June 23, 2010

Any general properties of these Green's function

scattering of EM wave   suppress $\mu, \nu, 4\pi, \epsilon, \ldots$ For a given J

$\Box A = J$, $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

$\Box A_h = 0$.   homogeneous solution

$A(x) = A_h(x) + \int dx' G(x, x') J(x')$, $\Box G(x, x') = \delta^4(x - x')$

* Itself an inhomogeneous Eq so G is not unique either. B.C. again.

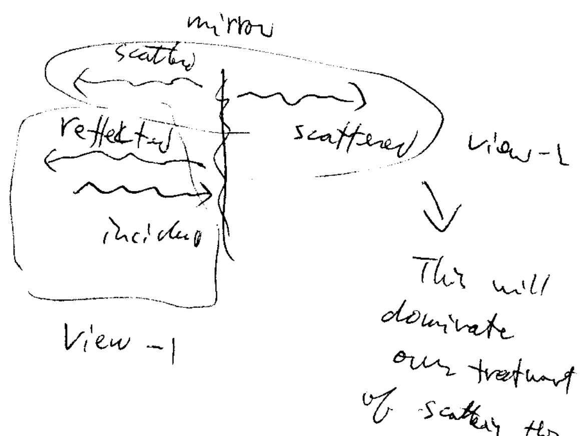
* The solution is never unique for an inhomogeneous Eq
* You can always add an arbitrary H. solution to it.
* physically, the unique solution is usually selected out by B.C.

But in practice, J is unknown before hand

say $A \to J \to A_{\text{radiated}}$

$\Rightarrow A_{\text{total}} = A_{\text{incident}} + A_{\text{radiated}}$

linear response $J = \underset{\sim}{\sigma} \underset{\sim}{A_{tot}}$
↑ some operator



$\Rightarrow \Box A = \sigma A$, or again $(\Box - \sigma) A = 0$

inhomogeneous   homogeneous.

(1) depends partly on our point of view

(2) $A_{tot} \Rightarrow$ self-consistent problem.

$(H_0 + V)\psi(r) = E\psi(r)$, $H_0 = \frac{p^2}{2m}$, $E > 0$

$\Rightarrow (E - H_0)\psi(r) = V\psi(r)$

Energy-dependent Green's function

$\Downarrow$

$(\Box A = \sigma A)$

free particle   linear response

Time-dependent Green's function in 3d

$$\left(i\hbar\frac{\partial}{\partial t} - H(t)\right)\psi(r,t) = \underbrace{S(r,t)}_{\text{"source" term}}$$

$$\left[i\hbar\frac{\partial}{\partial t} - H(t)\right]\underline{F(r,t;r',t')} = \underbrace{i\hbar}_{\text{convention}}\delta(t-t')\delta^3(r-r')$$

time-dep Green's function

r,t : field point, r',t' : source point.

If F is found

$$\psi(r,t) = \underbrace{\psi_h(r,t)}_{} + \frac{1}{i\hbar}\int_{-\infty}^{\infty}dt'\int d^3r'\; F(r,t;r't')\,S(r',t')$$

$$\left[i\hbar\frac{\partial}{\partial t} - H(t)\right]\psi_h(r,t) = 0$$

again the solution is not unique before we take B.C. into account.

$\Rightarrow$ if ~~known~~ F known, we can break the driving terms into a sum of δ-function contributions

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outgoing time-dep. Green's funct

$$K_+(r,t;r',t') = \theta(t-t')\langle r|U(t,t')|r'\rangle$$

$$\theta(\tau) = \begin{cases} 0 & \tau<0 \\ 1 & \tau>0.\end{cases} \qquad \frac{d\theta}{d\tau} = \delta(\tau)$$

$$\int_{-a}^{b} d\tau\,\frac{d\theta}{d\tau}F(\tau) = (\theta F(\tau)) - \underbrace{\int\theta\frac{d}{d\tau}F(\tau)}_{} = F(b) - \underbrace{\int_0^b\frac{d}{d\tau}F(\tau)}_{} = -F(b)+F(0) = F(0)$$

$$i\hbar\frac{\partial U(t,t')}{\partial t} = \hat H(t)\,U(t,t')$$

B.C. $\underline{U(t',t')=1}$

$$i\hbar\frac{\partial}{\partial t}K_+ = \underbrace{i\hbar\,\delta(t-t')\langle r|U(t,t')|r'\rangle}_{} + \theta(t-t')\langle \underbrace{H\hat H U}_{}|r'\rangle$$

$$\overset{\shortparallel}{i\hbar\,\delta(t-t')\langle r|r'\rangle}$$

$$\Downarrow$$

$$i\hbar\,\delta(t-t')\,\delta^3(r-r')$$

$\because \underbrace{H(t)}_{\text{x-repres}}\psi(r,t) = H(t)\langle r|\psi(t)\rangle = \langle r|\hat H|\psi(t)\rangle$

$\theta(t-t')H(t)\langle r|U(t,t')|r'\rangle$
$= H(t)K_+$

$$\Rightarrow \left[i\hbar\frac{\partial}{\partial t} - H(t)\right]K_+(r\,t;r'\,t') = i\hbar\,\delta(t-t')\,\delta^3(r-r')$$

In dealing the path-integral $K(r,t,r't') = \langle r | U(t,t') | r' \rangle$

we called K the propagator,

the outgoing Green's function $K_+ = \theta(t-t')K$

is also called propagator, (outgoing / advanced) propagator

but K is not a Green's function.

① K_+ vanish for $t<t'$.

②. K_+ is a solution of the t-dep Schrodinger Eq.

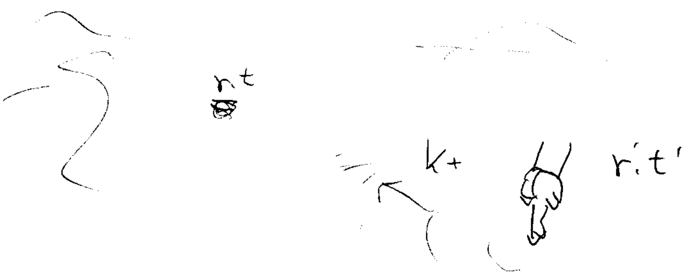
(RHS vanish for $t>t'$)

③ let $t \to t'$ from positive side, $K_+ \to \delta^3(r-r')$.

$\Rightarrow$ K_+ is a solution $\psi(r,t)$ of the t-dep Schrödinger Eq for $t>t'$

with the singular initial condition $\psi(r,t') = \delta^3(r-r')$ at $t=t'$

In a lake



a quiet lake

$t = -\infty$

until $t = t'$

for $t>t'$ K_+ is freely propagating

$$\psi(r,t) = \psi_h(r,t) + \int_{-\infty}^{\infty} dt' \int d^3r' \, K_+(r,t,r',t') \, S(r',t')$$

quiet $\psi_h = 0$, otherwise

the $\theta(t-t')$ function makes it

$$\int_{-\infty}^{t} dt' \int d^3r' \, K(r,t,r',t') \, S(r',t')$$

<Green's operator> $\to$ In Hilbert space, more general.

$$K_\pm(r,t,r't') = \langle r | \hat{K}_\pm(t,t') | r' \rangle$$

$$\hat{K}_\pm(t,t') = \pm \theta(\pm(t-t')) \, U(t,t')$$

satisfy

$$\left[i\hbar \frac{\partial}{\partial t} - \hat{H}(t) \right] \hat{K}_\pm(t,t') = i\hbar \, \delta(t-t')$$

Time-indept Hamiltonian

$U(t,t') \to U(t-t')$

$\hat{K}_\pm(t) = \pm\theta(\pm t)\,U(t)$ { out going, advanced / incoming, retarded

$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right)\hat{K}_\pm = i\hbar\,\delta(t)$

$K_\pm(r,r';t) = \langle r|\hat{K}_\pm(t)|r'\rangle$

free particle. $H = H_0 = \frac{p^2}{2m}$

$K_{0\pm}(r,r',t) = \pm\theta(\pm t)\left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}}\exp\left(\frac{i}{\hbar}\frac{m(r-r')^2}{2t}\right)$

Energy-dependent Green's function in QM

$(E-H)\psi(r) = S(r)$

H: time-independent; $S(r)$: source

eg. $H = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$, $\hat{H}$ the operator acting on kets

$(E-H)\,G(r,r',E) = \delta^3(r-r')$

E ↓ just a parameter → Energy-dependent Green's function

↓ not necessary an eigenvalue of H.

$\psi(r) = \underbrace{\psi_h(r)} + \int d^3r\, G(r,r';E)\,S(r')$

$(E-H)\psi_h(r) = 0$

→ an eigenfunction of H of energy E. (if E not eigenvalue, $\psi_h = 0$)

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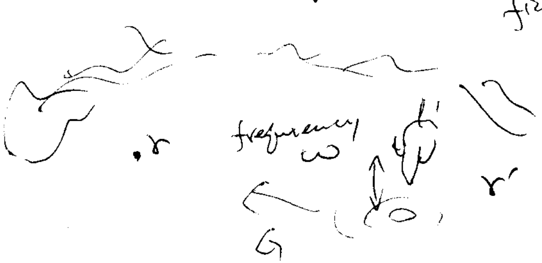
go to lake again



first lake was quiet for $t = -\infty \to 0$, dissipation ϵ

$t = 0 \to \infty$ repeatly periodic perturbation

⇒ to a stand wave with ω, $t\to\infty$, $\epsilon\to 0$

$\omega \leftrightarrow E$

when $\omega \approx \omega_n$ resonance. ~>

P. 2 : the propagator (is the Green's function)

$$K(x'',t;x',t_0) = \langle x'',t | x',t_0 \rangle$$

$$= \langle x'' | \exp\left(\frac{-iH(t-t_0)}{\hbar}\right) | x' \rangle$$

$$\left[-\left(\frac{\hbar^2}{2m}\right)\nabla''^2 + V(x'') - i\hbar\frac{\partial}{\partial t}\right] K(x'',t;x',t_0)$$

$$= -i\hbar\,\delta^3(x''-x')\,\delta(t-t_0)$$

and $K(x'',t;x',t_0) = 0$ for $t < t_0$

It can be worked out

$$K(x'',t;x',t_0) = \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}}\,\exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right]$$

Higher-order Born Approximation

$$V|\psi\rangle = T|\phi\rangle$$

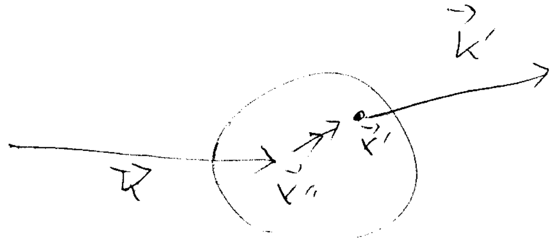
$$T = V + V\frac{1}{E-H_0+i\epsilon}V + V\frac{1}{E-H_0+i\epsilon}V\frac{1}{E-H_0+i\epsilon}V + \cdots$$

$$f^{(1)}(\vec{k}',\vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle \vec{k}'|V|\vec{k}\rangle$$

$$f^{(2)}(\vec{k}',\vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle \vec{k}'|V\frac{1}{E-H_0+i\epsilon}V|\vec{k}\rangle$$

$$f^{(2)} = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3 \int d^3x' \, d^3x'' \, \langle \vec{k}' | \vec{x}' \rangle V(x') \langle x' | \frac{1}{E - H_0 + i\epsilon} | x'' \rangle$$
$$V(x'') \langle x'' | \vec{k} \rangle$$

$$= -\frac{1}{4\pi}\frac{2m}{\hbar^2} \int d^3x' \, d^3x'' \, e^{-ik' \cdot x'} V(x') \underbrace{\frac{2m}{\hbar^2} G_+(\vec{x}', \vec{x}'')}_{} V(x'') e^{ik \cdot x''}$$



$$\frac{\hbar^2}{2m} \langle \vec{x} | \frac{1}{E - H_0 + i\epsilon} | \vec{x}' \rangle = \frac{\hbar^2}{2m} \int d^3p' \, d^3p'' \, \langle x | p' \rangle \langle p' | \frac{1}{E - \frac{p'^2}{2m} + i\epsilon} | p'' \rangle \langle p'' | x' \rangle$$

$$= \frac{\hbar^2}{2m} \int d^3p' \, d^3p'' \, \frac{e^{i\frac{p' \cdot x}{\hbar}}}{(2\pi\hbar)^{\frac{3}{2}}} \frac{\delta^3(p' - p'')}{E - \frac{p'^2}{2m} + i\epsilon} \frac{e^{-i\frac{p'' \cdot x'}{\hbar}}}{(2\pi\hbar)^{\frac{3}{2}}}$$

$$= \frac{\hbar^2}{2m} \int \frac{d^3p'}{(2\pi\hbar)^3} \frac{e^{i\frac{p' \cdot (x - x')}{\hbar}}}{E - \frac{p'^2}{2m} + i\epsilon} = -\frac{1}{4\pi} \frac{e^{ik|x - x'|}}{|x - x'|}$$

The G_+ is nothing but the Green's function

$$(\nabla^2 + k^2) \, G_+(\vec{x}, \vec{x}') = \delta^3(\vec{x} - \vec{x}')$$

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or the Lippmann-Schwinger Eq

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0 + i\epsilon} V |\psi\rangle$$

$H_0 |\phi\rangle = E |\phi\rangle$
$H \rightarrow H_0 + V$
$(H_0 + V)|\psi\rangle = E\psi$

Scattering, time-dep formulation

$$\left(i\hbar\frac{\partial}{\partial t} - H_0\right) |\psi; t\rangle = V |\psi; t\rangle$$

Like solving the D.E with an inhomogeneous term by the Green's function
→ This operator Schrödinger Eq is solved by introducing the Green's operator $\vec{G}_+(t, t')$, which satisfies $\left(i\hbar\frac{\partial}{\partial t} - H_0\right) \vec{G}_+(t, t') = \delta(t - t')$

The causality requires $\hat{G}_+(t,t') = 0$ for $t<t'$

$\Rightarrow$ The solution is

$$\hat{G}_+(t,t') = -\frac{i}{\hbar}\,\theta(t-t')\,e^{-i\frac{H_0(t-t')}{\hbar}}$$

for $t>t'$, RHS $\delta(t-t')=0$, LHS

$$\left(i\hbar\frac{\partial}{\partial t} - H_0\right)\hat{G}_+ = 0$$

For $t<t'$, RHS $=0$, LHS $=0$

at $t=t'$

$$\underbrace{-\frac{i}{\hbar}\theta(t-t')\left(i\hbar\frac{\partial}{\partial t} - H_0\right)e^{-i\frac{H_0(t-t')}{\hbar}}}_{0} \quad -\frac{i}{\hbar}\left[\left(i\hbar\frac{\partial}{\partial t} - H_0\right)\theta(t-t')\right]\underbrace{e^{-i\frac{H_0(t-t')}{\hbar}}}_{1} = \delta(t-t')$$

$\delta(t-t')$ X

$$\Rightarrow \quad |\psi;t\rangle = \underbrace{|\phi,t\rangle}_{\text{homogeneous solution}} + \int_{-\infty}^{\infty} G_+(t,t')\,V\,|\psi,t'\rangle\,dt'$$

$$\left(i\hbar\frac{\partial}{\partial t} - H_0\right)|\phi;t\rangle = 0$$

$\Rightarrow$ for $t<t'$ $\quad |\psi,t\rangle = |\phi,t\rangle$ as ~~requir~~ one desires.

for $t>t'$.

$$\left(i\hbar\frac{\partial}{\partial t} - H_0\right)|\psi,t\rangle = 0 + \int_{-\infty}^{\infty}\delta(t-t')\,V\,|\psi,t'\rangle\,dt'$$

$$= V\,|\psi,t\rangle \quad \Rightarrow \text{is the solution}$$

in general, we want to solve eq.

$$(\nabla^2+k^2)\,\psi(r) = V(r)\psi(r)$$

suppose there exists a function $G(r)$ such that

$$(\nabla^2+k^2)\,G(r) = \delta^3(r),$$ Green's function of the operator ∇^2+k^2. then any function

$$\varphi(r) = \underbrace{\varphi_0(r)}_{(\nabla^2+k^2)\varphi_0=0} + \int d^3r'\, G(r-r')\, U(r')\,\varphi(r')$$ is the solution.

$$G_\pm(r) = -\frac{1}{4\pi}\frac{e^{\pm i\vec{k}\vec{r}}}{r}$$

G_+ : out going, G_- : incoming Green's functions

$\nabla^2 = \cancel{\left(\frac{1}{r}\frac{\partial^2}{\partial r^2} r\right) \to \text{spherical coordinate}}$
$= \nabla\cdot\vec{\nabla} = \partial_x\partial_x + \partial_y^2 + \partial_z^2$

$$\nabla^2 G_\pm(r) = -\frac{1}{4\pi r}(-k^2)\,e^{\pm i\vec{k}\cdot\vec{r}} - \frac{e^{\pm i\vec{k}\cdot\vec{r}}}{4\pi}\,\nabla^2\!\left(\frac{1}{r}\right)$$

$\nabla^2(\frac{1}{r}) = -4\pi\delta^3(\vec{r})$

$\nabla\cdot\underbrace{\vec{\nabla}\left(\frac{1}{r}\right)}_{\propto\, \vec{E}(r) \text{ of a point charge at origin}}$

$\propto \nabla\cdot\vec{E} = \rho(r) = A\,\delta^3(r)$

$\int dV\, \nabla\cdot\vec{\nabla}\left(\frac{1}{r}\right) = A$

$= \oint_{da} -\frac{\hat{r}}{r^2}\cdot da$

$-\frac{4\pi R^2}{R^2} = A$

For $r \neq 0$

$$\nabla^2 G_\pm = -\frac{1}{4\pi}\partial_x\left(-\frac{x}{r^2}e^{\pm i\vec{k}\cdot\vec{r}} \pm i k_x \frac{e^{\pm i\vec{k}\cdot\vec{r}}}{r}\right)$$

$$= -\frac{1}{4\pi}\left[-\frac{1}{r^2}e^{\pm i\vec{k}\cdot\vec{r}} + \frac{2x^2 e^{\pm i k r}}{r^4} \mp \frac{i k_x\cdot x}{r^2}e^{\pm i\vec{k}\cdot\vec{x}} \mp \frac{i k_x\cdot x\, e^{\pm i\vec{k}\cdot\vec{r}}}{r^2} - k_x^2\frac{e^{\pm i\vec{k}\cdot\vec{r}}}{r}\right)$$

$$\partial_x^2\left(-\frac{1}{4\pi}\frac{e^{\pm i\vec{k}\cdot\vec{r}}}{r}\right)$$

$$= -\frac{1}{4\pi}\left[-\frac{3}{r^2} + \frac{2}{r^2} \mp \frac{2i\vec{k}\cdot\vec{r}}{r^2} - \frac{\vec{k}^2}{r}\right]e^{\pm i\vec{k}\cdot\vec{x}}$$

$$= -\frac{1}{4\pi}\partial_x\left(\frac{1}{4\pi}\partial_x\left(\frac{1}{r}\right)e^{\pm i\vec{k}\cdot\vec{r}} \pm i k_x \frac{e^{\pm i\vec{k}\cdot\vec{x}}}{r}\right)$$

$$= -\frac{1}{4\pi}\partial_x\left(-\frac{x}{r^2}e^{\pm i\vec{k}\cdot\vec{r}} \pm i k_x\frac{e^{i\vec{k}\cdot\vec{x}}}{r}\right)$$

$$= \cdots - \frac{1}{4\pi}\left(-\frac{x}{r^2}(\pm i k_x)e^{\pm i\vec{k}\cdot\vec{r}} - \frac{x}{r^2}(\pm i k_x)e^{i\vec{k}\cdot\vec{x}}\right) + \cdots$$

$$= \cdots + \frac{2}{4\pi}\frac{\pm i\vec{r}\cdot\vec{k}}{r^2}e^{i\vec{k}\cdot\vec{r}} + \cdots$$

9

when $r \neq 0$, $\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r$ in spherical coordinates

$$\nabla^2\left(\frac{e^{\pm ikr}}{r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(e^{\pm ikr}\right) = -k^2\frac{e^{\pm ikr}}{r}$$

$$\Rightarrow (\nabla^2 + k^2)\left(\frac{e^{\pm ikr}}{r}\right) = 0.$$

$r = \sqrt{x^2+y^2+z^2}$

In Cartesian coordinate.

$\nabla^2 = \sum_i \partial_x^2$

$$\cancel{\partial_x\left(-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} e^{\pm i\vec{k}\cdot\vec{r}} + \frac{\pm ik_x}{(x^2+y^2+z^2)^{\frac{1}{2}}} e^{\pm i\vec{k}\cdot\vec{r}}\right)}$$

$$\cancel{= \left(-\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} e^{\pm i\vec{k}\cdot\vec{r}} + \frac{3}{2}\cdot 2\frac{x^2 e^{\pm i\vec{k}\cdot\vec{r}}}{(x^2+y^2+z^2)^{\frac{5}{2}}} \mp \frac{ik_x x}{(x^2+y^2+z^2)^{\frac{3}{2}}} e^{\pm i\vec{k}\cdot\vec{r}} \mp \frac{ik_x x\, e^{\pm i\vec{k}\cdot\vec{r}}}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{k_x^2 e^{\pm i\vec{k}\cdot\vec{r}}}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right.}$$

$$\cancel{\Rightarrow -\frac{3}{r^3}e^{\pm i\vec{k}\cdot\vec{r}} + \frac{3r^2}{r^5}e^{\pm i\vec{k}\cdot\vec{r}} \mp \frac{2i\vec{k}\cdot\vec{r}}{r^3}e^{\pm i\vec{k}\cdot\vec{r}} - \frac{k^2 e^{\pm i\vec{k}\cdot\vec{r}}}{r}}$$

$$\partial_x^2\left(\frac{e^{\pm ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) = \partial_x\left(-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}e^{\pm ikr} \pm ik\frac{2x}{2\sqrt{x^2+y^2+z^2}}\frac{e^{\pm ik\sqrt{\ }}}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \partial_x\left(-\frac{x}{(r^2)^{\frac{3}{2}}}e^{\pm ikr} \pm ik\frac{x}{(\ )^{\frac{2}{2}}}e^{\pm ik\sqrt{\ }}\right)$$

$$= \left(-\frac{1}{(\ )^{\frac{3}{2}}}e^{\pm ikr} + 3\frac{x^2}{(\ )^{\frac{5}{2}}}e^{\pm ikr} - \frac{x}{(\ )^{\frac{3}{2}}}\frac{\pm ikx}{(\ )^{\frac{1}{2}}}e^{\pm ikr}\right.$$

$$\left.\pm\frac{ik}{(\ )^{\frac{2}{2}}}e^{\pm ik\sqrt{\ }} \mp \frac{ik\,2x^2}{(\ )^2}e^{\pm ik\sqrt{\ }} - \frac{k^2}{(\ )^1}\frac{x^2}{(\ )^{\frac{1}{2}}}e^{\pm ik\sqrt{\ }}\right)$$

$$\Rightarrow \left(\cancel{-\frac{3}{r^3}e^{\pm ikr}} + \cancel{\frac{3r^2}{r^5}e^{\pm ikr}} \mp \cancel{\frac{i2kr^2}{r^4}e^{\pm ikr}} \cancel{\pm \frac{i2kr^2}{r^4}e^{\pm ikr}}\right.$$

$$\left.\mp\cancel{\frac{ikr^2}{r^4}e^{\pm ikr}} \pm \cancel{\frac{i3k}{r^2}e^{\pm ikr}} - \frac{k^2r^2}{r^3}e^{\pm ikr}\right)$$

$$\Rightarrow \quad (\nabla^2 + k^2) \frac{e^{+ikr}}{r} = -4\pi \delta^3(\vec{r})$$

shift the origin to $\vec{r}'$

$$\Rightarrow \quad (\nabla^2 + k^2) \frac{e^{+ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} = -4\pi \delta^3(\vec{r}-\vec{r}')$$

The Schrödinger Eq can be written in a similar form

$$(\nabla^2 + k^2)\, \psi_k(\vec{r}) = \frac{2m}{\hbar^2} V(\vec{r})\, \psi_k(\vec{r})$$

$$\Rightarrow \quad \psi_k^+(r) = e^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi}\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \frac{2m}{\hbar} V(\vec{r}')\, \psi_k^+(\vec{r}')\, d^3r'$$

In Born approximation, Coulomb scattering

$$f(\theta) = -\frac{m}{2\pi\hbar} \tilde{V}(\underbrace{\vec{k}'-\vec{k}}_{\vec{q}}) \qquad V(r) = \frac{Ze^2}{r}$$

$$\tilde{V}(\vec{q}) = \int e^{-i\vec{q}\cdot\vec{r}}\, V(r)\, d^3r$$

since. $\nabla^2 V(r) = -4\pi Z e^2 \delta^3(r)$

in the k-space

$$-q^2 \tilde{V}(\vec{q}) = -4\pi Z e^2$$

$$\text{or} \quad \tilde{V}(\vec{q}) = \frac{4\pi Z e^2}{q^2}$$

$$\Rightarrow \quad f(\theta) = -\frac{Ze^2}{\frac{\hbar k^2}{2m} \frac{|q|^2}{|k|^2}} \qquad |\vec{q}| = 2k \sin\frac{\theta}{2}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{E}\right)^2 \frac{1}{16 \sin^4\frac{\theta}{2}}$$

$$(\nabla^2+k^2)\,G(\vec r-\vec r') = \delta(\vec r-\vec r')$$

their Fourier transforms are.

$$G(\vec r-\vec r') = \frac{1}{(2\pi)^3}\int e^{i\vec q\cdot(\vec r-\vec r')}\,\tilde G(\vec q)\,d^3q$$

$$\delta(\vec r-\vec r') = \frac{1}{(2\pi)^3}\int e^{i\vec q\cdot(\vec r-\vec r')}\,d^3q$$

$$\frac{1}{(2\pi)^3}\int(-\vec q^{\,2}+k^2)\,\tilde G(q)\,e^{i\vec q(\vec r-\vec r')}\,d^3q = \frac{1}{(2\pi)^3}\int e^{i\vec q(\vec r-\vec r')}\,d^3q$$

or $(-\vec q^{\,2}+k^2)\,\tilde G(q) = 1 \Rightarrow \tilde G(\vec q) = \frac{1}{\vec k^2-\vec q^2}$

Then $G(r-r') = \frac{1}{(2\pi)^3}\int \frac{e^{i\vec q\cdot(\vec r-\vec r')}}{k^2-q^2}\,d^3q$

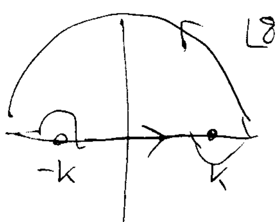
$$= \frac{1}{(2\pi)^3}\int_0^\infty \frac{q^2\,dq}{k^2-q^2}\int_0^{2\pi}d\varphi\int_{-1}^{1}d\cos\theta\; e^{iq|r-r'|\cos\theta}$$

$$= \frac{1}{4\pi^2}\int_0^\infty \frac{q^2\,dq}{k^2-q^2}\,\frac{e^{iq|r-r'|}-e^{-iq|r-r'|}}{iq|r-r'|}$$

$$= \frac{1}{i4\pi^2}\frac{1}{|\vec r-\vec r'|}\int_0^\infty \frac{q\,dq}{(k^2-q^2)}\left(e^{iq|r-r'|}-e^{-iq|r-r'|}\right)$$

$$= \frac{1}{4\pi^2 i}\,\frac{1}{|\vec r-\vec r'|}\int_{-\infty}^{\infty}\frac{q\,dq\;e^{iq|r-r'|}}{(k^2-q^2)}$$

(8)



outgoing

$\Downarrow$

$G_+ = -\frac{1}{4\pi}\frac{e^{ik|r-r'|}}{|r-r'|}$,

incoming $\Downarrow$

$G_- = -\frac{1}{4\pi}\frac{e^{-ik|r-r'|}}{|r-r'|}$