Optical theorem

Statement : $\mathrm{Im}\, f(\theta=0) = \frac{k}{4\pi}\sigma_{tot}$

$f(\theta=0) \equiv f(\vec{k},\vec{k})$: scattering in the forward direction

$\sigma_{tot} \equiv \int \frac{d\sigma}{d\Omega} d\Omega$

Proof :

$$f(\theta=0) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\int d^3x' \frac{e^{-i\vec{k}\cdot x'}}{(2\pi)^{\frac{3}{2}}} V(x') \langle x'|\psi\rangle$$

$$= -\frac{(2\pi)^3}{4\pi}\frac{2m}{\hbar^2}\langle\vec{k}|T|\vec{k}\rangle$$

from Lippman-Schwinger Eq.

$$|\psi\rangle = |\vec{k}\rangle + \frac{1}{E-H_0+i\epsilon}V|\psi\rangle$$

so $|\vec{k}\rangle = |\psi\rangle - \frac{1}{E-H_0+i\epsilon}V|\psi\rangle$

and $\langle\vec{k}| = \langle\psi| - \langle\psi|V\frac{1}{E-H_0-i\epsilon}$ ⇑

$\int \frac{f(x)\,dx}{x-i\epsilon}$

↓

$\frac{1}{x-i\epsilon} = Pr\left(\frac{1}{x}\right) + i\pi\delta(x)$

inside the integration

Therefore

$$\mathrm{Im}\langle\vec{k}|T|\vec{k}\rangle = \mathrm{Im}\langle\vec{k}|V|\psi\rangle$$

$$= \mathrm{Im}\left[\left(\langle\psi| - \langle\psi|V\frac{1}{E-H_0-i\epsilon}\right)V|\psi\rangle\right]$$

$$= \mathrm{Im}\langle\psi|V|\psi\rangle - \mathrm{Im}\langle\psi|V\,Pr\left(\frac{1}{E-H_0}\right)V|\psi\rangle - \pi\langle\psi|V\delta(E-\hat{H}_0)V|\psi\rangle$$

(the first two terms are 0) because V is hermition ⇒ real

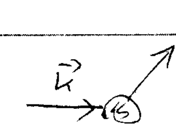
$$\mathrm{Im}\langle\vec{k}|T|\vec{k}\rangle = -\pi\langle\vec{k}|T^\dagger\delta(E-\hat{H}_0)T|\vec{k}\rangle$$

$$= -\pi\int d^3k'\, d^3k''\, \langle\vec{k}|T^\dagger|\vec{k}'\rangle\langle\vec{k}'|\delta(E-\hat{H}_0)|\vec{k}''\rangle\langle\vec{k}''|T|\vec{k}\rangle$$

$$= -\pi\int d^3k'\, d^3k''\, \langle\vec{k}|T^\dagger|\vec{k}'\rangle\delta^3(\vec{k}'-\vec{k}'')\,\delta\left(E-\frac{\hbar^2k''^2}{2m}\right)\langle\vec{k}''|T|\vec{k}\rangle$$

$$= -\pi\frac{m}{\hbar^2}\int\frac{d^3k'}{k'}\langle\vec{k}|T^\dagger|\vec{k}'\rangle\,\delta\left(k'-\sqrt{\frac{2mE}{\hbar^2}}\right)\langle\vec{k}'|T|\vec{k}\rangle$$

$$= -\frac{\pi m}{\hbar^2}\int_0^\infty\frac{k'^2dk'}{k'}d\Omega\;\delta\left(k'-\sqrt{\frac{2mE}{\hbar^2}}\right)|\langle\vec{k}'|T|\vec{k}\rangle|^2$$



$$\Rightarrow \text{Im} \langle \vec{k} | T | \vec{k} \rangle = -\frac{\pi m k}{\hbar^2} \int d\Omega \, | \langle \vec{k}' | T | \vec{k} \rangle |^2$$

$$= -\frac{\pi m k}{\hbar^2} \int d\Omega \left(-\frac{4\pi}{(2\pi)^3} \right) \frac{\hbar^2}{2m} \, | f(\theta) |^2$$

$$= \frac{k}{4\pi} \sigma_{total}$$

or

$$\boxed{\sigma_{total} = \frac{4\pi}{k} \text{Im} f(\theta)}$$

The meaning is clear

☆ The scattered wave takes away the prob to diff direct.

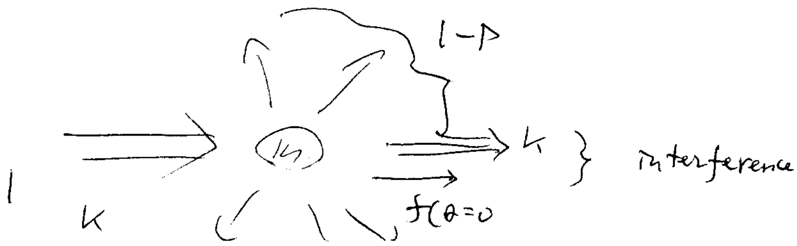
⇒ The total prob for the particle to go to the forward direction should decrease.

☆ The decrease is caused by the interference between the unscattered and scattered waves in the forward direction.

$\propto f(\theta = 0)$

☆ The amount of decrease in the forward direction

= the total prob at other directions = σ_{total}



* Note, in the case that not all scattering is elastic

$$\sigma_{tot} = \sigma_{elastic} + \sigma_{absoption}$$

Cancellation between the incident and scattered wave is relavent only for the scattered wave which can interfere with the incident wave,

$$\Rightarrow \sigma_{tot} = \frac{4\pi}{k} \text{Im} \, f_{k,el}(0)$$

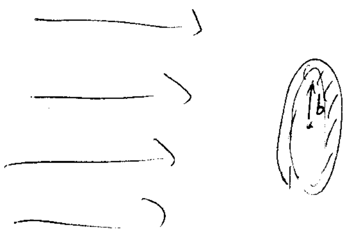
Scattering by a spherically sym potential $V(\vec{r}) = V(r)$

$\Rightarrow$ angular momentum is conserved.

$\Rightarrow$ how states with definite angular momentum are affected by the scatterer.

$\Rightarrow$ partial wave analysis.

Some reminding on the plane wave decomposition.



$L = mvb$

The Schrödinger Eq for the free particle is

$$\left(\frac{-d^2}{d\rho^2} + \frac{\ell(\ell+1)}{\rho^2}\right) u_\ell(\rho) = u_\ell(\rho), \quad \rho = kr$$

define $d_\ell = \frac{d}{d\rho} + \frac{\ell}{\rho}$ and $d_\ell^+ = -\frac{d}{d\rho} + \frac{\ell}{\rho}$

then

$$d_\ell^+ d_\ell = \frac{-d^2}{d\rho^2} + \frac{-\ell}{\rho^2} + \frac{\ell^2}{\rho^2} = \frac{-d^2}{d\rho^2} + \frac{\ell(\ell+1)}{\rho^2}$$

$$d_\ell d_\ell^+ = \frac{-d^2}{d\rho^2} + \frac{(\ell-1)\ell}{\rho^2} = d_{\ell-1}^+ d_{\ell-1}$$

and $d_\ell^+ d_\ell u_\ell(\rho) = u_\ell(\rho) \Rightarrow d_\ell d_\ell^+ d_\ell u_\ell(\rho) = d_\ell u_\ell(\rho)$

$$\Rightarrow (d_{\ell-1}^+ d_{\ell-1})\left[d_\ell u_\ell(\rho)\right] = \left[d_\ell u_\ell(\rho)\right]$$

$\Rightarrow d_\ell u_\ell(\rho) \cong u_{\ell-1}(\rho)$, similarly $d_\ell^+ u_{\ell-1}(\rho) \simeq u_\ell(\rho)$

$$\begin{cases} d_\ell^+ \rho j_{\ell-1}(\rho) = \rho j_\ell(\rho) \\ d_\ell \rho j_\ell(\rho) = \rho j_{\ell-1}(\rho) \end{cases} \qquad j_\ell(\rho) \equiv \frac{u_\ell(\rho)}{\rho}$$

with the $\rho j_0(\rho) = \sin\rho$, for $\ell = 0$, $j_0(\rho=0) = 0$

Also, rewrite

$$d_\ell^+ = -\frac{d}{d\rho} + \frac{\ell}{\rho} = -\rho^\ell \frac{d}{d\rho}\left(\frac{1}{\rho^\ell}\right)$$

$$\rho j_\ell(\rho) = (d_\ell^+ d_{\ell-1}^+ \cdots d_2^+ d_1^+) \sin\rho = (-1)^\ell \rho^{\ell+1} \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \frac{\sin\rho}{\rho}$$

$$\Rightarrow j_\ell(\rho) = (-\rho)^\ell \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \frac{\sin\rho}{\rho}$$

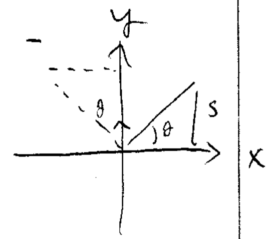
j_ℓ is the so called spherical Bessel function.

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x$$

$$\vdots$$



for $r \to \infty$ (or $\rho \to \infty$)

$$j_\ell(\rho) = (-\rho)^\ell \left(\frac{1}{\rho}\frac{d}{d\rho}\right) \cdots \left(\frac{1}{\rho}\frac{d}{d\rho}\right)\frac{\sin\rho}{\rho}$$

$$\sim (-\rho)^\ell \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{\ell-1}\left[\frac{1}{\rho^2}\frac{d}{d\rho}\sin\rho + \cancel{O\left(\frac{1}{\rho^3}\right)}\right]$$

$$\xrightarrow{\sim} (-1)^\ell \rho^\ell \frac{1}{\rho^{\ell+1}}\left(\frac{d}{d\rho}\right)^\ell \sin\rho = \frac{1}{\rho}\sin\left(\rho - \frac{\pi}{2}\ell\right)$$

$$= \frac{1}{kr}\sin\left(kr - \frac{\pi\ell}{2}\right) = \frac{1}{2ikr}e^{-\frac{\ell\pi}{2}i}\left(e^{ikr} - (-1)^\ell e^{-ikr}\right)$$

ℓ	j_ℓ
0	s
1	−c
2	−s
3	+c
4	s
5	−c
6	−s

Also, we know that

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty}(2\ell+1)\, i^\ell\, j_\ell(kr)\, P_\ell(\cos\theta)$$

$\cos\theta = \hat{k}\cdot\hat{r}$

$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^0$

⇒ no ϕ-dependence

$$\psi(\vec{x}) \sim e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

$$\xrightarrow{r\to\infty} \frac{1}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)\left(e^{ikr} - (-1)^\ell e^{-ikr}\right)P_\ell(\cos\theta) + f(\theta)\frac{e^{ikr}}{r}$$

⇒ plane wave contains both the wave converging to the origin (e^{-ikr}) and the wave emerging from the origin (e^{ikr})

⇒ similarly, the second term should be expanded in terms of the partial waves

$$f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\, f_\ell\, P_\ell(\cos\theta)$$

, emitted from the origin.

$(2\ell+1)$: just convention; f_ℓ: complex coefficient; $P_\ell(\cos\theta)$: also, no ϕ-depend

All physics information is contained in the complex number f_ℓ.

Optical theorem constraint.

$$\sigma_{tot} = \int d\Omega \, |f(\theta)|^2 = \frac{4\pi}{k} \, \mathrm{Im} \, f(\theta=0)$$

L.H.S. $\sigma_{tot} = 2\pi \int_{-1}^{1} d(\cos\theta) \sum_{\ell,\ell'} (2\ell+1)(2\ell'+1) f_\ell^* f_{\ell'} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$

$\because \int_{-1}^{1} d(\cos\theta) \, P_\ell(\cos\theta) P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1} \delta_{\ell\ell'}$

$$= 4\pi \sum_\ell (2\ell+1) \, |f_\ell|^2$$

R.H.S. $\frac{4\pi}{k} \sum_\ell (2\ell+1) \, \mathrm{Im} \, f_\ell \qquad \because P_\ell(\cos\theta=1) = 1$

$$\Rightarrow \quad |f_\ell|^2 = \frac{1}{k} \mathrm{Im} \, f_\ell$$

It can be written as $|1 + 2ik f_\ell|^2 = 1$

check : $1 = 1 + 4k \, \mathrm{Im} \, f_\ell + 4k^2 |f_\ell|^2$ ✓

In other words, $1+2ik f_\ell$ is just a phase. $\equiv e^{2i\delta_\ell}$

$$\Rightarrow \boxed{f_\ell = \frac{e^{2i\delta_\ell} - 1}{2ik} = \frac{e^{i\delta_\ell}}{k} \sin\delta_\ell}$$

δ_ℓ is called the phase shift.

with δ_ℓ, the WF can be expressed as :

$$\psi(x) = \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta) + \frac{e^{ikr}}{r} \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell P_\ell(\cos\theta)$$

$$\xrightarrow[r\to\infty]{} \frac{1}{kr} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta) \left[\frac{e^{\frac{i\pi}{2}\ell}}{2i} \left(e^{i(kr-\frac{\pi}{2}\ell)} - e^{-i(kr-\frac{\pi}{2}\ell)} \right) + \frac{e^{ikr} e^{i\delta_\ell}}{2i} \left(e^{i\delta_\ell} - e^{-i\delta_\ell} \right) \right]$$

$$= \frac{1}{2ikr} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta) \left[e^{2i\delta_\ell} e^{ikr} - (-1)^\ell e^{-ikr} \right]$$

In other words, the radial WF behaves as

$$R_\ell(r) \underset{r\to\infty}{\sim} \frac{1}{2ikr} \frac{1}{\sqrt{4\pi(2\ell+1)}} \left[\underbrace{e^{2i\delta_\ell} e^{ikr}}_{\text{outgoing}} - \underbrace{(-1)^\ell e^{-ikr}}_{\text{flow to origin}} \right]$$

($Y_\ell^0 = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$ is used)

Compare it with the unscattered incident wave

$$\Phi_\ell(r) \underset{r\to\infty}{\sim} \frac{1}{2ikr}\frac{1}{\sqrt{4\pi(2\ell+1)}}\left[1\times e^{ikr} - (-1)^\ell e^{-ikr}\right]$$

⇒ the outgoing scattered wave picks up an extra phase factor due to the interaction.

Why phase? ⇒ to <u>preserve Prob!!</u>

The phase factor is the so called S-matrix element

$S_\ell \equiv e^{2i\delta_\ell}$ it is related to the T-matrix by

$$S_\ell = 1 + i\,T_\ell$$

check $T_\ell = i(1-S) = i e^{i\delta_\ell}\left[e^{-i\delta_\ell} - e^{i\delta_\ell}\right] = 2e^{i\delta_\ell}\sin\delta_\ell$

$= \boxed{2k f_\ell}$ ✓ Same as the definition of T-matrix.

⇒ { S : includes the transmitted wave
T : remove the incident wave, keeps only the scattered wave.

In terms of the phase shift,

$$\boxed{\sigma_{tot} = \sum_\ell \sigma_\ell = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell}$$

① Exact calculation of phase shifts is basically solving the Schrödinger Eq for each partial wave

$$\left[-\frac{1}{r}\frac{d^2}{dr^2}r + \frac{\ell(\ell+1)}{r^2} + \frac{2m}{\hbar^2}V(r)\right]R_\ell(r) = k^2 R_\ell(r)$$

② take the asymptotic limit $r\to\infty$ ao write R_ℓ as

$$j_\ell(kr)\cos\delta_\ell + n_\ell(kr)\sin\delta_\ell$$

relative phase determines the phase shift δ_ℓ

⇒ cross section.

Unitarity Limit

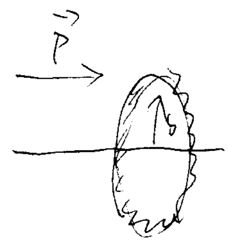
$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell \leq \boxed{\frac{4\pi}{k^2}(2\ell+1)} \qquad (\text{for } \delta_\ell = \pm\frac{\pi}{2})$$

This is called unitarity limit
as a consequence of the unitarity of the S-matrix

(* This also gives interesting bounds on SM Higgs mass!)

Classical picture

Inject a particle with p, impact parameter b



$L = pb$, on the otherhand, $L = \ell\hbar$

for $\hbar\ell < L < \hbar(\ell+1)$

$$\frac{\ell}{k} < b < \frac{\ell+1}{k}$$

Say, the particle gets scattered with 100% prob when entering this ring, the classical cross section would be

$$\sigma_\ell = \pi\left(\frac{\ell+1}{k}\right)^2 - \pi\left(\frac{\ell}{k}\right)^2 = \boxed{\frac{\pi}{k^2}(2\ell+1)}$$

We obtain the same result, except the factor 4

(the diff between QM & CM)

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e.g. **hard sphere scattering**

an impenetrable ball

$$V = \begin{cases} 0 & , r > a \\ \infty & , r < a \end{cases}$$

$\because$ the infinite potential within the radius a

$\Rightarrow$ Born approximation is not appropriate!

$\Rightarrow$ need to solve the Schrödinger Eq,

with B.C, $R_\ell(a) = 0$

At low energy $k << \frac{1}{a}$,

* the centrifugal barrier inhibits the particle from enerting the region of the scatter
* since $\lambda >> a$, only S-wave can "feel" the potential

$$\because a << \frac{1}{k} < b_1 < \frac{2}{k}$$

$$\frac{1}{k} = \frac{\lambda}{2\pi}$$

* The radial Schrödinger Eq. is

$$\left[-\frac{d^2}{dr^2} + \frac{2m}{\hbar^2} V(r)\right] (rR_0(r)) = k^2 (rR_0(r)), \text{ with } R_0(a) = 0$$

$r > a$, $V(r) = 0$, $\Rightarrow$ $rR_0(r) = Ae^{ikr} + Be^{-ikr}$

$r = a$ $\quad 0 = Ae^{ika} + Be^{-ika}$, $\quad B = -Ae^{2ika}$

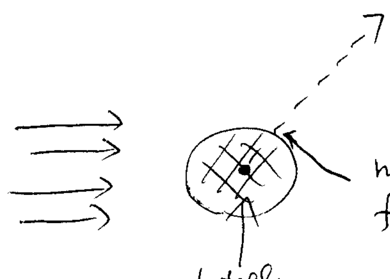
$$\Rightarrow rR_0(r) = Ae^{ika}\left(e^{ik(r-a)} - e^{-ik(r-a)}\right)$$

* To determine the phase shift, we compare this solution to the general expansion

$$R_0(r) \sim \frac{1}{2ikr}\frac{1}{\sqrt{4\pi}}\left[e^{2i\delta_0}e^{ikr} - e^{-ikr}\right]$$

$$\Rightarrow \boxed{\delta_0 = -ka}$$

* The reason is obvious



* The cross section for s-wave scattering

$$\sigma_0 = \frac{4\pi}{k}\sin^2\delta_0 = \frac{4\pi}{k^2}\sin^2 ka \Rightarrow \lim_{ka<<1} \frac{4\pi a^2}{k^2a^2}\sin^2(ka) = \boxed{4}\pi a^2$$

when $k \to 0$

$k >> \frac{1}{a} \to ka << 1$

We see the factor 4 again.

The cross section saturates at $ka = n\pi$

* For hard sphere scattering, $\Rightarrow$ 4 times larger then the classical one!