

The eigenvalue problem.

(1)

If $|v\rangle = A|u\rangle$, in general $|v\rangle$ is in a diff. direction than $|u\rangle$. However, there might exist vectors $|u_1\rangle, |u_2\rangle, \dots$ the "eigenvectors" of A , such that

$$A|u_i\rangle = \lambda_i |u_i\rangle$$

eigenvalues, $\Rightarrow$ collection of the λ_i is called the "spectrum" of A .

For a Hermitian operator A , the eigenvalues are all real and the eigenvectors corresponding to distinct eigenvalues are mutually orthogonal.

if $A|u_n\rangle = \lambda_n |u_n\rangle$

then $\langle u_n | A | u_n \rangle = \lambda_n \langle u_n | u_n \rangle$

c.c. $\downarrow$
 $\overline{\langle u_n | A | u_n \rangle} = \langle u_n | A^\dagger | u_n \rangle = \langle u_n | A | u_n \rangle$
" $\lambda_n^* \langle u_n | u_n \rangle$

$$\Rightarrow \boxed{\lambda_n = \lambda_n^*}$$

Moreover $\langle u_m | A | u_n \rangle = \lambda_n \langle u_m | u_n \rangle$

$$\overline{\langle u_m | A | u_n \rangle} = \lambda_m^* \overline{\langle u_n | u_m \rangle} = \lambda_m^* \langle u_m | u_n \rangle$$

$$\langle u_m | A | u_n \rangle$$

Subtract the two $\Rightarrow (\lambda_m - \lambda_n) \langle u_m | u_n \rangle = 0$

$$\Rightarrow \text{if } \lambda_m \neq \lambda_n, \langle u_m | u_n \rangle = 0$$

degenerate case, exist k linearly indep. degenerate eigenvectors

$\Rightarrow k$ -dim subspace S_k .

$\Rightarrow$ The basis of S_k can be made orthonormal by the Schmidt process.

$\Rightarrow$ All eigenvectors of a Hermitian operator are, or can be made, orthonormal.

Projection operators & completeness

(2)

A set of basis vectors is complete if any vector in the space can be expressed as a linear combination of the basis vectors.

- For finite-dim vector space, any set that spans the space is complete.

orthonormal $|u_i\rangle$, (assume to be eigenvectors of operator $\hat{Q}$)

$$|w\rangle = \sum |u_i\rangle \langle u_i | w \rangle$$

- The operator $P_i \equiv |u_i\rangle \langle u_i|$ (Hermitian) is a projection operator that acts on $|w\rangle$ to pick up the component of $|w\rangle$ in the direction $|u_i\rangle$.

- The sum of all these projections must be identity,

$$P_A = \sum_{i=1} |u_i\rangle \langle u_i| = \sum_i P_i = \mathbb{1} \Rightarrow \text{completeness relation.}$$

~~same~~

for continuously co-dim space

$$\mathbb{1} = \int d\lambda | \lambda \rangle \langle \lambda |$$

Representations,

$$|w\rangle = \sum |u_i\rangle \underbrace{\langle u_i | w \rangle}_{\text{complex \#}}$$

$\sim$ "represent" vector $|w\rangle$ with respect to the basis $|u_i\rangle$

These numbers can be arranged in ~~the~~

a column matrix

$$\begin{pmatrix} \langle u_1 | w \rangle \\ \langle u_2 | w \rangle \\ \vdots \end{pmatrix}$$

Similarly, the dual vector can be expressed in terms of the same basis:

$$\langle v | = \sum_i \langle v | u_i \rangle \langle u_i |$$

$\Rightarrow$ row matrix $(\langle v | u_1 \rangle, \langle v | u_2 \rangle, \langle v | u_3 \rangle, \dots)$

$\Rightarrow$ This represents $\langle v |$ with respect to basis $\langle u_i |$

The scalar product $\langle v | w \rangle$ is obtained by

$$(\langle v | u_1 \rangle, \langle v | u_2 \rangle, \dots) \begin{pmatrix} \langle u_1 | w \rangle \\ \langle u_2 | w \rangle \\ \vdots \end{pmatrix} = \langle v | w \rangle$$

Now consider an operator A

$$A = \mathbb{1} A \mathbb{1} = P_a A P_a$$

$$= \sum_{i,j} (|u_i\rangle \langle u_i|) A (|u_j\rangle \langle u_j|) = \sum_{i,j} |u_i\rangle \langle u_i | A | u_j \rangle \langle u_j |$$

$\Rightarrow$ The matrix $A_{ij} \equiv \begin{pmatrix} \langle u_1 | A | u_1 \rangle & \langle u_1 | A | u_2 \rangle & \dots \\ \langle u_2 | A | u_1 \rangle & \langle u_2 | A | u_2 \rangle & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$

represents A with respect to basis $|u_i\rangle$

Since $|u_i\rangle$ are eigenvectors of Q with eigenvalues g_i

$Q |u_i\rangle = g_i |u_i\rangle$, The matrix of Q in basis $|u_i\rangle$

is diagonal, $\{Q\} = \begin{pmatrix} g_1 & & 0 \\ & g_2 & \\ 0 & & \ddots \end{pmatrix}$

or $Q = P_a Q P_a = \sum_{i,j} |u_i\rangle \langle u_i| Q |u_j\rangle \langle u_j|$

$$= \sum_{i,j} g_j \delta_{ij} |u_i\rangle \langle u_j| = \sum_i g_i |u_i\rangle \langle u_i|$$

The rules of QM

- ✓ (1) Every physical system is associated with a Hilbert Space $\mathcal{E}$.
- ✓ (2) Every state of a physical system is associated with a definite ray in $\mathcal{E}$.

(3) Every measurement process, which can be carried out, on the system corresponds to a complete Hermitian operator A .

(4) The possible results of the measurement are the eigenvalues of A

a_i { discrete $a_1, a_2, \dots$
continuous $a(\nu)$

(statistic)

(5) In the discrete case.

$$\text{Prob}(A = a_n) = \frac{\langle \psi | P_n | \psi \rangle}{\langle \psi | \psi \rangle}$$

P_n is the projection operator onto the subspace $\mathcal{E}_n$ corresponding to a_n , $|\psi\rangle$ is any nonzero ket in the ray representing the state of the system.

In continuous case, $\text{Prob}(a_0 \leq A \leq a_1) = \frac{\langle \psi | P_I | \psi \rangle}{\langle \psi | \psi \rangle}$

(Collapse postulate) P_I is the projection operator corresponding to $I = [a_0, a_1]$

After the measurement with discrete outcome $A = a_n$, the system is represented by the ket $P_n |\psi\rangle$

after outcome

The system is in an eigenstate of A with eigenvalue a_n afterwards

for continuous case $\Rightarrow P_I |\psi\rangle$.