Linear vector space.

Vectors $|u\rangle$, $|v\rangle$,

complex c-# a, b, c.

(a) addition: exists another vector $|w\rangle = |u\rangle + |v\rangle$

(b) ~~addition:~~ $|u\rangle + |v\rangle = |v\rangle + |u\rangle$ for all $|u\rangle, |v\rangle$

(c) null vector $|u\rangle + |0\rangle = |0\rangle + |u\rangle = |u\rangle$ for all $|u\rangle$

(d) $|u'\rangle = a|u\rangle$ is in the same direction as $|u\rangle$

(e) distributive law $a(|u\rangle + |v\rangle) = a|u\rangle + a|v\rangle$

☆ $|u_1\rangle, |u_2\rangle, \ldots |u_n\rangle$ are linear-independent

iff $a_1|u_1\rangle + \cdots a_n|u_n\rangle = 0$ possesses no solution except $a_1 = \cdots = a_n = 0$

⇒ max # of ~~indep~~ linear-indep.

(any n+1 vector are linear dependent) ⇒ n-dimensional

finite,

denumerably infinite

continously infinite

☆ n-dim space, ~~is~~ linearly-indep. vectors $|u_1\rangle \ldots |u_n\rangle$ are said to span the space, or form a basis for the space

⇒ any vector can be

$$|w\rangle = \sum_{i=1}^{n} a_i |u_i\rangle \qquad a_i : \text{complex c-\#}$$

☆ Unitary space: one can define the scalar product $\langle u|v\rangle$ for any $|u\rangle, |v\rangle$ with the properties

$$\langle u|v\rangle = \overline{\langle v|u\rangle}$$

$$\langle u|av\rangle = a\langle u|v\rangle \quad \text{so} \quad \langle au|v\rangle = \overline{\langle v|au\rangle} = a^*\overline{\langle v|u\rangle} = a^*\langle u|v\rangle$$

$$\langle u|v+w\rangle = \langle u|v\rangle + \langle u|w\rangle$$

$$\langle u|u\rangle > 0 \quad \text{unless} \quad |u\rangle = 0$$

☆ $\langle u|u\rangle \equiv$ "norm" of $|u\rangle$, $\sqrt{\langle u|u\rangle} \equiv$ "length" of $|u\rangle$

- If $|w\rangle \neq 0$, $|v\rangle \neq 0$, but $\langle w|v\rangle = 0$. $\Rightarrow$ orthogonal.

☆ A unitary space with an infinite # of dimension is a "Hilbert space"

However, you can see the finite-dim Hilbert space in QM.

$\Rightarrow$ ~~The first few rules of QM.~~

⊕ ~~Every physical system is associated with a Hilbert space~~ $\mathcal{E}$.

orthonormal basis

Schmidt process — given linearly-independent $|u_1\rangle \cdots |u_n\rangle$ in a unitary space.

$$|\varphi_1\rangle = \frac{|u_1\rangle}{\sqrt{\langle u_1|u_1\rangle}}$$

$$|\varphi_2\rangle = a|u_2\rangle + b|\varphi_1\rangle \Rightarrow \overset{\text{such that}}{\langle \varphi_1|\varphi_2\rangle = 0}, \quad \langle \varphi_2|\varphi_2\rangle = 1$$

$$b + a\langle \varphi_1|u_2\rangle = 0 \qquad b = -a\langle \varphi_1|u_2\rangle$$

$$\cancel{|a|^2\langle u_2|u_2\rangle + |b|^2 \quad 1}$$

$$|\varphi_2\rangle = a\left(|u_2\rangle - |\varphi_1\rangle\langle \varphi_1|u_2\rangle\right)$$

$\Rightarrow$ $\cancel{1 = |a|^2\left[\langle u_2|u_2\rangle - \langle u_2|\varphi_1\rangle\langle \varphi_1|u_2\rangle\right.}$

$$\langle \varphi_2|\varphi_2\rangle = a^*a\left[\langle u_2| - \langle u_2|\varphi_1\rangle\langle \varphi_1|\right]\left[|u_2\rangle - |\varphi_1\rangle\langle \varphi_1|u_2\rangle\right]$$

$$= |a|^2\left(\langle u_2|u_2\rangle + \langle u_2|\varphi_1\rangle\langle \varphi_1|u_2\rangle - \langle u_2|\varphi_1\rangle\langle \varphi_1|u_2\rangle - \langle u_2|\varphi_1\rangle\langle \varphi_1|u_2\rangle\right)$$

$$= |a|^2\left[\langle u_2|u_2\rangle - \langle u_2|\varphi_1\rangle\langle \varphi_1|u_2\rangle\right] \Rightarrow \text{'a' can be determined.}$$

$\Rightarrow$ next

$$|\varphi_3\rangle = a'\left[|u_3\rangle - |\varphi_1\rangle\langle \varphi_1|u_3\rangle - |\varphi_2\rangle\langle \varphi_2|u_3\rangle\right] \quad \cdots$$

$\Rightarrow$ $|\varphi_1\rangle, |\varphi_2\rangle \cdots |\varphi_n\rangle$ orthonormal basis.

* Expansion of $|w\rangle$ in terms of an orthonormal basis $|\phi_i\rangle$

$$|w\rangle = \sum_i a_i |\phi_i\rangle$$

then $\langle \phi_j | w \rangle = \sum_i a_i \langle \phi_j | \phi_i \rangle = \sum_i a_i \delta_{ij} = a_j$

$$\Rightarrow \boxed{|w\rangle = \sum_i |\phi_i\rangle \langle \phi_i | w\rangle}$$

can also be viewed as

$$\underbrace{\left(\sum_i |\phi_i\rangle\langle\phi_i| \right)}_{\mathbb{1}} |w\rangle \Rightarrow$$

complete: any vector in the space can be expressed as a linear combination of the basis vectors.

The scalar product of any ~~two~~ $|v\rangle, |w\rangle$

$$|v\rangle = \sum_i a_i |\phi_i\rangle \qquad |w\rangle = \sum_j b_j |\phi_j\rangle$$

$$\langle v | w \rangle = \sum_{i,j} a_i^* b_j^* \langle \phi_i | \phi_j \rangle = \sum_i a_i^* b_i$$

* <u>Schwarz inequality</u>.

any two $|u\rangle, |v\rangle$ $\boxed{\langle u|u\rangle \cdot \langle v|v\rangle \geq |\langle u|v\rangle|^2}$

equality $\Leftrightarrow$ $|u\rangle = a|v\rangle$

let $|w\rangle = |u\rangle + h|v\rangle$, h: complex

$$\langle w|w\rangle = \langle u|u\rangle + |h|^2\langle v|v\rangle + h\langle u|v\rangle + h^*\langle v|u\rangle$$

writing $\langle u|v\rangle = \alpha + i\beta$ & $h = a + bi$

$$\langle u|u\rangle + (a^2+b^2)\langle v|v\rangle + 2(a\alpha - b\beta)$$

extremum of $\langle w|w\rangle$

$$0 = \frac{\partial}{\partial a}\langle w|w\rangle = 2a\langle v|v\rangle + 2\alpha, \quad 0 = \frac{\partial}{\partial b}\langle w|w\rangle = 2b\langle v|v\rangle - 2\beta$$

since $\langle v|v\rangle > 0$. $\Rightarrow$ $\langle v|v\rangle = -\frac{\alpha}{a} = +\frac{\beta}{b}$

or $a = -\frac{\alpha}{\langle v|v\rangle}$, $b = \frac{\beta}{\langle v|v\rangle}$

$$\Rightarrow \langle w|w\rangle = \langle u|u\rangle + \frac{\alpha^2+\beta^2}{(\langle v|v\rangle)^2}\langle v|v\rangle + 2\left(\frac{-\alpha^2}{\langle v|v\rangle} - \frac{\beta^2}{\langle v|v\rangle}\right)$$

$$= \langle u|u\rangle - \frac{(\alpha^2+\beta^2)}{\langle v|v\rangle} \geq 0$$

namely.

$$\langle u|u\rangle\langle v|v\rangle \geq (\alpha^2+\beta^2) = |\langle u|v\rangle|^2$$

equality $\leadsto \langle w|w\rangle = 0 \quad \Rightarrow \quad |u\rangle = -h|v\rangle$

~~☆ first 2 rules: ① Every physical system is associated with a Hilbert space $\mathcal{E}$ (the vector of this space is call kets) ② Every state of a physical system is associated with a definite ray in $\mathcal{E}$~~

Linear operators: map a vector $|u\rangle$ into another vector $|v\rangle$

we write $|v\rangle = A|u\rangle$

if A is a linear operator

$$A(a|u\rangle) = a(A|u\rangle)$$

$$A[|u\rangle + |v\rangle] = A|u\rangle + A|v\rangle$$

$|v\rangle = AB|u\rangle$ means $|w\rangle = B|u\rangle$, $A|w\rangle = |v\rangle$

if ~~for~~ all $|u\rangle$ $AB|u\rangle = BA|u\rangle$. we write $AB = BA$

or $[A, B] = AB - BA = 0$

A, B are said to commute. (in general, not the case e.g. 3D rotation)

☆ Adjoint operator

with the use of scalar product, we could introduce the idea of a dual vector space in 1−1 correspondence.

$$\langle u| \overset{1-1}{\longleftrightarrow} |u\rangle$$

suppose. $|w\rangle = A|u\rangle$ the dual of $|w\rangle$ is $\langle w|$

the dual of $|u\rangle$ is $\langle u|$

There exists some operator maps $\langle u|$ into $\langle w|$.
call it $A^\dagger$, the <u>adjoint</u> or <u>Hermitian conjugate</u> of A

$$\langle u|A^\dagger = \langle w| \; ; \quad \Longleftrightarrow \quad |w\rangle = A|u\rangle$$

since $\langle v|w\rangle = \overline{\langle w|v\rangle}$

$$\langle v|A|u\rangle = \overline{(\langle u|A^\dagger)|v\rangle} \qquad \text{for any } A,\ |u\rangle,\ |v\rangle$$

even $A^\dagger$ was defined as an operator on the dual space we may consider it as an operator acting on the original space

$$\langle v|A|u\rangle = \overline{\langle u|\underbrace{A^\dagger|v\rangle}}$$

let $|w\rangle = A^\dagger|u\rangle$, $|x\rangle = B|v\rangle$

$$\langle w|x\rangle = \langle u|AB|v\rangle = \overline{\langle x|w\rangle} = \overline{\langle v|B^\dagger A^\dagger|u\rangle}$$

$$(AB)^\dagger = B^\dagger A^\dagger$$

Hermitian operator

If A is self-adjoint or Hermitian $\Rightarrow$ $\boxed{A^\dagger = A}$

For a Hermitian operator A

$$\langle v|A|u\rangle = \overline{\langle u|A^\dagger|v\rangle} = \overline{\langle u|A|v\rangle}$$

e.g. $|u\rangle\langle u|$ is a Hermitian operator

for any $|w\rangle, |v\rangle$

$$\langle w|(|u\rangle\langle u|)|v\rangle = \overline{\langle u|w\rangle}\,\overline{\langle v|u\rangle} = \overline{\langle v|u\rangle\langle u|w\rangle}$$