Determinate state. $\sigma_Q = 0$, $Q = Q^\dagger$
(is the eigenstate of Q)

$\sigma^2 = \langle\psi|(Q-\bar{Q})^2|\psi\rangle = \langle\psi|(Q^\dagger-\bar{Q})(Q-\bar{Q})|\psi\rangle$

$= \langle(Q-\bar{Q})\psi|(Q-\bar{Q})\psi\rangle = 0$ $\qquad \hat{Q}\psi = \bar{Q}\psi$

e.g. $\hat{H}\psi_1 = E_1\psi_1$, $\hat{H}\psi_2 = E_2\psi_2$

ψ_1 and ψ_2 are determinate state of $\hat{H}$

but $\alpha\psi_1 + \beta\psi_2$ is <u>not</u>

$\hat{H}|\alpha\psi_1+\beta\psi_2\rangle = \alpha E_1|\psi_1\rangle + \beta E_2|\psi_2\rangle$

$\langle\hat{H}\rangle = |\alpha|^2E_1 + |\beta|^2E_2$ $\qquad (\langle H\rangle)^2 = (|\alpha|^2E_1 + |\beta|^2E_2)^2$

$(\hat{H})^2|\alpha\psi_1+\beta\psi_2\rangle = \alpha E_1^2|\psi_1\rangle + \beta E_2^2|\psi_2\rangle$

$\langle(\hat{H}^2)\rangle = |\alpha|^2E_1^2 + |\beta|^2E_2^2$

if $|\psi\rangle$ is normalized $|\alpha|^2+|\beta|^2=1$. let $|\alpha|^2=p$, then $|\beta|^2=1-p$

$pE_1^2 + (1-p)E_2^2 = (pE_1 + (1-p)E_2)^2$
$= p^2E_1^2 + (1-p)^2E_2^2 + 2p(1-p)E_1E_2$

$\Rightarrow p(p-1)E_1^2 - p(1-p)E_2^2 + 2p(1-p)E_1E_2 = 0$

$\Rightarrow p(1-p)[(E_1-E_2)^2] = 0 \Rightarrow$ (i) $E_1 = E_2$
(ii) $p=0$, $p=1$.

Back to postulates of QM

(1) Σ

(2) ray $|\psi\rangle$

(3) Hermitian complete operator A

(4) $A = a_n$ $(A = a(\lambda))$

(5) $\mathrm{Prob}(A=a_n) = \dfrac{\langle\psi|\mathbb{P}_n|\psi\rangle}{\langle\psi|\psi\rangle}$

$\mathbb{P}_n = |n\rangle\langle n|$

$\mathbb{P}_\lambda = \int d\lambda\, |\lambda\rangle\langle\lambda|$

(6) $|\psi\rangle \rightarrow \mathbb{P}_n|\psi\rangle$

example. 2-state system

$|1\rangle, |2\rangle$. $\mathcal{E}$ is 2-dimensional

$|1\rangle \sim \begin{pmatrix}1\\0\end{pmatrix}$. $|2\rangle \sim \begin{pmatrix}0\\1\end{pmatrix}$.

any state can be $|\psi\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix}a\\b\end{pmatrix}$

A physical operator A must be hermitian!

$A = A^{\dagger}$, for instant, H

the most general $H \sim \begin{pmatrix}\alpha & \beta\\ \delta & \gamma\end{pmatrix} = \begin{pmatrix}\alpha^* & \delta^*\\ \beta^* & \gamma^*\end{pmatrix}$

α and γ are real. $\beta = \delta^*$

$\Rightarrow \hat{H} \sim \begin{pmatrix}\alpha & \beta\\ \beta^* & \gamma\end{pmatrix}$

what is its eigenstate?

$\begin{pmatrix}\alpha & \beta\\ \beta^* & \gamma\end{pmatrix}\begin{pmatrix}u_{1,2}\\ d_{1,2}\end{pmatrix} = E_{1,2}\begin{pmatrix}u_{1,2}\\ d_{1,2}\end{pmatrix}$

The question is equivalent to the diagonalization of $\hat{H}$

$(\alpha-\lambda)(\gamma-\lambda) = |\beta|^2$

$\lambda^2 - \lambda(\alpha+\gamma) - |\beta|^2 = 0 \quad \lambda = \frac{1}{2}\left(\alpha+\gamma \pm \sqrt{(\alpha+\gamma)^2+4|\beta|^2}\right)$

Consider a simple case $\alpha = \gamma$ (real). β real!

$\hat{H} \sim \begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix}$. $(\alpha-\lambda)^2 = \beta^2$ or $\lambda = \alpha \pm \beta$

$\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix}\begin{pmatrix}u_1\\ d_1\end{pmatrix} = (\alpha \pm \beta)\begin{pmatrix}u_1\\ d_1\end{pmatrix} \Rightarrow \left[\alpha u_1 + \beta d_1 = (\alpha+\beta)u_1 \atop (\alpha-\beta)u_1\right. \Rightarrow \begin{matrix}(u_1 = d_1)\\ (u_1 = -d_1)\end{matrix}$

two normalized energy eigenstate $|e_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|e_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

Such that $\langle e_1|e_1\rangle = \langle e_2|e_2\rangle = 1$, $\langle e_1|e_2\rangle = \langle e_2|e_1\rangle = 0$

and $\hat{H}|e_1\rangle = (\alpha+\beta)|e_1\rangle$, $\hat{H}|e_2\rangle = (\alpha-\beta)|e_2\rangle$

$$\exp\left(-\frac{i}{\hbar}\hat{H}\Delta t\right)|\psi(t)\rangle = |\psi(t+\Delta t)\rangle$$

$$\left(\mathbb{1} - \frac{i}{\hbar}\hat{H}\epsilon\right)|\psi(t)\rangle = |\psi(t+\epsilon)\rangle$$ Note: there is no x-depend.

namely $$-\frac{i\epsilon}{\hbar}\hat{H}|\psi(t)\rangle = |\psi(t+\epsilon)\rangle - |\psi(t)\rangle$$

$$\Rightarrow \hat{H}|\psi(t)\rangle = i\hbar\,\frac{|\psi(t+\epsilon)\rangle - |\psi(t)\rangle}{\epsilon} \xrightarrow[\epsilon\to 0]{} i\hbar\frac{d}{dt}|\psi(t)\rangle$$

$$\Rightarrow \boxed{\hat{H}|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle}$$

and $|1\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle)$

$|2\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle - |e_2\rangle)$

So, $$i\hbar\frac{d}{dt}|e_1(t)\rangle = (\alpha+\beta)|e_1(t)\rangle$$

$$\Rightarrow \begin{cases} |e_1(t)\rangle = \exp\left(-\frac{i}{\hbar}(\alpha+\beta)t\right)|e_1(t=0)\rangle \\ |e_2(t)\rangle = \exp\left(-\frac{i}{\hbar}(\alpha-\beta)t\right)|e_2(t=0)\rangle \end{cases} \longrightarrow b)$$

For a given state $|\psi(t=0)\rangle = a|1\rangle + b|2\rangle = \frac{(a+b)}{\sqrt{2}}|e_1\rangle + \frac{(a-b)}{\sqrt{2}}|e_2\rangle$

$$\Rightarrow |\psi(t)\rangle = \frac{(a+b)}{\sqrt{2}}\exp\left(-\frac{i}{\hbar}(\alpha+\beta)t\right)|e_1\rangle + \frac{a-b}{\sqrt{2}}\exp\left(-\frac{i}{\hbar}(\alpha-\beta)t\right)|e_2\rangle$$

$$= e^{-\frac{i}{\hbar}\alpha t}\left(\frac{(a+b)}{\sqrt{2}}e^{-\frac{i}{\hbar}\beta t}\underbrace{|e_1\rangle}_{=\frac{|1\rangle+|2\rangle}{\sqrt{2}}} + \frac{a-b}{\sqrt{2}}e^{+\frac{i}{\hbar}\beta t}\underbrace{|e_2\rangle}_{=\frac{|1\rangle-|2\rangle}{\sqrt{2}}}\right)$$

$$= e^{-\frac{i\alpha t}{\hbar}}\left(\frac{1}{2}\right)\left(\begin{array}{l} 2a\cos\left(\frac{\beta t}{\hbar}\right)|1\rangle - i2a\sin\left(\frac{\beta t}{\hbar}\right)|2\rangle \\ -2bi\sin\left(\frac{\beta t}{\hbar}\right)|1\rangle + 2b\cos\left(\frac{\beta t}{\hbar}\right)|2\rangle \end{array}\right)$$

If at $t=0$, $a=1$, $b=0$,

$$|\psi(t)\rangle = e^{-\frac{i\alpha t}{\hbar}}\begin{pmatrix}\cos\left(\frac{\beta t}{\hbar}\right) \\ -i\sin\left(\frac{\beta t}{\hbar}\right)\end{pmatrix} \longrightarrow \boxed{\text{oscillation between } |1\rangle \,\&\, |2\rangle}$$

$$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\langle\psi|Q|\psi\rangle$$

$$= \left(\frac{d}{dt}\langle\psi|\right)\hat{Q}|\psi\rangle + \langle\psi|\frac{\partial Q}{\partial t}|\psi\rangle + \langle\psi|Q\left(\frac{d}{dt}|\psi\rangle\right)$$

also we know

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle \Rightarrow \frac{d}{dt}|\psi\rangle = \frac{-i}{\hbar}\hat{H}|\psi\rangle$$

and

$$-i\hbar\frac{d}{dt}\langle\psi| = \langle\psi|\hat{H}$$

$$\Rightarrow \frac{d}{dt}\langle\psi| = \frac{i}{\hbar}\langle\psi|\hat{H}$$

$$= \frac{i}{\hbar}\langle\psi|HQ|\psi\rangle - \frac{i}{\hbar}\langle\psi|QH|\psi\rangle + \langle\psi|\frac{\partial Q}{\partial t}|\psi\rangle$$

$$= \frac{i}{\hbar}\langle[H,Q]\rangle + \left\langle\frac{\partial Q}{\partial t}\right\rangle$$

$$\Rightarrow \boxed{\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[H,Q]\rangle + \left\langle\frac{\partial Q}{\partial t}\right\rangle}$$

e.g. $Q = x$, $H = \frac{p^2}{2m}$, $\frac{\partial Q}{\partial t} = 0$, $[H,Q] = \frac{1}{2m}[p^2, x]$

$$= \frac{1}{2m}(p^2x - xp^2)$$

$$= \frac{1}{2m}(ppx - pxp + pxp - xpp)$$

$$= \frac{1}{2m}(-p[x,p] - [x,p]p)$$

$$= -\frac{i\hbar}{m}p$$

$$\frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\left\langle -\frac{i\hbar}{m}p\right\rangle$$

$$= \frac{1}{m}\langle p\rangle$$

Sol. $E_1 = \alpha+\beta$, $\quad E_2 = \alpha-\beta$

$$\mathbb{P}_1 = |e_1\rangle\langle e_1| = \frac{1}{\sqrt{2}}\Big(|1\rangle + |2\rangle\Big)\frac{1}{\sqrt{2}}\Big(\langle 1| + \langle 2|\Big)$$

$$= \frac{1}{2}\Big(|1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 2|\Big)$$

its matrix representation is $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}$

similarly $\mathbb{P}_2 = |e_2\rangle\langle e_2| = \left(\frac{1}{\sqrt{2}}\right)^2\Big(|1\rangle - |2\rangle\Big)\Big(\langle 1| - \langle 2|\Big)$

$$\sim \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix}$$

we see that $\displaystyle\sum_{i=1}^{2}\mathbb{P}_{e_i} = \mathbb{P}_1 + \mathbb{P}_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} = \mathbb{1}$$

For a state $|\psi\rangle = a|1\rangle + b|2\rangle \sim \begin{pmatrix} a \\ b\end{pmatrix}$

If we measure $H = E_1$

$$|\psi\rangle \to \mathbb{P}_1|\psi\rangle \Rightarrow \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix} = \frac{a+b}{2}\begin{pmatrix} 1 \\ 1\end{pmatrix} = \frac{(a+b)}{\sqrt{2}}|e_1\rangle$$

$$\text{Prob}(H = E_1) = \frac{\langle\psi|\mathbb{P}_1|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{(a^*, b^*)\begin{pmatrix} 1 \\ 1\end{pmatrix}\frac{a+b}{2}}{|a|^2+|b|^2} = \frac{\frac{1}{2}|a+b|^2}{|a|^2+|b|^2}$$

Why? Let's understand

$$|\psi\rangle = a|1\rangle + b|2\rangle = a\frac{1}{\sqrt{2}}\Big(|e_1\rangle + |e_2\rangle\Big) + b\,\frac{|e_1\rangle - |e_2\rangle}{\sqrt{2}} = \frac{(a+b)}{\sqrt{2}}|e_1\rangle + \frac{(a-b)}{\sqrt{2}}|e_2\rangle$$

So the prob. finding $|e_1\rangle$ in $|\psi\rangle$ is

$$\frac{\left|\frac{a+b}{\sqrt{2}}\right|^2}{\left|\frac{a+b}{\sqrt{2}}\right|^2 + \left|\frac{a-b}{\sqrt{2}}\right|^2} = \frac{\frac{1}{2}|a+b|^2}{|a|^2+|b|^2} \quad ✓$$

Consider 2 operator $A = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $B = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$[A,B] = \left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} - \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$$

$$= \begin{pmatrix}0&-1\\1&0\end{pmatrix} - \begin{pmatrix}0&1\\-1&0\end{pmatrix} = 2\begin{pmatrix}0&-1\\1&0\end{pmatrix} \neq 0$$

A has eigenvalue ± 1 with $\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \pm\begin{pmatrix}a\\b\end{pmatrix}$

$+1$: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\quad -1$: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ $\quad \mathbb{P}_{A-} = \frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$

$\mathbb{P}_{A+} = \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$ $\quad \hookrightarrow |e_1\rangle$ $\quad \hookrightarrow |e_2\rangle$ in the previous case

B has eigenvalue ± 1 with $+1$: $\begin{pmatrix}1\\0\end{pmatrix}$, -1: $\begin{pmatrix}0\\1\end{pmatrix}$

Given a state $|\psi\rangle = \begin{pmatrix}a\\b\end{pmatrix}$. $\quad \mathbb{P}_{B+} = \begin{pmatrix}1&0\\0&0\end{pmatrix}$, $\mathbb{P}_{B-} = \begin{pmatrix}0&0\\0&1\end{pmatrix}$

① measure A $\to +1$ $\quad |\psi\rangle \to \frac{a+b}{2}\begin{pmatrix}1\\1\end{pmatrix} \propto |e_1\rangle$ $\quad$ Prob $= \frac{\frac{1}{2}|a+b|^2}{|a|^2+|b|^2}$

$\hookrightarrow -1$ $\quad |\psi\rangle \to \frac{a-b}{2}\begin{pmatrix}1\\-1\end{pmatrix} \propto |e_2\rangle$ $\quad$ Prob $= \frac{\frac{1}{2}|a-b|^2}{|a|^2+|b|^2}$

next measure B

A+ $\to$ $B=+1$ $\quad \frac{a+b}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \to \frac{a+b}{\sqrt{2}}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{a+b}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}$

$\hookrightarrow$ $B=-1$ $\quad \frac{a+b}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \to \frac{a+b}{\sqrt{2}}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{a+b}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}$

A− $\to$ $B=+1$ $\quad \frac{a-b}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \to \frac{a-b}{\sqrt{2}}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{a-b}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}$

$\hookrightarrow$ $B=-1$ $\quad \frac{a-b}{\sqrt{2}} \to \frac{a-b}{\sqrt{2}}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{a-b}{\sqrt{2}}\begin{pmatrix}0\\-1\end{pmatrix}$

$A=+1 \Rightarrow |\psi\rangle \to \frac{a+b}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \xrightarrow{\text{再測一次}} A=+1$ $\quad \frac{a+b}{\sqrt{2}}\cdot\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{a+b}{2\cdot 2}\begin{pmatrix}2\\2\end{pmatrix} = \frac{a+b}{2}\begin{pmatrix}1\\1\end{pmatrix}$ 一樣

100%