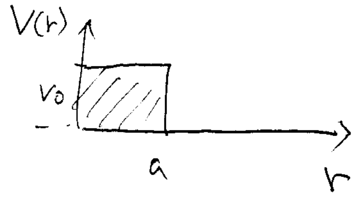
consider a 3-D square well potential

$$V = \begin{cases} 0 & r > a \\ V_0 & r < a \end{cases}$$



$$\left[-\frac{d^2}{dr^2} + \frac{2m}{\hbar^2} V(r) \right] r R_0(r) = k^2 (r R_0(r))$$

$$\mathbb{K}^2 \equiv \frac{2m V_0}{\hbar^2}$$

for $r < a$ $\quad -\frac{d^2}{dr^2}(r R_0(r)) = (k^2 - \mathbb{K}^2)(r R_0(r))$

if $k > \mathbb{K}$ $\quad r R_0(r) = A \sin(\sqrt{k^2 - \mathbb{K}^2}\, r) \quad \because \quad R_0(r=0) = 0$

for $r > a$, $\quad r R_0(r) = B \sin(kr + \delta_0)$

We do the matching at $r = a$

$$\left.\frac{(rR)'}{rR}\right|_{r=a} = \sqrt{k^2 - \mathbb{K}^2} \cot(\sqrt{k^2 - \mathbb{K}^2}\, a) = k \cot(ka + \delta_0)$$

$$\Rightarrow \quad \delta_0 = \tan^{-1}\left[\frac{k}{\sqrt{k^2 - \mathbb{K}^2}} \tan(\sqrt{k^2 - \mathbb{K}^2}\, a) \right] - ka$$

If $k \gg \mathbb{K}$

$$\delta_0 \sim \tan^{-1}[1 \times \tan ka] - ka = 0$$

Energy is too large to care the sallow potential

$\Rightarrow$ No scattering at all.

The partial wave cross section doesnot saturate the unitarity limit at $k \gg \mathbb{K}$ and asymptotes to zero.

On the other hand, for $k < \mathbb{K}$, the WF is

$$rR \sim \begin{cases} A \sinh(\sqrt{\mathbb{K}^2 - k^2}\, r) & r < a \\ B \sin(kr + \delta_0) & r > a \end{cases}$$

From Matching at $r = a$

$$\delta_0 = \tan^{-1}\left[\frac{k}{\sqrt{\mathbb{K}^2 - k^2}} \tanh(\sqrt{\mathbb{K}^2 - k^2}\, a) \right] - ka$$

equivalent to the previous result for $k > \mathbb{K}$

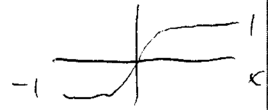
$\sqrt{k^2 - \mathbb{K}^2} \to i\sqrt{\mathbb{K}^2 - k^2}$

(see Fig. 1)

$r > a$, $V = 0$, $\quad rR_0 \sim A \sin(kr + \delta_0)$

if $\frac{K}{k} \gg 1$ $\quad \delta_0 \to \tan^{-1}\left[\frac{ka}{Ka}\underbrace{\tanh Ka}_{\ll 1}\right] - ka$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \sim x$ near $x=0$



$\sim ka\left[\underbrace{\frac{1}{Ka}\tanh Ka}_{\in[-1,0]} - 1\right]$

namely,
$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \xrightarrow[k\to 0]{} 4\pi a^2\left[\frac{1}{Ka}\tanh Ka - 1\right]^2$$
$\leq$ hard sphere scattering

Interesting points at small k:

① negative slope ② linear $\Rightarrow$ general result for a repulsive potential

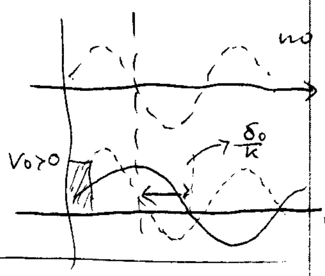
* A convenient quantity, scattering length $\quad [a_0] \sim [\frac{1}{k}] \sim [L]$

$a_0 \equiv -\lim_{k\to 0}\frac{d\delta_0}{dk}$ $\quad$ (it is positive here (repulsive))

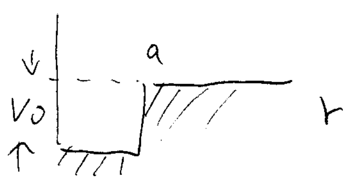
basically measures how big the scatterer is.

(recall that $\lim_{k\to 0}\sigma_0 = 4\pi a_0^2$)

* For hard sphere potential, $a_0 = a$



$\boxed{\text{Attractive potential}}$ if $V_0 < 0$.



$r>a$, $V=0$, $\quad rR_0 \sim A\sin(kr+\delta_0)$

$r<a$, $V=-V_0$ $\quad rR_0 \sim B\sin(\sqrt{k^2+K^2}\,r)$

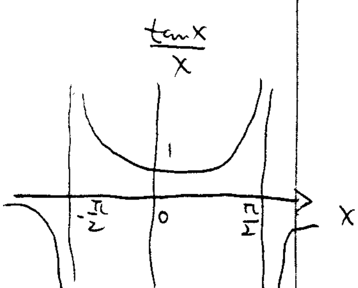
$K^2 \equiv \frac{2m|V_0|}{\hbar^2}$

matching at $r=a$

$\left(-\frac{d^2}{dr^2} + \frac{2m}{\hbar}V(r)\right)(rR) = k^2(rR)$

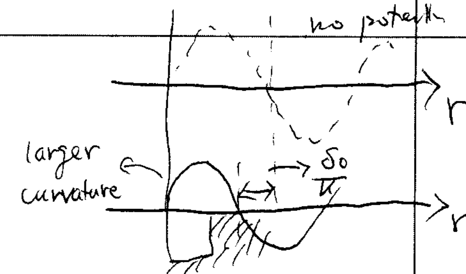
$\Rightarrow \frac{1}{k}\tan(ka+\delta_0) = \frac{1}{\sqrt{k^2+K^2}}\tan(\sqrt{k^2+K^2}\,a)$

or $\quad \delta_0 = \tan^{-1}\left[\frac{k}{\sqrt{k^2+K^2}}\tan(\sqrt{k^2+K^2}\,a)\right] - ka$



$\frac{k}{K} \ll 1$ $\quad \delta_0 \to \tan^{-1}\left[\frac{k}{K}\tan Ka\right] - ka$

$\sim ka\left[\frac{\tan Ka}{Ka} - 1\right]$



The scattering length is

$$a_0 = -\frac{d\delta_0}{dk}\Big|_{k=0} \simeq a\left[-\frac{\tan \mathbb{K}a}{\mathbb{K}a} + 1\right]$$

$a_0 < 0$, for small $\mathbb{K}$ (shallow dip) $\Leftrightarrow$ $a_0 > 0$ for repulsive
been pulling in

If we make V more attractive, (larger $\mathbb{K}$), the scattering length grows and even becomes ∞ at $\mathbb{K} = \frac{\pi}{2a}$!! (see Fig 2)

What's going on ??

---

from $\delta_0 = \underbrace{\tan^{-1}\left[\frac{k}{\sqrt{k^2+\mathbb{K}^2}} \tan\left(\sqrt{k^2+\mathbb{K}^2}\, a\right)\right]}_{X} - ka$

$$\tan X = \frac{k}{\sqrt{\ }} \tan \sqrt{\ }\, a$$

$$\frac{e^{iX}-e^{-iX}}{i(e^{iX}+e^{-iX})} = \frac{k}{\sqrt{\ }} \tan\sqrt{\ }\, a$$

$$\Rightarrow \frac{e^{2iX}-1}{e^{2iX}+1} = \frac{ik}{\sqrt{\ }} \tan\sqrt{\ }\, a$$

The S-matrix element

$$S_0 = e^{2i\delta_0} = e^{-2ika}\, e^{2iX}$$

$$= e^{-2ika} \frac{1 + i\frac{k}{\sqrt{k^2+\mathbb{K}^2}} \tan\sqrt{k^2+\mathbb{K}^2}\, a}{1 - i\frac{k}{\sqrt{k^2+\mathbb{K}^2}} \tan\sqrt{k^2+\mathbb{K}^2}\, a}$$

So can have a pole if

$$1 - i\frac{k}{\sqrt{k^2+\mathbb{K}^2}} \tan\sqrt{k^2+\mathbb{K}^2}\, a = 0$$ which is impossible for real k,

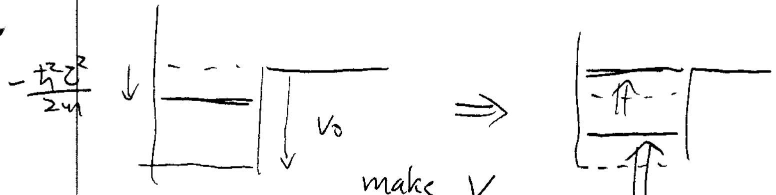
However, in the complex k plane, let $k = i\tau$ ① $E = \frac{\hbar^2 k^2}{2m} = -\frac{\hbar^2\tau^2}{2m} < 0$)

$$1 - i\frac{i\tau}{\sqrt{\mathbb{K}^2-\tau^2}} \tan\left(\sqrt{\mathbb{K}^2-\tau^2}\, a\right) = 0$$

② WF $\sim e^{i(i\tau)r} = e^{-\tau r}$
$\rightarrow$ confined in a finite region

or $-\sqrt{\mathbb{K}^2-\tau^2} = \tau \tan\left(\sqrt{\mathbb{K}^2-\tau^2}\, a\right)$

$$\tau = -\frac{\sqrt{\mathbb{K}^2-\tau^2}}{\tan\left(\sqrt{\mathbb{K}^2-\tau^2}\, a\right)} \Rightarrow$$ the condition for bound states.



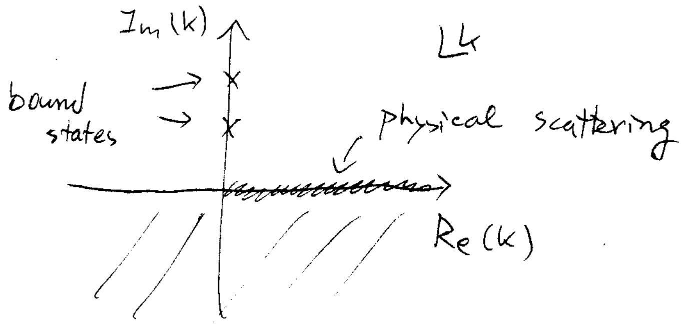
make V shallower and lift the bound state energy

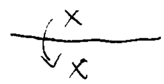
△ when $\mathbb{K}a = (n+\frac{1}{2})\pi$, $\tan \mathbb{K}a = \infty$

$a_0 \rightarrow$ div, ∞ (see Fig 2)
(↑ the first time)

☆ In other words, $\delta_0 = \infty$ at $k=0$ happens because there is a bound state exactly at $k=0$.

☆ If you further decrease $|k|$, bound state disappears
σ for small k is still very large, $\gg 4\pi^2 a^2$





the sinking pole still give large contribution

↳ unphysical
$\psi \sim e^{+kr}$

☆ when $\delta_0 = n\pi$, $\sigma = 0$, Ramsauer - Townsend effect.
e^- on rare gas. (Fig. 5)

☆ good example of large cross section close to the threshold
neutron scattering off large nuclei,
$\sigma \gg 4\pi a^2$ → slow neutron can be effectively absorbed by Uranium in nuclear power plants → chain reaction

or in <u>atom bombs</u>

Phase Shift

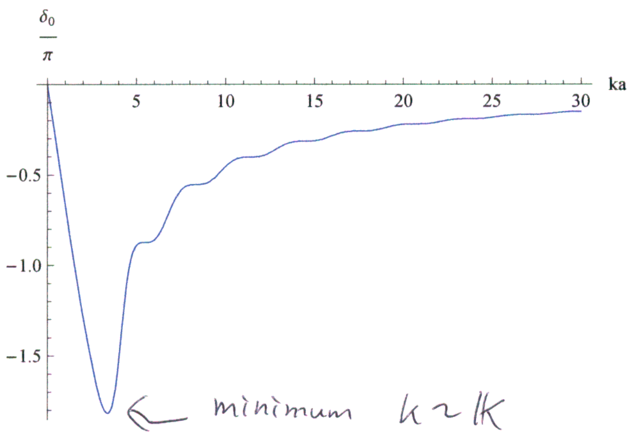
```
(* a=1  *)
```

Repulsive potential

In[1]:=
```
deltaR[{k_, BK_}] := ArcTan[k / Sqrt[k^2 - BK^2] * Tan[Sqrt[k^2 - BK^2]]] - k
```

In[3]:=
```
Plot[Mod[deltaR[{x, 3}], -Pi], {x, 0, 30}, AxesLabel → {ka, δ0 / π}]
```

Out[3]=



← minimum $k \approx K$

Fig. 1

Attrractive Potential

In[4]:=
```
deltaA[{k_, BK_}] := ArcTan[k / Sqrt[k^2 + BK^2] * Tan[Sqrt[k^2 + BK^2]]] - k
```

In[33]:=
```
Plot[deltaA[{0.01, KB}] / Pi, {KB, 0.1, 5.0}]
```

Out[33]=

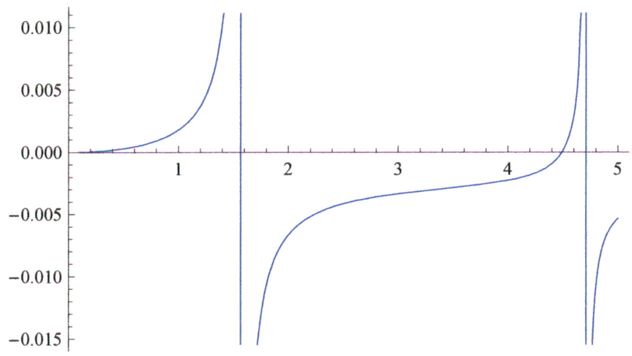


Fig. 2

↑ $Ka = \frac{\pi}{2}$

$\sigma_0 \Rightarrow \infty$

↑ $Ka = (1+\frac{1}{2})\pi$

$\sigma_0 \Rightarrow \infty$

In[11]:=
```
Plot[{ deltaA[{x, 0.5}] / Pi, deltaA[{x, 1.0}] / Pi, deltaA[{x, Pi / 2 - 0.0001}] / Pi,
  deltaA[{x, Pi / 2 + 0.0001}] / Pi, deltaA[{x, 1.6}] / Pi, deltaA[{x, 2.0}] / Pi },
 {x, 0, 4}, AxesLabel → {ka, δ0 / π}, PlotRange → All]
```

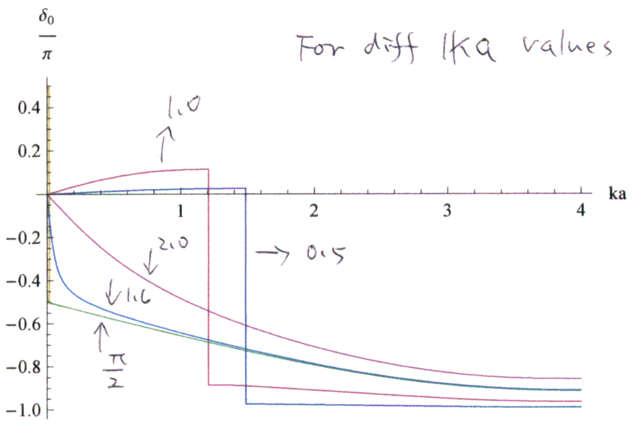
Out[11]=


Fig. 3

In[14]:=
```
Plot[{deltaA[{x, Pi / 2 - 0.001}] / Pi, deltaA[{x, Pi / 2 + 0.001}] / Pi},
 {x, 0, 0.5}, AxesLabel → {ka, δ0 / π}, PlotRange → All]
deltaA[{0.02, Pi / 2 - 0.001}] / Pi
```

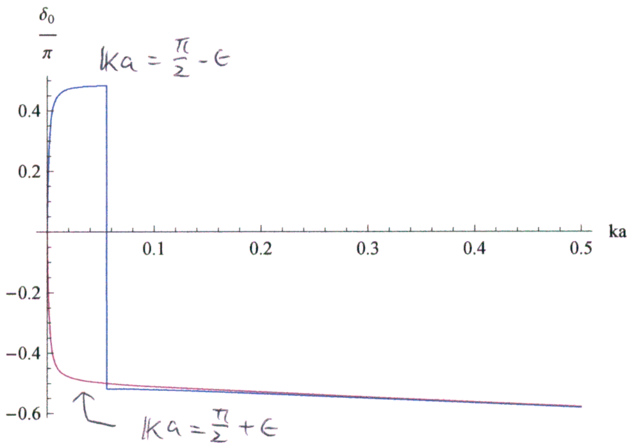
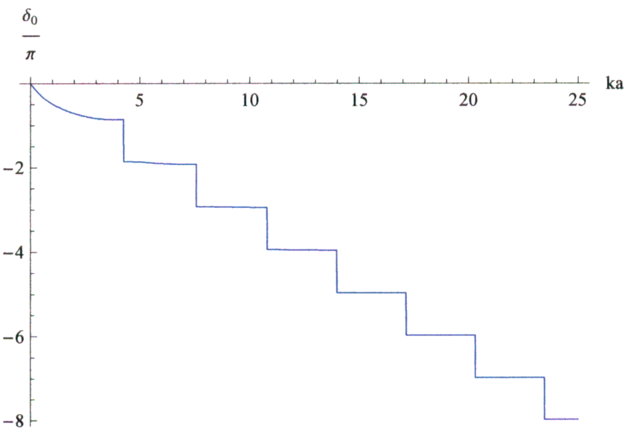
Out[14]=


Fig. 4

Out[15]= 0.471865

In[9]:=
```
Plot[deltaA[{x, 2}] / Pi, {x, 0, 25}, AxesLabel → {ka, δ0 / π}]
```

Out[9]=


at some points

$\delta_0 = n\pi \Rightarrow \sigma_0 = 0$

no cross section (no interaction)

This is called

Ramsauer-Townsend effect

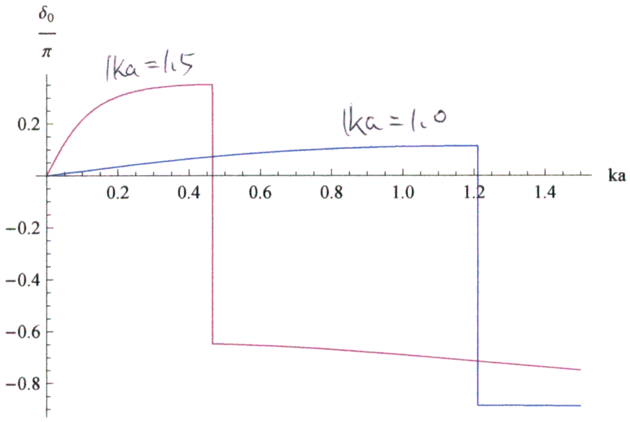
Scattering e^- on rare gas (1923)

was a big mystery ~

Fig. 5

In[17]:=
```
Plot[{deltaA[{x, 1.0}] / Pi, deltaA[{x, 1.5}] / Pi},
  {x, 0, 1.5}, AxesLabel → {ka, δ0 / π}, PlotRange → All]
```

Out[17]=



In[48]:=
```
Plot3D [Abs[I  * deltaA[{x + I  y, 4 * Pi}]] , {x, -5, 5},

  {y, 0, 4 Pi}, PlotPoints → 47  , MaxRecursion → 3, Mesh → None]
```

Out[48]=

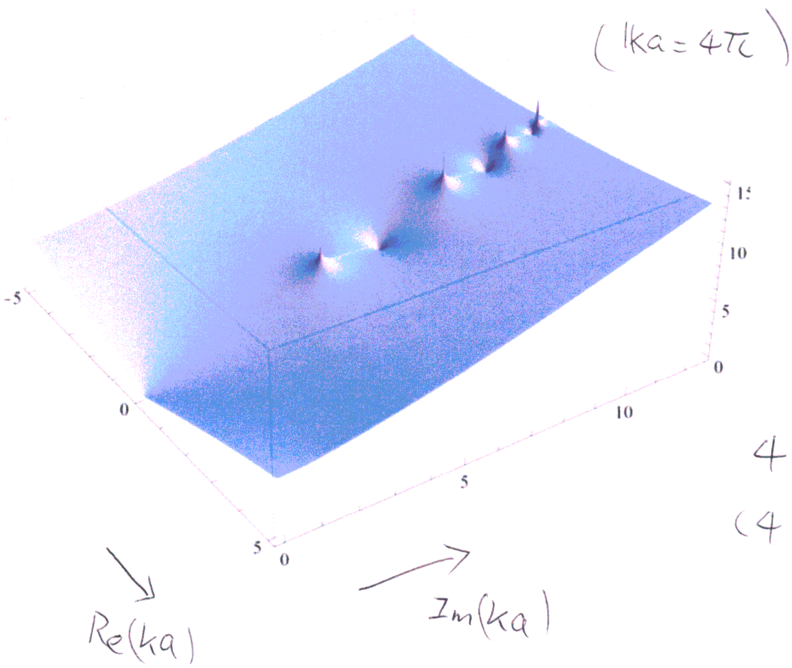


Fig. 6

4 poles on the +i axis
(4 bound states)