} + \underbrace{\frac{1}{E-H_0+i\epsilon} V(x)\,|\phi\rangle}_{②} + \underbrace{\frac{1}{E-H_0+i\epsilon} V(x) \frac{1}{E-H_0+i\epsilon} V(x)\,|\phi\rangle}_{③}$$

$$+ \frac{1}{E-H_0+i\epsilon} V(x) \frac{1}{E-H_0+i\epsilon} V(x) \frac{1}{E-H_0+i\epsilon} V(x)\,|\phi\rangle + \cdots$$

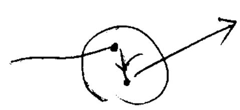
This is called Born expansion.

① : doesn't get scattered $\longrightarrow$

② : get scattered at a point in the potential, then propagate outward by the $\frac{1}{E-H_0}$ operator



③ get scattered twice in the potential



More formally, an operator called T-matrix is used in scattering problems. The definition is

$$V|\psi\rangle = T\,|\phi\rangle, \qquad |\phi\rangle : \text{plane wave}$$

$$f(\vec{k}',\vec{k}) = -4\pi^2 m\hbar \langle \hbar\vec{k}'|V(x)|\psi\rangle = -4\pi^2 m\hbar \langle \hbar\vec{k}'|T|\hbar\vec{k}\rangle$$

T : transition matrix from $\hbar\vec{k}$ to $\hbar\vec{k}'$.

Multiply $V(x)$ on both sides of the Lippmann-Schwinger Eq.

$$V(x)|\psi\rangle = V(x)|\phi\rangle + V(x)\frac{1}{E-H_0+i\epsilon}V(x)|\psi\rangle$$

hence

$$T|\phi\rangle = V(x)|\phi\rangle + V(x)\frac{1}{E-H_0+i\epsilon}T|\phi\rangle$$

or

$$T = V(x) + V(x)\frac{1}{E-H_0+i\epsilon}T.$$

In other word, a formal solution to the T-matrix is

$$T = \frac{1}{1-V(x)\frac{1}{E-H_0+i\epsilon}}V(x)$$

$$= \left(1 + V(x)\frac{1}{E-H_0+i\epsilon} + V(x)\frac{1}{E-H_0+i\epsilon}V(x)\frac{1}{E-H_0+i\epsilon} + \cdots\right)V(x)$$

$$= V(x) + V(x)\frac{1}{E-H_0+i\epsilon}V(x) + V(x)\frac{1}{E-H_0+i\epsilon}V(x)\frac{1}{E-H_0+i\epsilon}V(x) + \cdots$$

And we obtain the Born expansion again.

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e.g. In the x space, the 2nd Born term is given by

$$\langle\vec{x}|\frac{1}{E-H_0+i\epsilon}V(x)\frac{1}{E-H_0+i\epsilon}V(x)|\phi\rangle$$

$$= \int dx'\,dx''\,\langle\vec{x}|\frac{1}{E-H_0+i\epsilon}V|x'\rangle\langle x'|\frac{1}{E-H_0+i\epsilon}|x''\rangle\langle x''|V|\phi\rangle$$

$$= \int dx'\,dx''\left(\frac{-2m}{\hbar^2}\right)\frac{e^{ik|x-x'|}}{4\pi|x-x'|}V(x')\left(\frac{-2m}{\hbar^2}\right)\frac{e^{ik|x'-x''|}}{4\pi|x'-x''|}V(x'')\underbrace{\phi(x'')}_{=\frac{e^{i\vec{k}\cdot\vec{x}''}}{(2\pi\hbar)^{3/2}}}$$

Validity of Born Approximation

☆ $$|\psi(x) - \phi(x)| << |\phi(x)|$$

☆ Namely,
$$\left| \frac{2m}{\hbar^2} \int dx' \frac{e^{ik|x-x'|}}{4\pi|x-x'|} V(x') e^{i\vec{k}\cdot\vec{x}'} \right| << 1$$

☆ In particular, near $\vec{x} \simeq 0$, where the potential is the strongest.

☆ Smooth central potential, with magnitude $\sim O(V_0)$, range $\sim O(a)$
taking $\vec{k}$ alone $\vec{z}$

$$\frac{2m}{\hbar^2} \left| \int d\vec{x} \frac{e^{ikr}}{4\pi r} V(x) e^{ikz} \right| << 1$$

○ When $k << \frac{1}{a}$, the 2 phases can be ignored.

$$\Rightarrow \quad \frac{2m}{\hbar^2} |V_0| a^2 << 1, \quad \text{for } k << a^{-1}$$

○ for $k >> a^{-1}$, phase factor oscillates rapidly $\Rightarrow$ Stationary approximation

expand $ik(r+z)$ around $z = -r$

$$\rightarrow i \frac{k(x^2+y^2)}{r} + O(x^3, y^3)$$

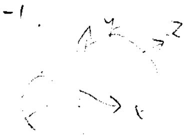
, The Gaussian integral over x & y $\rightarrow \frac{\pi r}{k}$,
z integral from $-a \to 0$.

$$\Rightarrow \quad \frac{2m}{\hbar^2} \frac{a}{4k} |V_0| << 1, \quad \text{for } k >> a^{-1}$$

On the other hand,

for $q << a^{-1}$,
$$f^{(1)}(\vec{q}) = -\frac{1}{4\pi} \frac{2m}{\hbar^2} \int dx \, V(x) e^{i\vec{q}\cdot\vec{x}}$$
$$\sim -\frac{1}{4\pi} \frac{2m}{\hbar^2} V_0 \frac{4\pi}{3} a^3$$

for $q >> a^{-1}$.



$x, y \rightarrow a$
$z \rightarrow \frac{1}{q}$
$$\Rightarrow f^{(1)}(\vec{q}) \sim -\frac{1}{4\pi} \frac{2m}{\hbar^2} V_0 \frac{\pi a^2}{q}$$

$$\Rightarrow \sigma \sim \begin{cases} \frac{1}{4\pi}\left(\frac{2m}{\hbar^2} V_0 \frac{4\pi}{3} a^3\right)^2, & k << a^{-1} \\ \frac{1}{4\pi}\left(\frac{2m}{\hbar^2} V_0 \frac{\pi a^2}{q}\right)^2, & k >> a^{-1} \end{cases} \Rightarrow << \begin{cases} \frac{4}{9}\pi a^2 \\ 4\pi a^2 \end{cases}$$

much smaller then the geometrial area $4\pi a^2$