

WF $\Rightarrow \psi(x, t)$, $1 = \int \psi^* \psi dx$, probability $P = |\psi|^2$

S.E $\Rightarrow -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) + V(x) \psi(x, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t)$

Solution $\Rightarrow$ stationary state $\psi_n(x, t) = \psi_n(x) e^{-\frac{i\bar{E}_n}{\hbar} t}$

and $-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_n(x) + V(x) \psi_n(x) = \bar{E}_n \psi_n(x)$

$\Rightarrow \psi(x, t) = \sum_n c_n \psi_n e^{-\frac{i\bar{E}_n}{\hbar} t}$, in 1D $\int \psi_n^* \psi_m dx = \delta_{nm}$

$\langle \hat{Q} \rangle = \int dx \psi^*(x, t) \hat{Q} \psi(x, t)$

$1 = \sum_{n, m} c_n^* c_m e^{\frac{i(\bar{E}_n - \bar{E}_m)}{\hbar} t} \int \psi_n^* \psi_m dx$

$= \sum_{n, m} c_n^* c_m \delta_{nm} e^{\frac{i(\bar{E}_n - \bar{E}_m)}{\hbar} t}$

$= \sum_n |c_n|^2$

$= (c_1^*, c_2^*, c_3^*, \dots)$

$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$

$= \sum_{n, m} c_n^* c_m e^{\frac{i(\bar{E}_n - \bar{E}_m)}{\hbar} t} \int \psi_n^* \hat{Q} \psi_m dx$

Taking S.H.O as example , $\psi(x, t)$ can be represent by $\{c_n\}$

$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}} (a^+ + a)$ $\hat{P} = i\sqrt{\frac{\hbar m\omega}{2}} (a^+ - a)$

$a^+ \psi_n = \sqrt{n+1} \psi_{n+1}$

$a \psi_n = \sqrt{n} \psi_{n-1}$

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$$\hat{X} \psi(x,t) = \sqrt{\frac{\hbar}{2m\omega}} (a^- + a) \sum_{n=0}^{\infty} C_n(t) \psi_n(x)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left\{ \sum_n \left(C_n(t) \sqrt{n+1} \psi_{n+1} \right) + \sum_n \left(C_n(t) \sqrt{n} \psi_{n-1} \right) \right\}$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \sum_n \left(C_{n-1}(t) \sqrt{n} + C_{n+1}(t) \sqrt{n+1} \right) \psi_n$$

we see that

$$\hat{X} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ \vdots \end{pmatrix} \longrightarrow \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} C_1 \\ C_0 + C_2\sqrt{2} \\ C_1\sqrt{2} + C_3\sqrt{3} \\ C_2\sqrt{3} + C_4\sqrt{4} \\ \vdots \end{pmatrix}$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ \vdots \end{pmatrix}$$

Similarly,

$$\hat{P} = i \sqrt{\frac{\hbar m \omega}{2}} (a^+ - a)$$

$$\Rightarrow \hat{P} \psi(x,t) = i \sqrt{\frac{\hbar m \omega}{2}} (a^+ - a) \sum_{n=0}^{\infty} C_n(t) \psi_n(x)$$

$$= i \sqrt{\frac{\hbar m \omega}{2}} \left\{ \sum_n C_n \sqrt{n+1} \psi_{n+1} - \sum_n C_n \sqrt{n} \psi_{n-1} \right\}$$

$$= i \sqrt{\frac{\hbar m \omega}{2}} \sum_n \left(C_{n-1} \sqrt{n} - C_{n+1} \sqrt{n+1} \right) \psi_n$$

$$\Rightarrow \hat{P} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} \Rightarrow i\sqrt{\frac{\hbar m \omega}{2}} \begin{pmatrix} -c_1\sqrt{1} \\ c_0\sqrt{1} - c_2\sqrt{2} \\ c_1\sqrt{2} - c_3\sqrt{3} \\ c_2\sqrt{3} - c_4\sqrt{4} \\ \vdots \end{pmatrix}$$

$$= i\sqrt{\frac{\hbar m \omega}{2}} \begin{pmatrix} 0 & -\sqrt{1} & & & \\ \sqrt{1} & 0 & -\sqrt{2} & & \\ & \sqrt{2} & 0 & -\sqrt{3} & \\ & & \sqrt{3} & 0 & -\sqrt{4} \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

operator $\leadsto$ a matrix acting on the vector made of $\{c_n\}$

$$\hat{X} \leadsto i\sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & \sqrt{1} & & & \\ \sqrt{1} & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \sqrt{3} & 0 & \sqrt{4} \\ 0 & & & \sqrt{4} & 0 \\ & & & & \ddots \end{pmatrix} = M_X = M_X^+ \quad (\equiv (M_X^T)^*)$$

also $\hat{P} \leadsto M_P = i\sqrt{\frac{\hbar m \omega}{2}} \begin{pmatrix} 0 & -\sqrt{1} & & & \\ \sqrt{1} & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \sqrt{3} & 0 & \sqrt{4} \\ & & & \sqrt{4} & 0 \\ & & & & \ddots \end{pmatrix}$

$$M_P^+ = (M_P^T)^* = \left(i\sqrt{\frac{\hbar m \omega}{2}} \begin{pmatrix} 0 & +\sqrt{1} & & & \\ -\sqrt{1} & 0 & +\sqrt{2} & & \\ & -\sqrt{2} & 0 & +\sqrt{3} & \\ & & -\sqrt{3} & 0 & +\sqrt{4} \\ & & & -\sqrt{4} & 0 \\ & & & & \ddots \end{pmatrix} \right)^*$$

$$= -i\sqrt{\frac{\hbar m \omega}{2}} \begin{pmatrix} 0 & +\sqrt{1} & & & \\ -\sqrt{1} & 0 & +\sqrt{2} & & \\ & -\sqrt{2} & 0 & +\sqrt{3} & \\ & & -\sqrt{3} & 0 & +\sqrt{4} \\ & & & -\sqrt{4} & 0 \\ & & & & \ddots \end{pmatrix} = \underline{\underline{M_P}}$$

Is that special?

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Let's check $(\hat{X})^2 = \left(\frac{\hbar}{\sqrt{2m\omega}}\right)^2 (a^\dagger + a)^2$ $\{a, a^\dagger\} = 1$
 $aa^\dagger - a^\dagger a = 1$

$$= \frac{\hbar}{2m\omega} \left((a^\dagger)^2 + a^2 + \underbrace{a^\dagger a + a a^\dagger}_{= 2a^\dagger a + 1} \right)$$

$$(\hat{X})^2 \sum c_n \psi_n = \frac{\hbar}{2m\omega} \left(\sum_n c_n \sqrt{n+1}\sqrt{n+2} \psi_{n+2} + \sum_n c_n \sqrt{n}\sqrt{n-1} \psi_{n-2} \right. \\ \left. + 2 \sum_n c_n \sqrt{n} \cdot \sqrt{n} \psi_n + \sum_n c_n \psi_n \right)$$

$$= \frac{\hbar}{2m\omega} \sum_n \left(c_{n-2} \sqrt{n}\sqrt{n-1} + c_{n+2} \sqrt{n+2}\sqrt{n+1} + 2\sqrt{n} \cdot \sqrt{n} c_n \right) \psi_n + c_n$$

$$\Rightarrow \hat{X}^2 \leadsto M_{\hat{X}^2} = \frac{\hbar}{2m\omega} \begin{pmatrix} 1 & 0 & \sqrt{1}\sqrt{2} & 0 & 0 \\ 0 & 1+2 & 0 & \sqrt{2}\sqrt{3} & 0 \\ \sqrt{2}\sqrt{1} & 0 & 1+2\sqrt{2} & 0 & \sqrt{3}\sqrt{4} \\ 0 & \sqrt{2}\sqrt{1} & 0 & 1+2 & 0 \\ 0 & 0 & \sqrt{3}\sqrt{2} & 0 & 1+2 \end{pmatrix}$$

$$(M_X)^2 = \frac{\hbar}{2m\omega} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ 0 & 0 & \sqrt{4} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ 0 & 0 & \sqrt{4} & 0 & 0 \end{pmatrix}$$

$$= \frac{\hbar}{2m\omega} \begin{pmatrix} 1 & 0 & \sqrt{1}\sqrt{2} & 0 & 0 \\ 0 & 1+2 & 0 & \sqrt{2}\sqrt{3} & 0 \\ \sqrt{2}\sqrt{1} & 0 & 1+2\sqrt{2} & 0 & \sqrt{3}\sqrt{4} \\ 0 & \sqrt{2}\sqrt{1} & 0 & 1+2 & 0 \\ 0 & 0 & \sqrt{3}\sqrt{2} & 0 & 1+2 \end{pmatrix} = M_{X^2}$$

$$\Rightarrow \hat{X} \hat{P} = \sqrt{\frac{\hbar}{2m\omega}} i \sqrt{\frac{\hbar m\omega}{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}$$

$$= \frac{i\hbar}{2} \begin{pmatrix} 1 & 0 & -\sqrt{1}\sqrt{2} & 0 & 0 \\ 0 & -1+2 & 0 & -\sqrt{2}\sqrt{3} & 0 \\ \sqrt{2} & 0 & 2+3 & 0 & -\sqrt{3}\sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{i\hbar}{2} \begin{pmatrix} 1 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & -\sqrt{3} & 0 \\ \sqrt{2} & 0 & 5 & 0 & -2\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{P} \hat{X} = \frac{i\hbar}{2} \begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}$$

$$= \frac{i\hbar}{2} \begin{pmatrix} -1 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 1-2 & 0 & -\sqrt{2}\sqrt{3} & 0 \\ \sqrt{2} & 0 & 2-3 & 0 & -\sqrt{3}\sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \hat{X} \hat{P} - \hat{P} \hat{X} = \frac{i\hbar}{2} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} = i\hbar \mathbb{1}$$

Proved $\{\hat{X}, \hat{P}\} = i\hbar$

$$SA \sin B = \frac{1}{2}(C(A+B) - C(A-B))$$

$$= \frac{1}{2}(C(A+B) - C(A-B))$$

$$2. \infty \text{ S.W. } \int_0^L \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \cdot x \sin \frac{m\pi x}{L} dx$$

$$= \frac{1}{2} \left(\frac{2}{L} \right) \int_0^L x \left(\cos \frac{(n-m)\pi x}{L} - \cos \frac{(n+m)\pi x}{L} \right) dx$$

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_n(x) + V(x) \psi_n(x) = E_n \psi_n(x)$$

$$\psi_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \phi_n(k) e^{ikx}$$

let then

$$\Rightarrow +\frac{\hbar^2}{2m} k^2 \int_{-\infty}^{\infty} dk \phi_n(k) e^{ikx}$$

R.H.S.

$$\Rightarrow E_n \int_{-\infty}^{\infty} dk \phi_n(k) e^{ikx}$$

$$V(x) \psi_n(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} dq dp e^{ipx} \tilde{V}(p) e^{iqx} \phi_n(q)$$

$$k = \frac{p+q}{2}$$

$$w = \frac{p-q}{2}$$

$$p = z+w$$

$$q = z-w$$

$$\begin{vmatrix} \frac{\partial p}{\partial z} & \frac{\partial p}{\partial q} \\ \frac{\partial w}{\partial z} & \frac{\partial w}{\partial q} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\Rightarrow dp dq = -2 dz dw$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} dz dw e^{izx} \tilde{V}(z+w) \phi_n(z-w)$$

$$= -\left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} dk dw e^{ikx} \tilde{V}\left(\frac{k}{2}+w\right) \phi_n\left(\frac{k}{2}-w\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dq' e^{i\tilde{V}(\frac{k}{2}-q')} \phi_n\left(\frac{k}{2}+q'\right) \right] e^{ikx}$$

$\Rightarrow$ The "S.E" for $\phi_n(k)$ is: $q' \Rightarrow q + \frac{k}{2}$

$$\left[\frac{\hbar^2 k^2}{2m} \phi_n(k) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dq \tilde{V}(q) \phi_n(k+q) \right] = E_n \phi_n(k)$$

$$\left[E_n - \frac{\hbar^2 k^2}{2m} \right] \phi_n(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dq \tilde{V}(k-q) \phi_n(q)$$

Becomes an integral equation!