① quick review of the previous lecture.

複習. ① $|1\rangle, |2\rangle \ldots$ $\langle s'|s\rangle = \delta_{ss'}$

are orthonormal basis

② $|s\rangle$ is just a naming protocal (Arabic number), if like.

they can be called $|甲\rangle, |乙\rangle, |丙\rangle$ ... or $|壹\rangle, |貳\rangle, |叁\rangle$ ...

$|Do\rangle, |Re\rangle, |Mi\rangle$ ...

bottom line is the ordering $\rightarrow$ integer system

③. In addition to $|s\rangle$,

$|k_n\rangle$ is also a good basis for the N-dim Hilbert space.

$$|k_0\rangle = A\left(|1\rangle + |2\rangle + |3\rangle\right) \qquad k_n = \frac{2\pi n}{3}$$

$$|k_1\rangle = A\left(e^{\frac{2\pi i}{3}}|1\rangle + e^{\frac{4\pi i}{3}}|2\rangle + e^{\frac{6\pi i}{3}}|3\rangle\right)$$

$$|k_2\rangle = A\left(e^{\frac{4\pi i}{3}}|1\rangle + e^{\frac{8\pi i}{3}}|2\rangle + e^{\frac{12\pi i}{3}}|3\rangle\right)$$

$$\langle k_n|k_{n'}\rangle = A^2\delta_{n,n'}$$

④. you can obtain $|s\rangle$ from $|k_n\rangle$

e.g.

$$|1\rangle = \frac{1}{3A}\left[\,|k_0\rangle + e^{-\frac{2\pi i}{3}}|k_1\rangle + e^{-\frac{4\pi i}{3}}|k_2\rangle\,\right]$$

$$|2\rangle = \frac{1}{3A}\left[\,|k_0\rangle + e^{-\frac{4\pi i}{3}}|k_1\rangle + e^{-\frac{8\pi i}{3}}|k_2\rangle\,\right]$$

$$|3\rangle = \frac{1}{3A}\left[\,|k_0\rangle + |k_1\rangle + |k_2\rangle\,\right]$$

In the $N\to\infty$ limit

$$|x\rangle \propto \int dk\, e^{-ikx}|k\rangle$$

or $$|x\rangle = \int dk\, |k\rangle\langle k|x\rangle = \frac{1}{\sqrt{2\pi}}\int dk\, e^{-ikx}|k\rangle$$

② $\hat{P}$ 及 $\hat{A}$ 的引進.

③ uncertainty principle.

$A_1 = A - \bar{A}$, $\quad B_1 = B - \bar{B}$, $\quad A^\dagger = A$, $B^\dagger = B$.

therefore $A_1^\dagger = A_1$, $B_1^\dagger = B_1$

let $|\alpha\rangle = A_1|\psi\rangle$, $\quad |\beta\rangle = B_1|\psi\rangle$

by Schwarz inequality

$$\langle A_1^2\rangle = \langle A^2 - 2A\bar{A} + \bar{A}^2\rangle = \langle A^2\rangle - \bar{A}^2 = \sigma_A^2$$

$$\underbrace{\langle\alpha|\alpha\rangle}_{\Downarrow\ \sigma_A^2}\underbrace{\langle\beta|\beta\rangle}_{\Downarrow\ \sigma_B^2} \geq |\langle\alpha|\beta\rangle|^2$$

$$\sigma_A^2\ \sigma_B^2 \geq |\langle\psi|A_1B_1|\psi\rangle|^2$$

$$A_1B_1 = \frac{A_1B_1 + B_1A_1}{2} + \frac{A_1B_1 - B_1A_1}{2} = \frac{1}{2}\{A_1, B_1\} + \frac{1}{2}[A_1, B_1]$$

anticommutator → Hermitian → real

commutator → antiHermitian → purely imaginary

$$\overline{\langle\psi|O|\psi\rangle} = \langle\psi|O^\dagger|\psi\rangle = \langle\psi|O|\psi\rangle$$

$$\overline{\langle\psi|A|\psi\rangle} = \langle\psi|A^\dagger|\psi\rangle = -\langle\psi|A|\psi\rangle$$

$$\Rightarrow |\langle\psi|A_1B_1|\psi\rangle|^2 = \Big|\frac{1}{2}\underbrace{\langle\psi|\{A_1,B_1\}|\psi\rangle}_{\text{Real}} + \frac{1}{2}\underbrace{\langle\psi|[A_1,B_1]|\psi\rangle}_{\text{Imaginary}}\Big|^2$$

$$= \frac{1}{4}|\langle\{\ \}\rangle|^2 + \frac{1}{4}|\langle[\ ]\rangle|^2 \geq \frac{1}{4}|\langle[\ ]\rangle|^2$$

$$[A_1, B_1] = [A - \bar{A}, B - \bar{B}] = (AB - A\bar{B} - \bar{A}B + \bar{A}\bar{B}) - (BA - B\bar{A} - \bar{B}A + \bar{A}\bar{B}) = [A, B]$$

$$\Rightarrow \boxed{\sigma_A^2\sigma_B^2 \geq \frac{1}{4}|\langle[A,B]\rangle|^2}$$

$$\sigma_x^2 \sigma_p^2 \geq \frac{1}{4} \left| \langle \underbrace{[x, p]}_{= i\hbar} \rangle \right|^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\Rightarrow \boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$

It works for any pair of operators! → It's mathematical outcome, not put in by hand

if $[A, B] = 0$, A, B are called compatible or commuting observables

★ An important property ⇒ A, B admit complete sets of simultaneous eigenstate.

without going into the proof, see appendix

They can be simultaneously diagonalized;

an example $[L_z, L^2] = 0$,

any state can be labeled by $|\ell, m\rangle$

★ comment $\Delta E \Delta t \geq \frac{\hbar}{2}$ is not fundamental, because t is NOT an operator in QM. The relation is derivative —

★ 考试有时间，強調 Midterm 很多人犯错的概念

① current vs density ρ, J — but not $m\langle X\rangle = \langle p\rangle$

② time evolution vs. Ehrenfest relation, $\psi = \sum c_n e^{-iE_n t/\hbar} \phi_n$, $m\frac{d}{dt}\langle x\rangle = \langle p\rangle$

③ W.F. matching, ψ continuous, ψ' may / may not.

④. scattering state 不能归一. vs. bound state