① $\mathcal{E}$

② $|\psi\rangle$ is a ray

③ Hermitian, complete A

④ $\hat{A} \to a_n \ (a(\lambda))$

⑤ $\text{Prob}(A \to a_n (a(\lambda))) = \dfrac{\langle\psi|\mathbb{P}_a|\psi\rangle}{\langle\psi|\psi\rangle}$

⑥ $|\psi\rangle \to \mathbb{P}_a|\psi\rangle$.

Stern - Gerlach exp

Ag : $Z = 47$

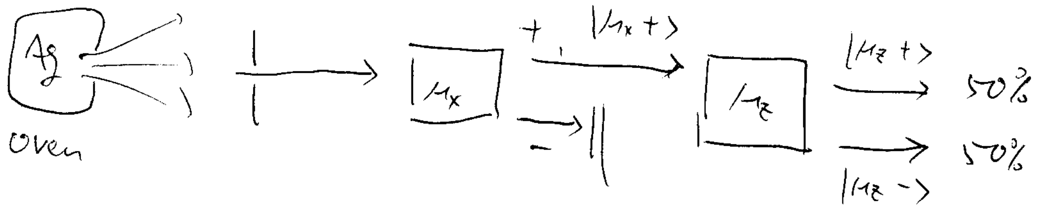
$(Kr)4d^{10}\underline{\underline{5s}}$

one outter electron

magnetic moment $\pm\mu_0$

$\mu_0 = \dfrac{e\hbar}{2m_e c}$, Bohr magneton.

- We only talk about magnetic moment, not spin at this moment



- $\hat{\mu}_x$ has 2 possible outcomes, by rule 3.4, $\hat{\mu}_x$ has eigenvalues: $\pm\mu_0$
  (also true for $\hat{\mu}_y$, $\hat{\mu}_z$)
- $\mathcal{E}$ must be 2-dim, at least: it can be spanned by the eigenkets of $\hat{\mu}_x$ with eigenvalues $\pm\mu_0$.
- Since it's only 2-dim, it can <u>also</u> be spanned by the eigenkets of $\hat{\mu}_y$ & $\hat{\mu}_z$.
- ~~any~~ eigenkets of any of $\hat{\mu}_{x,y,z}$ must be expressible as linear combinations of the eigenkets of any other operators.
- Denote the normalized eigenvectors (upto a phase) as

$$|\mu_x, \pm\rangle, \quad |\mu_y, \pm\rangle, \quad |\mu_z, \pm\rangle$$

and

$$\langle\mu_x, +|\mu_x, -\rangle = \langle\mu_y, +|\mu_y, -\rangle = \langle\mu_z, +|\mu_z, -\rangle = 0$$

- by rule-2, the state of Ag at various stages ~ some state vector.

by rule-6, after $\hat{\mu}_x$ with $+\mu_0$, the state is $|\mu_x, +\rangle$

- The state entering the 2nd magnet is a linear combination of $|\mu_z, \pm\rangle$.

$$\Rightarrow \quad |\mu_x +\rangle = c_+ |\mu_z +\rangle + c_- |\mu_z -\rangle$$

and $$c_\pm = \langle \mu_z, \pm | \mu_x, + \rangle$$

- by rule-5.

$$\text{Prob}\,(\hat{\mu}_z = +\mu_0) = \langle \mu_x + | \, \mathbb{P}_{z+} \, | \mu_x + \rangle$$

$$= \langle \mu_x + | \left(|\mu_z, + \rangle\langle \mu_z, + | \right) | \mu_x + \rangle$$

$$= \left| \langle \mu_z + | \mu_x + \rangle \right|^2 = |c_+|^2 = \frac{1}{2}$$

$$\Rightarrow \quad |\mu_x +\rangle = \frac{1}{\sqrt{2}} \left(|\mu_z, +\rangle + e^{i\alpha} |\mu_z, -\rangle \right)$$

with an overall phase to be absorbed by the kets.

- similarly,

$$|\mu_x, -\rangle = \frac{1}{\sqrt{2}} \left(|\mu_z, +\rangle + e^{i\beta} |\mu_z, -\rangle \right)$$

- But $\langle \mu_x, + | \mu_x, - \rangle = \frac{1}{2} \left(1 + e^{i(\beta - \alpha)} \right) = 0$

$$\Rightarrow \quad \boxed{e^{i\beta} = -e^{i\alpha}}$$

$$\Rightarrow \quad |\mu_x, \pm\rangle = \frac{1}{\sqrt{2}} \left(|\mu_z +\rangle \pm e^{i\alpha} |\mu_z, -\rangle \right)$$

- similarly,

$$|\mu_y, \pm\rangle = \frac{1}{\sqrt{2}}\left(|\mu_z, +\rangle \pm e^{i\theta} |\mu_z, -\rangle \right)$$

θ is another phase

- The operator can be constructed by their projection operator and eigenvalues:

$$\hat{\mu}_z = +\mu_0 \mathbb{P}_{z+} + (-\mu_0) \cdot \mathbb{P}_{z-}$$
$$= +\mu_0 |\mu_z, +\rangle\langle \mu_z, +| - \mu_0 |\mu_z, -\rangle \langle \mu_z, -|$$

reminder $\hat{A} \approx \begin{pmatrix} \langle 1|A|1\rangle & \langle 1|A|2\rangle & \cdots \\ \langle 2|A|1\rangle & \langle 2|A|2\rangle & \\ \vdots & & \ddots \end{pmatrix}$ in basis $|1\rangle, |2\rangle, |3\rangle$ ...

~~This is the simple case for eigenbasis~~

~~for~~ the case of eigenbasis, $\hat{A}$ is diagonal.

$$\hat{A}|\psi\rangle = \hat{A}\left(c_1|1\rangle + c_2|2\rangle + \cdots \right) \quad c_n = \langle n|\psi\rangle$$
$$= \left(c_1 \hat{A}|1\rangle + c_2 \hat{A}|2\rangle + \cdots \right)$$
$$= \left(c_1 a_1|1\rangle + c_2 a_2|2\rangle + \cdots \right)$$

The operation of $\hat{A} \approx \sum_n a_n |n\rangle\langle n| = \sum a_n \mathbb{P}_n$

and $$\hat{\mu}_x = +\mu_0 |\mu_x +\rangle\langle \mu_x +| - \mu_0 |\mu_x -\rangle\langle \mu_x -|$$

$$= \mu_0 \Big[\frac{1}{\sqrt{2}}\left(|\mu_z +\rangle + e^{i\alpha}|\mu_z -\rangle \right) \frac{1}{\sqrt{2}}\left(\langle \mu_z +| + e^{-i\alpha} \langle \mu_z, -| \right)$$
$$- \frac{1}{\sqrt{2}}\left(|\mu_z +\rangle - e^{i\alpha}|\mu_z -\rangle \right) \frac{1}{\sqrt{2}}\left(\langle \mu_z +| - e^{-i\alpha} \langle \mu_z, -| \right) \Big]$$

$$\Rightarrow \hat{\mu}_x = \frac{\mu_0}{2}\Big\{\Big(|\mu_z+\rangle\langle\mu_z+| + |\mu_z-\rangle\langle\mu_z-| + e^{i\alpha}|\mu_z-\rangle\langle\mu_z+| + e^{-i\alpha}|\mu_z+\rangle\langle\mu_z-|\Big)$$
$$-\Big(|\mu_z+\rangle\langle\mu_z+| + |\mu_z-\rangle\langle\mu_z-| - e^{i\alpha}|\mu_z-\rangle\langle\mu_z+| - e^{-i\alpha}|\mu_z+\rangle\langle\mu_z-|\Big)\Big\}$$
$$= \mu_0\left[e^{i\alpha}|\mu_z-\rangle\langle\mu_z+| + e^{-i\alpha}|\mu_z+\rangle\langle\mu_z-|\right]$$

(you can check that $\hat{\mu}_x = (\hat{\mu}_x)^+$)

Similarly

$$\hat{\mu}_y = \mu_0\left[e^{i\theta}|\mu_z-\rangle\langle\mu_z+| + e^{-i\theta}|\mu_z+\rangle\langle\mu_z-|\right]$$

Now, we do $\hat{\mu}_x$ first $\Rightarrow$ $\hat{\mu}_y$,

still 50% $\hat{\mu}_y = +\mu_0$, and 50% $\hat{\mu}_y = -\mu_0$

therefore $\frac{1}{2} = |\langle\mu_x+|\mu_{y},\pm\rangle|^2$

$$= \frac{1}{4}\left|\left(\langle\mu_z+| + e^{-i\alpha}\langle\mu_z-|\right)\left(|\mu_z+\rangle \pm e^{i\theta}|\mu_z-\rangle\right)\right|^2$$
$$= \frac{1}{4}\left|1 \pm e^{i(\theta-\alpha)}\right|^2 = \frac{1}{4}\left|1 \pm \cos(\theta-\alpha) \pm i\sin(\theta-\alpha)\right|^2$$
$$= \frac{1}{4}\left((1\pm\cos(\theta-\alpha))^2 + \sin^2(\theta-\alpha)\right)$$
$$= \frac{1}{2}\left(1 \pm \cos(\theta-\alpha)\right) \quad \Rightarrow \quad \theta-\alpha = \pm\frac{\pi}{2}$$

or $\quad e^{i\theta} = \underline{\underline{\pm i}}\, e^{i\alpha}$

Now it's convection.
Traditionally, $\hat{\mu}_x$ is choosen to be real.

In the $\hat{\mu}_z$ basis.

$$\hat{\mu}_x = \mu_0 \left[|\mu_z +\rangle\langle \mu_z -| + |\mu_z -\rangle\langle \mu_z +| \right]$$

$$\hat{\mu}_y = \mu_0 \left[-i\, |\mu_z +\rangle\langle \mu_z -| + i\, |\mu_z -\rangle\langle \mu_z +| \right]$$

$$\hat{\mu}_z = \mu_0 \left[+|\mu_z +\rangle\langle \mu_z +| - |\mu_z -\rangle\langle \mu_z -| \right]$$

In the matrix representation

$$|\mu_z \pm\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{\mu}_x \sim \mu_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\mu}_y \sim \mu_0 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\mu}_z \sim \mu_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli matrices are rediscovered
by solely using the rules of QM and the experimental results!