Quantum Mechanics − 2

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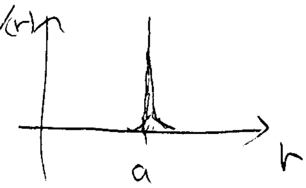
June 21, 2010

Resonance scattering

Illustate the physics by studing the scattering of a potential shell.

$$V(r) = \gamma\,\delta(r-a)$$



The radial WF $R = \frac{\chi}{r}$

and $\left(-\frac{d^2}{dr^2} + \frac{2m}{\hbar^2}V(r)\right]\chi = k^2\chi$

$$\begin{cases} r<a, & \chi_< = A\sin kr \\ r>a, & \chi_> = B\sin(kr+\delta) \end{cases}$$

matching $\chi_<$ and $\chi_>$ at $r=a$ $\Rightarrow$ $A = \frac{\sin(ka+\delta)}{\sin ka}B$

and their derivative by integrating the D.E around the shell.

$$\int_{a-\epsilon}^{a+\epsilon} dr\left(-\frac{d^2}{dr^2} + \frac{2m\gamma}{\hbar^2}\delta(r-a)\right]\chi(r) = \int_{a-\epsilon}^{a+\epsilon} k^2\chi(r)\,dr$$

$$\underset{\epsilon\to 0}{\Rightarrow} -\chi'\Big|_{a-\epsilon}^{a+\epsilon} + \frac{2m\gamma}{\hbar^2}\chi(a) \simeq k^2\chi(a)\,2\epsilon = 0$$

$$\Rightarrow \boxed{\chi'_<\Big|_a - \chi'_>\Big|_a + \frac{2m\gamma}{\hbar^2}\chi(a) = 0}$$

$$kA\cos ka - kB\cos(ka+\delta) + \frac{2m\gamma}{\hbar^2}\gamma A\sin ka = 0$$

$$\Rightarrow \tan(ka+\delta) = \frac{\sin ka}{\cos ka + \sin ka\frac{2m\gamma}{\hbar^2 k}}$$

$$\delta = \underbrace{\tan^{-1}\left[\frac{\sin ka}{\cos ka + \sin ka\frac{2m\gamma}{\hbar^2 k}}\right]}_{= x} - ka$$

$$\tan x = \frac{\sin ka}{c+s[\;]} = \frac{e^{ix}-e^{-ix}}{i(e^{ix}+e^{-ix})} = -\frac{(e^{2ix}-1)}{i(e^{2ix}+1)}$$

$$S = e^{2i\delta} = e^{-2ika}e^{2ix}$$

$$= e^{-2ika}\frac{e^{ika}+e^{-ika}+(e^{ika}-e^{-ika})\left(1-\frac{2im\gamma}{\hbar^2 k}\right)}{e^{ika}+e^{-ika}+(e^{ika}-e^{-ika})\left(-1-\frac{2im\gamma}{\hbar^2 k}\right)}$$

Headquarters
Hogil Kim Memorial Building #501, POSTECH, San 31, Hyoja-dong, Nam-gu
Pohang, Gyeongbuk 790-784, Korea
Tel : +82-54-279-8661~4 Fax : +82-54-279-8679

Branch Office
The Korean Federation of Science and Technology Societies Building 11th floor
635-4 Yoksam-dong, Kangnam-gu, Seoul, 135-703, Korea
Tel : +82-2-561-7641~2 Fax : +82-2-561-7140

or

$$S = e^{-2ika}\,\frac{e^{2ika}\left(1-\frac{im\gamma}{\hbar^2 k}\right)+\frac{im\gamma}{\hbar^2 k}}{e^{2ika}\left(-\frac{im\gamma}{\hbar^2 k}\right)+\left(1+\frac{im\gamma}{\hbar^2 k}\right)}$$

$$= e^{-2ika}\,\frac{\sin ka + e^{ika}\frac{\hbar^2 k}{2m\gamma}}{\sin ka + e^{-ika}\frac{\hbar^2 k}{2m\gamma}}$$

Looking for the pole of S

$$\Rightarrow e^{2ika} = \frac{\hbar^2 k}{im\gamma}\left(1+\frac{im\gamma}{\hbar^2 k}\right) = 1 - \frac{i\hbar^2 k}{m\gamma}$$

* when $\gamma \to \infty$, $2ka = 2n\pi$ or $k = \frac{\pi n}{a}$, discrete states in the cavity
* For large (but finite γ), expanding around the solution, $k \approx \frac{\pi n}{a}$, in terms of $\frac{1}{\gamma}$, $\left(\ln(1-x) \approx \ln(1) - x - \frac{x^2}{2} \cdots\right)$

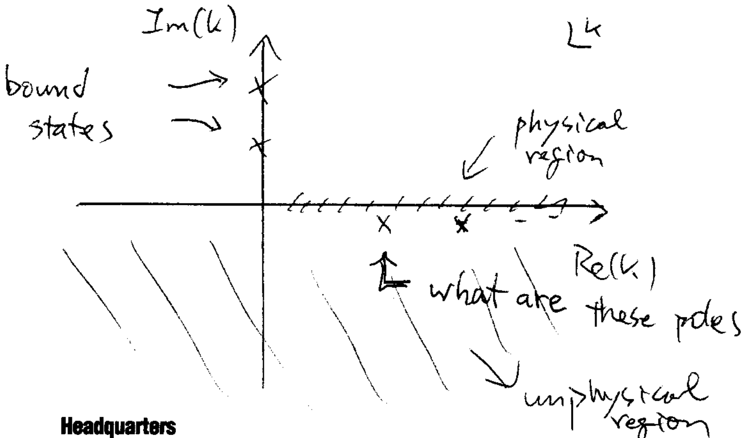
$$\Rightarrow 2ika = \ln\left(1-\frac{i\hbar^2 k}{m\gamma}\right) \simeq 2\pi n i - \frac{i\hbar^2 k}{m\gamma} + \frac{1}{2}\left(\frac{\hbar^2 k}{m\gamma}\right)^2 + O\left(\frac{1}{\gamma^3}\right)$$

$$\Rightarrow ka \simeq n\pi - \frac{\hbar^2 k}{2m\gamma} - \frac{i}{4}\left(\frac{\hbar^2 k}{m\gamma}\right)^2$$

$$\text{or } ka\left(1+\frac{\hbar^2}{2m\gamma a}\right) \simeq n\pi - \frac{i}{4}\left(\frac{\hbar^2 k}{m\gamma}\right)^2 \simeq n\pi - \frac{i}{4}\left(\frac{\hbar^2 n\pi}{m\gamma a}\right)^2$$

$$\Rightarrow k \approx \frac{n\pi}{a\left(1+\frac{\hbar^2}{2ma\gamma}\right)} - \frac{i}{4}\left(\frac{\hbar^2 n\pi}{2m\gamma}\right)^2\frac{1}{a^3} + O\left(\frac{1}{\gamma^3}\right)$$

* The pole posits in the lower half of complex plane



* Observation: when $\gamma \to$ very Large
The poles are very close to the real axis
$\to$ large scattering amplitude.

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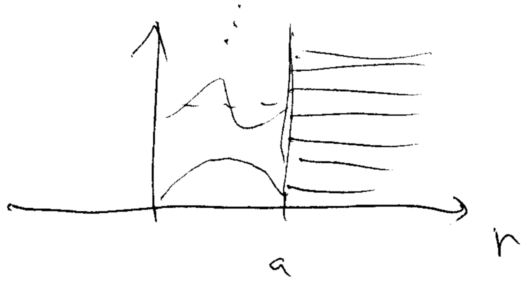
* In the limit $\gamma \to \infty$, the inside / outside regions decouple

Inside : discrete states with positive energy
outside : continuous spherical waves, $E > 0$



They become the poles on the real axis.

* Finite γ : coupling between two.
* It's instructive to solve Schrödinger Eq for these poles.

Note: the outgoing wave $e^{ikr} \propto e^{ir(-i\epsilon)} = e^{+\epsilon r}$
is enhanced by infinite amount related to the incoming wave $e^{-ikr} \propto e^{-ir(+i\epsilon)} = e^{-\epsilon r}$

$\Rightarrow$ Namely, the wave function is "purely outgoing".
We only keep the e^{ikr} piece.
c.f. the bound state solution is also "purely outgoing" e^{ikr}
since they locate at the positive imaginary axis, $+i\lambda$
$\Rightarrow e^{ir(i\lambda)} = e^{-r\lambda}$ gives localized WF.

$$\Rightarrow \quad \chi_0 = \begin{cases} \sin kr & (r<a) \quad \leftarrow \text{nothing changed} \\ \sin ka \, e^{ik(r-a)} & (r>a) \end{cases}$$

matching $\underbrace{A e^{i(ka+\delta)}}_{\text{complex}} = \underbrace{\sin ka}_{\text{real}} \quad \Rightarrow \delta = -ka, \quad A = \sin ka$

* Due to the $e^{+\epsilon r}$ factor, the solution is not a regular normalizable solution.

Headquarters
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However, let's proceed, for large γ, the pole is

$$ak \simeq n\pi\left(1+\frac{\hbar^2}{2ma\gamma}\right)^{-1} - i\left(\frac{\hbar^2}{2ma\gamma}\right)^2 (n\pi)^2 + O\left(\frac{1}{r^3}\right)$$

so $\sin ka \sim \sin(\pi n - O(\frac{1}{\gamma})) \sim O(\frac{1}{\gamma}) \ll 1$

$\Rightarrow \chi_<$ almost vanish at the shell $(r=a)$
(also $r=0$) $\Rightarrow$ discrete states.

Outside the shell, the WF oscillates at small amplitude
$\sin ka \sim O(\frac{1}{\gamma})$, but grows due to $e^{ik(r-a)}$

Now, put in the time-dependence

$$E = \frac{\hbar^2 k^2}{2m}, \qquad k = k_R - i k_I$$

$$= E_R - i\frac{\Gamma}{2} = \frac{\hbar^2}{2m}k_R^2 - i\frac{\hbar^2 k_R k_I}{m} + O(k_I^2)$$

For large γ limit

$$E_{nR} \simeq \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2\left(1+\frac{\hbar^2}{2ma\gamma}\right)^{-2}, \qquad \Gamma_n \simeq E_{nR}\frac{2n\pi}{a\gamma^2}$$

$$\Rightarrow \gamma R_0(r,t) = \gamma R_0(r)\, e^{-\frac{iEt}{\hbar}} = \gamma R_0(r)\, e^{-\frac{iE_{nR}t}{\hbar}}\, e^{-\frac{\Gamma_n t}{2\hbar}}$$

$$\begin{cases} r<a & \sin kr\, e^{-\frac{iE_{nR}t}{\hbar}}\, e^{-\frac{\Gamma_n t}{2\hbar}} \Rightarrow \text{Prob} = e^{-\frac{\Gamma_n t}{\hbar}}, \text{ decresing} \\ r>a & \sin ka\, e^{ik(r-a)}\, e^{-\frac{iE_{nR}t}{\hbar}}\, e^{-\frac{\Gamma_n t}{2\hbar}} \Rightarrow \text{Prob} = e^{-\frac{\Gamma_n t}{\hbar}}\, \underbrace{e^{2k_I r}}_{\text{flowing outward}} \end{cases}$$

$$\chi_> \sim e^{i(k_R - ik_I)(r-a)} = e^{ik_R(r-a)}\, e^{+k_I(r-a)}$$

$$\chi_>^* \sim e^{-ik_R(r-a)}\, e^{+k_I(r-a)}$$

$$\chi_>^* \chi_>' - \chi_> \chi_>^{*\prime} = e^{2k_I(r-a)}(2ik_R)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d}{dr}(\chi_>^*\chi_>' - \chi_>^{*\prime}\chi_>) = i\hbar\frac{\partial}{\partial t}(\chi_>^*\, \chi_>)$$

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Hogil Kim Memorial Building #501, POSTECH, San 31, Hyoja-dong, Nam-gu
Pohang, Gyeongbuk 790-784, Korea
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635-4 Yoksam-dong, Kangnam-gu, Seoul, 135-703, Korea
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$$\Rightarrow -\frac{i\hbar}{m}k_R\frac{d}{dr}\left(e^{2k_I(r-a)}e^{-\frac{\Gamma t}{\hbar}}\right) = i\frac{d}{dt}\left(e^{2k_I(r-a)}e^{-\frac{\Gamma t}{\hbar}}\right)$$

$V = \frac{\hbar k_R}{m}$ flowing out to ∞

$$\Rightarrow \frac{\hbar}{m}k_R k_I = \frac{\Gamma}{2\hbar} \quad \Rightarrow \underline{V = \frac{\hbar k_R}{m} = \frac{\Gamma}{2\hbar k_I}}$$

The WF describes a bound state inside the shell, which decays into a continuous state outside the shell moving away at the expected velocity,

☆ The resonance can be viewed as quasi-bound states which decay into continuous states,

$\Rightarrow$ The life time of the quasi-bound state is $\boxed{\tau = \frac{\hbar}{\Gamma}}$

again we see that $\Delta E\,\Delta t \sim \hbar$

☆ But how is the complex energy eigenvalue possible?

Doesn't Hermition operator $\rightarrow$ real eigenvalues?

☆ It works only for the normalizable WF !!

Here, we have exponentially growing WF, The usual statement breaks down.

is $H = H^\dagger$

$$H|\psi\rangle = E|\psi\rangle \Rightarrow \langle\psi|H|\psi\rangle = E\langle\psi|\psi\rangle$$

d.c.

$$\langle\psi|H = \langle\psi|E^* \qquad \langle\psi|H|\psi\rangle = E^*\underbrace{\langle\psi|\psi\rangle}_{\text{ill-defined}}$$

* In fact, all excited states of an atom appear as resonances in the photon-atom scattering. If there is no photons $\rightarrow$ all excited states are stable bound states. The EM coupling lets the excited state decay into the continuum states of photons.

Another look of our solution

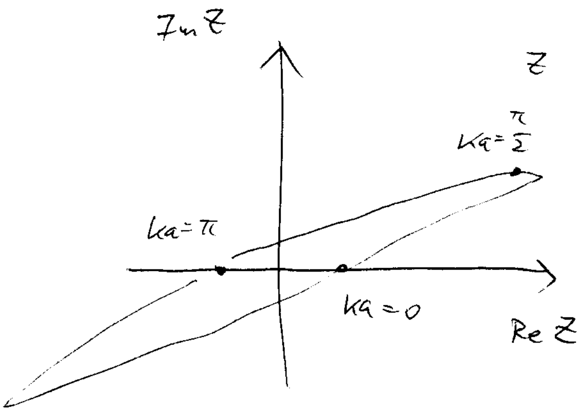
$$rR_0 = \begin{cases} \dfrac{\sin(ka+\delta_0)}{\sin(ka)} \sin(kr) \, , & r < a \\ \sin(kr+\delta_0) \, , & r > a \end{cases}$$

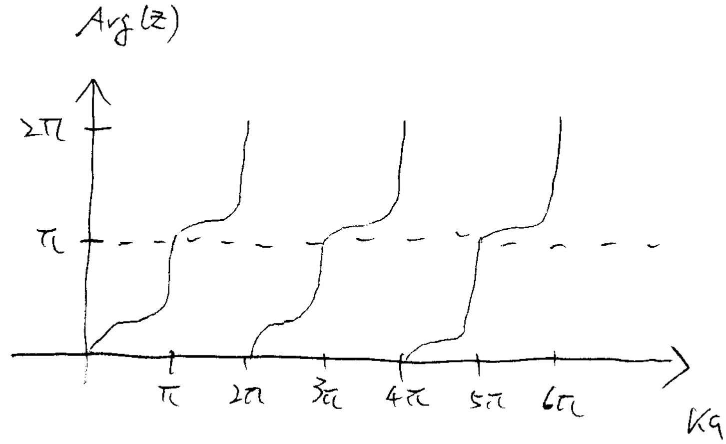
$$e^{2i\delta_0} = e^{-2ika} \frac{\sin ka + e^{ika} \frac{\hbar^2 k}{2m\gamma}}{\sin ka + e^{-ika} \frac{\hbar^2 k}{2m\gamma}} \quad \text{c.c.}$$

Take Arg on both sides:

$$\Rightarrow 2\delta_0 = -2ka + 2\,\mathrm{Arg}\left(\sin ka + e^{ika} \frac{\hbar^2 k}{2m\gamma}\right)$$

$$\text{or} \quad \delta_0 = -ka + \mathrm{Arg}\Big[\underbrace{\sin ka + \cos ka \left(\tfrac{\hbar^2 k}{2m\gamma}\right) + i \sin ka \tfrac{\hbar^2 k}{2m\gamma}}_{\equiv z} \Big]$$





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✱ for $\gamma \gg 1$, 2nd term in the Arg is negligible.

$\Rightarrow \delta_0 \simeq -ka$

Inside the shell, $rR_0 \propto \frac{\sin(ka-ka)}{\sin ka} \sin ka \sim 0$

$\Rightarrow$ wave doesn't enter the shell !!

✱ On the other hand, for special values,

$$ka = \begin{cases} 2\pi, 4\pi \cdots \\ \pi, 3\pi \cdots \end{cases} \qquad \text{Arg} \begin{cases} \pi \to 2\pi \\ 0 \to \pi \end{cases} \text{ changes rapidly}$$

✱ Only for these values of k, the prefactor

$\frac{\sin(ka+\delta_0)}{\sin(ka)}$ can be sizable (but at most unity)

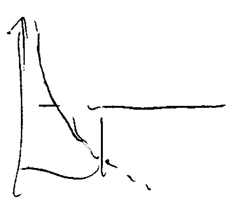
$\Rightarrow$ The wave enter the shell only for the resonant values of k!

✱ Energy shift is a general phenomenon not special to the δ-shell potential,

The energies of the excited states of an atom are different from the energies calculated without considering the decay and the difference has to be included given the high accuracy of atomic physics exp. (✱ same for high energy physic!)

✱ Resonance also appears in many attractive potential

ex. spherical well, $\left(\frac{-d^2}{dr^2} + \frac{2mV}{\hbar^2} + \frac{\hbar^2 \ell(\ell+1)}{r^2}\right)\chi_\ell = E_\ell \chi_\ell$



$\ell = 1$, has a pole at $ka \simeq \epsilon - i\frac{\epsilon^2}{3}$

when $ka = \pi - \frac{3}{2\pi}\epsilon^2$

Coulomb potential.

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☆ For the δ-potential, the pole is at

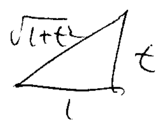
$$E_n \simeq E_{n,R} - \frac{i}{2}\Gamma_n \quad \text{where } E_{n,R} \simeq \frac{\hbar^2 n^2 \pi^2}{2ma^2}\left(1+\frac{\hbar^2}{2ma\gamma}\right)^{-2}$$

$$\Gamma_n \simeq E_{n,R}\,\frac{2n\pi}{a\gamma^2}$$

$E_R - \frac{i}{2}\Gamma$ behaves like a complex pole.

In the narrow width limit,

$$e^{2i\delta_\ell} \simeq \frac{E - E_R - \frac{i}{2}\Gamma}{E - E_R + \frac{i}{2}\Gamma}$$



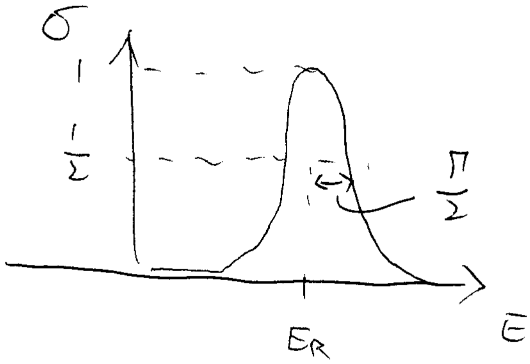
$$\Rightarrow \tan\delta_\ell \simeq \frac{-\frac{\Gamma}{2}}{E - E_R} \quad \text{and} \quad \sin^2\delta_\ell = \frac{t^2}{1+t^2} = \frac{\frac{\Gamma^2}{4(E-E_R)^2}}{1+\frac{\left(\frac{\Gamma}{2}\right)^2}{(E-E_R)^2}}$$

$$= \frac{\Gamma^2/4}{(E-E_R)^2+\left(\frac{\Gamma}{2}\right)^2}$$

$$\Rightarrow \sigma_\ell = \frac{4\pi}{k^2}\sin^2\delta_\ell\,(2\ell+1) = \frac{4\pi}{k^2}(2\ell+1)\frac{\Gamma^2/4}{(E-E_R)^2+\frac{\Gamma^2}{4}}$$

This is called Breit-Wigner formula

Γ is the "Full-width-half-maximum"



$e^+e^- \to Z^0 \to f\bar{f}$

$m_Z = 91.1875(21)$ GeV

$\Gamma_Z = 2.4952(23)$ GeV

↳ saturates the unitary limit ($\sin^2\delta = 1$)

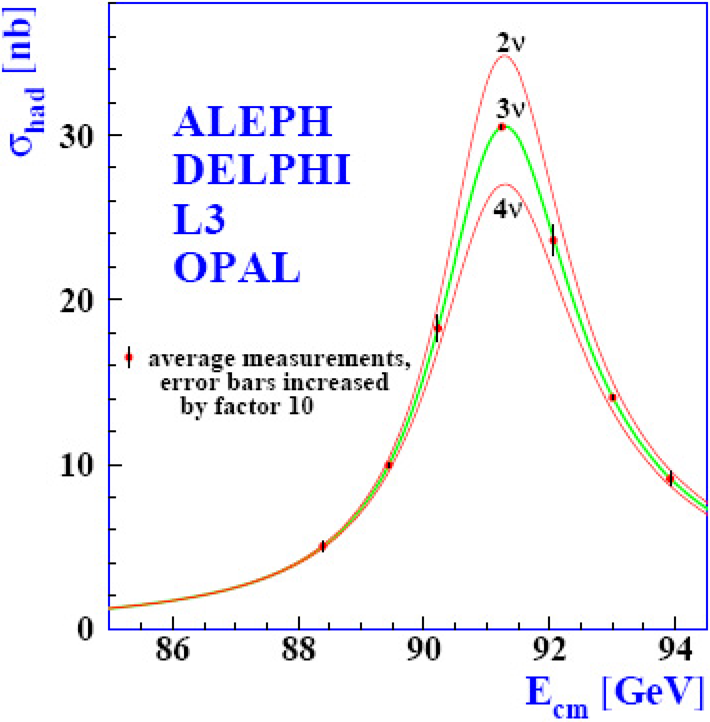


Figure 1.13: Measurements of the hadron production cross-section around the Z resonance. The curves indicate the predicted cross-section for two, three and four neutrino species with SM couplings and negligible mass.

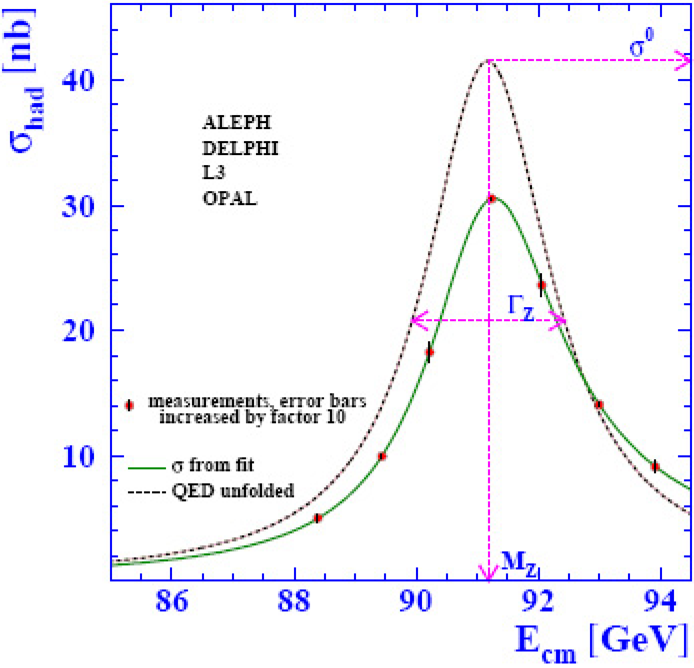


Figure 1: *Average over measurements of the hadronic cross-sections by the four experiments, as a function of centre-of-mass energy. The dashed curve shows the QED deconvoluted cross-section, which defines the Z parameters described in the text.*