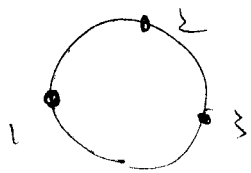


An example,



3-site ring; the particle can only live on the sites, $|1\rangle, |2\rangle, |3\rangle$

It's a 3-dim Hilbert space

$$\hat{S}_{\text{site}} |s\rangle = s |s\rangle \quad s = 1, 2, 3$$

eg. a state vector

$$|\psi\rangle = 4|1\rangle + 2|2\rangle + 5e^{i37^\circ}|3\rangle$$

$$\hat{S}_{\text{site}} |\psi\rangle = 1 \times 4|1\rangle + 2 \times 2|2\rangle + 3 \times 5e^{i37^\circ}|3\rangle$$

$$\hat{X} = \left(\frac{L}{S} \right) \times \hat{S}_{\text{site}} \approx \left(\frac{L}{S} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = (\hat{X})^\dagger$$

consider a 1-step translation

$$|s\rangle \rightarrow |s+1\rangle$$

$$\hat{D}|s\rangle = |s+1\rangle \quad \hat{D} \sim \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

then for a state $|\psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$
not necessary

$$\hat{D}|\psi\rangle = \frac{1}{\sqrt{3}}(|2\rangle + |3\rangle + |1\rangle) = |\psi\rangle, \text{ It's the eigenstate of } \hat{D}$$

Let's ~~define~~ construct a similar state

$$|\psi_k\rangle = \frac{1}{\sqrt{3}} \left(e^{ik \cdot 1} |1\rangle + e^{ik \cdot 2} |2\rangle + e^{ik \cdot 3} |3\rangle \right)$$

$$\begin{aligned} \text{then } \hat{D}|\psi_k\rangle &= \frac{1}{\sqrt{3}} \left(e^{ik \cdot 1} |2\rangle + e^{ik \cdot 2} |3\rangle + e^{ik \cdot 3} |1\rangle \right) \\ &= e^{-ik \cdot 1} \left(e^{ik \cdot 2} |2\rangle + e^{ik \cdot 3} |3\rangle + e^{ik \cdot 4} |1\rangle \right) \end{aligned}$$

If we require that ψ_k is a eigenstate of $\hat{D}$ with eigenvalue $e^{-ik \cdot 1}$

Then $e^{ik_1 t} = e^{ik} \Rightarrow 3k = 2\pi\pi$

or $k_n = \frac{2n\pi}{3}$, n integer.

(2) $(D)^3 = \mathbb{I}$
 $= e^{-i3k}$

we write $|k_s\rangle = |s\rangle$. It's the eigenstate of $\hat{X}$

and $|k_n\rangle = \sum_{s=1}^3 \frac{1}{\sqrt{3}} (e^{ik_n \cdot s} |s\rangle)$

(since $x_s = (\frac{L}{3}) \cdot s$) $= \sum_{s=1}^3 \frac{1}{\sqrt{3}} e^{ik_n \frac{3x_s}{L}} |x_s\rangle$
 $= \sum_{s=1}^3 \frac{1}{\sqrt{3}} e^{i \frac{2n\pi}{L} x_s} |x_s\rangle$

$n = 0, 1, 2$.

let $3 \rightarrow N$. $k_n = \frac{2\pi n}{L}$, $\hat{X} = (\frac{L}{N}) \text{ site}$

$\langle x_s | k_n \rangle = \left(\frac{1}{\sqrt{N}} \right) e^{ik_n x_s}$
not necessary

$|k_n\rangle$ is the eigenstate of $\hat{P}$ with eigenvalue $e^{-ik_n (\frac{L}{N})}$

momentum operator is define

$\hat{P} |k_n\rangle = \hbar k_n |k_n\rangle$, the eigenvalue is $\hbar k_n$

dimension $\left[\frac{ML}{T} \right]$ $\left[\frac{E \cdot T}{\hbar} \right] = \left[\frac{ML^2}{T^2 \cdot T} \right]$ $\left[\frac{1}{L} \right]$

$\Rightarrow \hat{P} |k_n\rangle = e^{-ik_n (\frac{L}{N})} |k_n\rangle = e^{-i \frac{\hat{P}}{\hbar} (\frac{L}{N})} |k_n\rangle$

so $\hat{P}$ can be viewed as $\hat{P} = \exp(-i \frac{\hat{P}}{\hbar} (\frac{L}{N}))$

Note that we can define 2-step, 3-step ... displacement

$$\hat{D}_2 = (\hat{D})^2 = e^{-i \frac{\hat{p}}{\hbar} \left(\frac{2L}{N} \right)}$$

$$\hat{D}_3 = (\hat{D})^3 = \exp\left(-i \frac{\hat{p}}{\hbar} \left(\frac{3L}{N} \right)\right)$$

⋮

$\hat{p}$ is called the generator of translation

$$\Rightarrow \hat{D}(\Delta x) = \exp\left(-i \frac{\hat{p}}{\hbar} \Delta x\right)$$

Now the continuous limit, $N \rightarrow \infty$

$\rightarrow$ N -dim Hilbert space

$$\begin{array}{c} \xleftrightarrow{\epsilon} \\ \text{---} \end{array} \quad \epsilon = \frac{L}{N}$$

$$\text{and } X_s = s \times \epsilon$$

$$\text{therefore } \sum_s \Rightarrow \sum_{X_s} \frac{1}{\epsilon} \Rightarrow \int \frac{dx}{\epsilon}$$

$$\hat{X} |x_s\rangle = x_s |x_s\rangle$$

$$|\psi\rangle = \sum_s \psi_s |s\rangle = \sum_{X_s} \frac{1}{\epsilon} \psi_s(x_s) |x_s\rangle = \int dx \left(\frac{\psi(x)}{\sqrt{\epsilon}} \right) \frac{|x_s\rangle}{\sqrt{\epsilon}}$$

$$\text{we write } \psi(x) \equiv \frac{1}{\sqrt{\epsilon}} \psi_s(x_s)$$

$$\text{and } |x\rangle \equiv \frac{1}{\sqrt{\epsilon}} |x_s\rangle (= \frac{1}{\sqrt{\epsilon}} |s\rangle)$$

$$\Rightarrow |\psi\rangle = \int dx \psi(x) |x\rangle$$

$$\text{apparently, } \langle x' | x \rangle = \frac{1}{\epsilon} \langle s' | s \rangle = \frac{1}{\epsilon} \delta_{s',s}$$

$$\lim_{N \rightarrow \infty} = \delta(x' - x)$$

the Dirac delta function

$$\begin{array}{c} \updownarrow \\ \text{---} \\ \epsilon \end{array}$$

and $\psi(x) = \langle x | \psi \rangle \Rightarrow$ this ~~is~~ ^{is} the Schrödinger wave function

$$\langle \psi | \psi \rangle = \sum_s |\psi_s|^2 = \int \frac{dx}{\epsilon} |\sqrt{\epsilon} \psi(x)|^2 = \int dx |\psi(x)|^2$$

$$= \begin{cases} 1 & \text{if normalized.} \\ \text{finite} & \text{if physical.} \end{cases}$$

Wave function is the coefficient the state projected onto a basis.

$$\psi(x) = \langle x | \psi \rangle, \quad \langle x | x' \rangle = \delta(x - x')$$

$$\begin{aligned} |\psi\rangle &= \mathbb{1}_x |\psi\rangle = \int dx (|x\rangle \langle x|) |\psi\rangle \\ &= \int dx \psi(x) |x\rangle \end{aligned}$$

Momentum eigenstate can also be used as a basis

$$|k_n\rangle \propto \sum_s e^{ik_n x_s} |x_s\rangle \quad (\text{with undetermined normalization})$$

$$\begin{aligned} \langle k_2 | k_1 \rangle &= \sum_{x_s} \langle k_2 | x_s \rangle \langle x_s | k_1 \rangle \\ &\propto \sum_s e^{-ik_2 x_s} e^{ik_1 x_s} = \int dx e^{i(k_1 - k_2)x} \end{aligned}$$

if $k_1 \neq k_2$, oscillations cancel ~~the~~ with each other.
only contribution is when $k_1 = k_2$

$$\Rightarrow \langle k_2 | k_1 \rangle \propto \delta_{k_2, k_1} \Rightarrow \text{orthogonal}$$

In the continuous limit

$$\phi_k(x) = \langle x | k \rangle = A e^{ikx}, \quad \text{we already have } \langle x | x' \rangle = \delta(x - x')$$

$$\begin{aligned} \text{then } \langle x | x' \rangle &= \int dk \langle x | k \rangle \langle k | x' \rangle \\ &= |A|^2 \int dk e^{ikx} e^{-ikx'} = |A|^2 2\pi \delta(x - x') \end{aligned}$$

$$\Rightarrow A = \frac{1}{\sqrt{2\pi}}, \quad \phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

In the momentum basis,

$$\begin{aligned} |\psi\rangle &= \int dk |k\rangle \langle k | \psi \rangle = \int dk |k\rangle \langle k | x \rangle \langle x | \psi \rangle \\ &= \int dk dx |k\rangle \frac{1}{\sqrt{2\pi}} e^{-ikx} \psi(x) \end{aligned}$$

with that normalization fixed

$$\langle k | k' \rangle = \int dx \langle k | x \rangle \langle x | k' \rangle$$

$$= \int dx \left(\frac{1}{\sqrt{2\pi}} \right)^2 e^{-ikx} e^{ik'x}$$

$$= \frac{1}{2\pi} \int dx e^{i(k' - k)x} = \left(\frac{1}{2\pi} \right) (2\pi) \delta(k' - k)$$

$$= \delta(k' - k)$$

We see that the continuous basis satisfies

$$\langle \lambda | \lambda' \rangle = \delta(\lambda - \lambda') \quad \text{orthonormal}''$$

$\Rightarrow$ Momentum wave function. $\psi(k) = \langle k | \psi \rangle$

$$= \frac{1}{\sqrt{2\pi}} \int dx \quad \cancel{\frac{1}{\sqrt{2\pi}}} e^{-ikx} \psi(x)$$

nothing but the Fourier transform as we discussed before.

Now momentum operator in $|x\rangle$ basis?

consider an infinitesimal translation.
(1-step)

$$\begin{aligned} \hat{D}(\epsilon) |\psi\rangle &= \sum_s \psi_s \hat{D}(\epsilon) |s\rangle \\ &= \sum_s \psi_s |s+1\rangle \end{aligned}$$

$$\langle x_s | \hat{D}(\epsilon) |\psi\rangle = \boxed{\psi_{s-1}}$$

$$\hat{D}(\epsilon) = e^{i \frac{\hat{p}}{\hbar} \epsilon} = \mathbb{1} - \frac{i\epsilon}{\hbar} \hat{p}$$

$$\langle x_s | \mathbb{1} - \frac{i\epsilon}{\hbar} \hat{p} | \psi \rangle$$

$$= \psi_s - \frac{i\epsilon}{\hbar} \langle x_s | \hat{p} | \psi \rangle = \psi_{s-1}$$

$$\Rightarrow \boxed{\langle x_s | \hat{p} | \psi \rangle = \frac{\hbar}{i} \frac{\psi_s - \psi_{s-1}}{\epsilon}}$$

$$\Rightarrow \hat{p} \text{ in the } x\text{-basis} = \boxed{\frac{\hbar}{i} \frac{d}{dx} \psi(x)}$$

How about the evolution.

$$T(0) = \mathbb{1}$$

$$T(a)T(b) = T(b)T(a) = T(a+b)$$

$$\begin{aligned} \Rightarrow T(\Delta t) &= \lim_{N \rightarrow \infty} \left(T\left(\frac{\Delta t}{N}\right) \right)^N \\ &= \lim_{N \rightarrow \infty} \left(\mathbb{1} + \frac{i}{\hbar} \hat{H}\left(\frac{\Delta t}{N}\right) \right)^N = \exp\left(-\frac{i}{\hbar} \hat{H} \Delta t\right) \end{aligned}$$

$\downarrow \quad \quad \downarrow \quad \uparrow$
 $[\mathbb{E}, T] \quad [E] \quad [T]$

$$\langle x | \underbrace{T(\epsilon)}_{\text{"}} | \psi(t) \rangle = \langle x | \mathbb{1} - \frac{i}{\hbar} \hat{H} \cdot \epsilon | \psi(t) \rangle$$

$$\underbrace{\langle x | \psi(t+\epsilon) \rangle}_{\psi(x, t+\epsilon)} = \psi(x, t) - \frac{i\epsilon}{\hbar} \langle x | \hat{H} | \psi(t) \rangle$$

$$\Rightarrow \langle x | \hat{H} | \psi(t) \rangle = \frac{i}{\epsilon} \underbrace{\psi(x, t+\epsilon) - \psi(x, t)}_{\epsilon}$$

$$= i \frac{\partial}{\partial t} \psi(x, t)$$

$$\Rightarrow \boxed{i \hbar \frac{\partial}{\partial t} \psi(x, t) = \hat{H} \psi(x, t)}$$

Schrodinger Eq.