Determinate state. $\sigma_Q = 0$, $Q = Q^\dagger$

(is the eigenstate of Q)

$$\sigma^2 = \langle\psi|(Q-\bar{Q})^2|\psi\rangle = \left(\langle\psi|(Q^\dagger-\bar{Q})\right)\left((Q-\bar{Q})|\psi\rangle\right)$$

$$= \langle(Q-\bar{Q})\psi\,|\,(Q-\bar{Q})\psi\rangle = 0 \qquad \hat{Q}\psi = \bar{Q}\psi$$

e.g. $\hat{H}\psi_1 = E_1\psi_1$, $\hat{H}\psi_2 = E_2\psi_2$

ψ_1 and ψ_2 are determinate state of $\hat{H}$

but $\alpha\psi_1 + \beta\psi_2$ is <u>not</u>

$$\hat{H}\,|\alpha\psi_1+\beta\psi_2\rangle = \alpha E_1|\psi_1\rangle + \beta E_2|\psi_2\rangle$$

$$\langle\hat{H}\rangle = |\alpha|^2E_1 + |\beta|^2E_2 \qquad (\langle H\rangle)^2 = (|\alpha|^2E_1 + |\beta|^2E_2)^2$$

$$(\hat{H})^2|\alpha\psi_1+\beta\psi_2\rangle = \alpha E_1^2|\psi_1\rangle + \beta E_2^2|\psi_2\rangle$$

$$\langle(\hat{H}^2)\rangle = |\alpha|^2E_1^2 + |\beta|^2E_2^2$$

if $|\psi\rangle$ is normalized $|\alpha|^2+|\beta|^2 = 1$. let $|\alpha|^2 = p$. then $|\beta|^2 = 1-p$

$$pE_1^2 + (1-p)E_2^2 = (pE_1 + (1-p)E_2)^2$$
$$= p^2E_1^2 + (1-p)^2E_2^2 + 2p(1-p)E_1E_2$$

$$\Rightarrow \quad p(p-1)E_1^2 - p(1-p)E_2^2 + 2p(1-p)E_1E_2 = 0$$

$$\Rightarrow \quad p(1-p)\left[(E_1-E_2)^2\right] = 0 \quad \Rightarrow \quad \text{(i) } E_1 = E_2$$
$$\text{(ii) } p=0,\ p=1.$$

Back to postulates of QM

(1) Σ

(2) ray $|\psi\rangle$

(3) Hermitian complete operator A

(4) $A = a_n \quad (A = a(\lambda))$

(5) $\text{Prob}(A = a_n) = \dfrac{\langle\psi|P_n|\psi\rangle}{\langle\psi|\psi\rangle}$

$P_n = |n\rangle\langle n|$

$P_\lambda = \int d\lambda\, |\lambda\rangle\langle\lambda|$

(6) $|\psi\rangle \to P_n|\psi\rangle$

example. 2-state system

$|1\rangle, |2\rangle$. $\quad \mathcal{E}$ is 2-dimensional

$|1\rangle \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $|2\rangle \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

any state can be $|\psi\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

~~A~~ physical operator A must be hermitian.

$A = A^\dagger$, for instant, H

the most general $H \sim \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} = \begin{pmatrix} \alpha^* & \delta^* \\ \beta^* & \gamma^* \end{pmatrix}$

α and γ are real. $\beta = \delta^*$

$\Rightarrow \hat{H} \sim \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix}$

what is its eigenstate ?

$\begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \begin{pmatrix} u_{1,2} \\ v_{1,2} \end{pmatrix} = E_{1,2} \begin{pmatrix} u_{1,2} \\ v_{1,2} \end{pmatrix}$

The question is equivalent to the diagonalization of $\hat{H}$

$(\alpha-\lambda)(\gamma-\lambda) = |\beta|^2$

~~$\lambda^2 - \lambda(\alpha+\gamma) - |\beta|^2 = 0 \quad \lambda = \frac{1}{2}\left(\alpha+\gamma \pm \sqrt{(\alpha+\gamma)^2+4|\beta|^2}\right)$~~

consider a simple case $\alpha = \gamma$ (real), β real !

$\hat{H} \sim \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$. $\quad (\alpha-\lambda)^2 = \beta^2$ or $\lambda = \cancel{\beta}\ \alpha \pm \beta$

$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = (\alpha \pm \beta) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \Rightarrow \left\{ \alpha u_1 + \beta v_1 = \begin{matrix} (\alpha+\beta) u_1 \\ (\alpha-\beta) u_1 \end{matrix} \right. \Rightarrow \begin{matrix} (u_1 = v_1) \\ (u_1 = -v_1) \end{matrix}$

two normalized energy eigenstate $|e_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|e_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

such that $\langle e_1|e_1\rangle = \langle e_2|e_2\rangle = 1$, $\langle e_1|e_2\rangle = \langle e_2|e_1\rangle = 0$

and $\hat{H}|e_1\rangle = (\alpha+\beta)|e_1\rangle$, $\hat{H}|e_2\rangle = (\alpha-\beta)|e_2\rangle$

$$\exp\left(-\frac{i}{\hbar}\hat{H}\Delta t\right)|\psi(t)\rangle = |\psi(t+\Delta t)\rangle$$

$$\left(\mathbb{1} - \frac{i}{\hbar}\hat{H}\epsilon\right)|\psi(t)\rangle = |\psi(t+\epsilon)\rangle$$ Note: there is no x-depend.

namely $\frac{-i\epsilon}{\hbar}\hat{H}|\psi(t)\rangle = |\psi(t+\epsilon)\rangle - |\psi(t)\rangle$

$$\Rightarrow \hat{H}|\psi(t)\rangle = i\hbar\frac{|\psi(t+\epsilon)\rangle - |\psi(t)\rangle}{\epsilon} \xrightarrow[\epsilon\to 0]{} i\hbar\frac{d}{dt}|\psi(t)\rangle$$

$$\Rightarrow \boxed{\hat{H}|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle}$$

and $|1\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle)$

$|2\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle - |e_2\rangle)$

So, $i\hbar\frac{d}{dt}|e_1(t)\rangle = (\alpha+\beta)|e_1(t)\rangle$

$$\Rightarrow \begin{cases} |e_1(t)\rangle = \exp\left(-\frac{i}{\hbar}(\alpha+\beta)t\right)|e_1(t=0)\rangle \\ |e_2(t)\rangle = \exp\left[\frac{-i}{\hbar}(\alpha-\beta)t\right]|e_2(t=0)\rangle \end{cases} \longrightarrow \text{b)}$$

For a given state $|\psi(t=0)\rangle = a|1\rangle + b|2\rangle = \frac{(a+b)}{\sqrt{2}}|e_1\rangle + \frac{(a-b)}{\sqrt{2}}|e_2\rangle$

$$\Rightarrow |\psi(t)\rangle = \frac{(a+b)}{\sqrt{2}}\exp\left(\frac{-i}{\hbar}(\alpha+\beta)t\right)|e_1\rangle + \frac{(a-b)}{\sqrt{2}}\exp\left(\frac{-i}{\hbar}(\alpha-\beta)t\right)|e_2\rangle$$

$$= e^{-\frac{i}{\hbar}\alpha t}\left(\frac{(a+b)}{\sqrt{2}}e^{-\frac{i}{\hbar}\beta t}\underbrace{|e_1\rangle}_{\frac{|1\rangle+|2\rangle}{\sqrt{2}}} + \frac{(a-b)}{\sqrt{2}}e^{+\frac{i}{\hbar}\beta t}\underbrace{|e_2\rangle}_{\frac{|1\rangle-|2\rangle}{\sqrt{2}}}\right)$$

$$= e^{-\frac{i\alpha t}{\hbar}}\left(\frac{1}{2}\right)\begin{pmatrix} 2a\cos\left(\frac{\beta t}{\hbar}\right)|1\rangle & -i2a\sin\left(\frac{\beta t}{\hbar}\right)|2\rangle \\ -2bi\sin\left(\frac{\beta t}{\hbar}\right)|1\rangle & +2b\cos\left(\frac{\beta t}{\hbar}\right)|2\rangle \end{pmatrix}$$

If at $t=0$, $a=1$, $b=0$,

$$\boxed{|\psi(t)\rangle = e^{-\frac{i\alpha t}{\hbar}}\begin{pmatrix}\cos\frac{\beta t}{\hbar} \\ -i\sin\left(\frac{\beta t}{\hbar}\right)\end{pmatrix}} \longrightarrow \boxed{\text{oscillation between } |1\rangle \text{ \& } |2\rangle}$$

$$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\langle \psi | Q | \psi\rangle$$

$$= \left(\frac{d}{dt}\langle\psi|\right)\hat{Q}|\psi\rangle + \langle\psi|\frac{\partial Q}{\partial t}|\psi\rangle + \langle\psi|Q\left(\frac{d}{dt}|\psi\rangle\right)$$

also we know

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle \Rightarrow \frac{d}{dt}|\psi\rangle = \frac{-i}{\hbar}\hat{H}|\psi\rangle$$

and

$$-i\hbar\frac{d}{dt}\langle\psi| = \langle\psi|\hat{H}$$

$$\Rightarrow \frac{d}{dt}\langle\psi| = \frac{i}{\hbar}\langle\psi|\hat{H}$$

$$= \frac{i}{\hbar}\langle\psi|HQ|\psi\rangle - \frac{i}{\hbar}\langle\psi|QH|\psi\rangle + \langle\psi|\frac{\partial Q}{\partial t}|\psi\rangle$$

$$= \frac{i}{\hbar}\langle[H,Q]\rangle + \left\langle\frac{\partial Q}{\partial t}\right\rangle$$

$$\Rightarrow \boxed{\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[H,Q]\rangle + \left\langle\frac{\partial Q}{\partial t}\right\rangle}$$

e.g. $Q = x$, $H = \frac{p^2}{2m}$, $\frac{\partial Q}{\partial t} = 0$, $[H,Q] = \frac{1}{2m}[p^2, x]$

$$= \frac{1}{2m}(p^2x - xp^2)$$

$$= \frac{1}{2m}(ppx - pxp + pxp - xpp)$$

$$= \frac{1}{2m}(-p[x,p] - [x,p]p)$$

$$= -\frac{i\hbar}{m}p$$

$$\frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\left\langle -\frac{i\hbar}{m}p\right\rangle$$

$$= \frac{1}{m}\langle p\rangle$$

Say $E_1 = \alpha+\beta$, $\quad E_2 = \alpha-\beta$

$$\mathbb{P}_1 = |e_1\rangle\langle e_1| = \frac{1}{\sqrt{2}}\Big(|1\rangle+|2\rangle\Big)\frac{1}{\sqrt{2}}\Big(\langle 1|+\langle 2|\Big)$$

$$= \frac{1}{2}\Big(|1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 2|\Big)$$

its matrix representation is $\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$

similarly $\mathbb{P}_2 = |e_2\rangle\langle e_2| = \left(\frac{1}{\sqrt{2}}\right)^2\Big(|1\rangle-|2\rangle\Big)\Big(\langle 1|-\langle 2|\Big)$

$$\sim \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$$

we see that $\displaystyle\sum_{i=1}^{2}\mathbb{P}_{e_i} = \mathbb{P}_1+\mathbb{P}_2 = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

$$= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbb{1}$$

For a state $|\psi\rangle = a|1\rangle + b|2\rangle \sim \begin{pmatrix}a\\ b\end{pmatrix}$

If we measure $H = E_1$

$$|\psi\rangle \to \mathbb{P}_1|\psi\rangle \Rightarrow \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{a+b}{2}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{(a+b)}{\sqrt{2}}|e_1\rangle$$

$$\text{Prob}(H=E_1) = \frac{\langle\psi|\mathbb{P}_1|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{(a^*, b^*)\begin{pmatrix}1\\ 1\end{pmatrix}\frac{a+b}{2}}{|a|^2+|b|^2} = \frac{\frac{1}{2}|a+b|^2}{|a|^2+|b|^2}$$

Why? Let's understand

$$|\psi\rangle = a|1\rangle + b|2\rangle = a\frac{1}{\sqrt{2}}\Big(|e_1\rangle+|e_2\rangle\Big) + b\,\frac{|e_1\rangle-|e_2\rangle}{\sqrt{2}} = \frac{(a+b)}{\sqrt{2}}|e_1\rangle + \frac{(a-b)}{\sqrt{2}}|e_2\rangle$$

So the prob. finding $|e_1\rangle$ in $|\psi\rangle$ is.
$$\frac{\left|\frac{a+b}{\sqrt{2}}\right|^2}{\left|\frac{a+b}{\sqrt{2}}\right|^2+\left|\frac{a-b}{\sqrt{2}}\right|^2} = \boxed{\frac{\frac{1}{2}|a+b|^2}{|a|^2+|b|^2}}\ \checkmark$$

9/10

consider 2 operator $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$[A, B] = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq 0$$

A has eigenvalue ± 1 with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\begin{pmatrix} a \\ b \end{pmatrix}$

$+1: \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad -1: \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \mathbb{P}_{A-} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$\mathbb{P}_{A+} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \simeq |e_1\rangle$ $\quad \simeq |e_2\rangle$ in the previous case

B has eigenvalue ± 1 with $+1: \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $-1: \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Give a state $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$. $\quad \mathbb{P}_{B+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbb{P}_{B-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

① measure A $\to +1$ $\quad |\psi\rangle \to \frac{a+b}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \propto |e_1\rangle$ $\quad$ Prob $= \frac{\frac{1}{2}|a+b|}{|a|^2+|b|}$

$\qquad\qquad \to -1$ $\quad |\psi\rangle \to \frac{a-b}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \propto |e_2\rangle$ $\quad$ Prob $= \frac{\frac{1}{2}|a-b|^2}{|a|^2+|b|^2}$

next measure B

$|A+\rangle$ $\to$ $B=+1$ $\quad \frac{a+b}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \frac{a+b}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{a+b}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\qquad\quad \to$ $B=-1$ $\quad \frac{a+b}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \frac{a+b}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{a+b}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|A-\rangle$ $\to$ $B=+1$ $\quad \frac{a-b}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \to \frac{a-b}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{a-b}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\qquad\quad \to$ $B=-1$ $\quad \frac{a-b}{\sqrt{2}} \to \frac{a-b}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{a-b}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$A=+1 \Rightarrow |\psi\rangle \to \frac{a+b}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{再測一次}} A=+1$ $\quad \frac{a+b}{(\sqrt{2})^2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{a+b}{2\sqrt{2}}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \frac{a+b}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 一樣

100%

$$\text{Prob}(A_+ \to B_+) = \left(\frac{\langle \psi | \mathbb{P}_{A+} | \psi \rangle}{\langle \psi | \psi \rangle} \right) \cdot \left(\frac{\langle \psi | \mathbb{P}_{A+} \mathbb{P}_{B+} \mathbb{P}_{A+} | \psi \rangle}{\langle \psi | \mathbb{P}_{A+} \mathbb{P}_{A} | \psi \rangle} \right) \qquad \mathbb{P}_{A+} = \mathbb{P}_{A+}^2$$

$$\left(\begin{aligned} \mathbb{P}_n &= |n\rangle\langle n| = (\mathbb{P}_n)^\dagger \\ \mathbb{P}_n^m &= (|n\rangle\langle n|)^m = |n\rangle\langle n| \end{aligned} \right.$$

拿光的 polarization as an example. $\mathbb{P} = \mathbb{P}^2$

$$= \frac{\langle \psi | \mathbb{P}_{A+} \mathbb{P}_{B+} \mathbb{P}_{A+} | \psi \rangle}{\langle \psi | \psi \rangle}$$

$$\text{Prob}(B_+ \to A_+) = \left(\frac{\langle \psi | \mathbb{P}_{B+} | \psi \rangle}{\langle \psi | \psi \rangle} \right) \cdot \left(\frac{\langle \psi | \mathbb{P}_{B+} \mathbb{P}_{A+} \mathbb{P}_{B+} | \psi \rangle}{\langle \psi | \mathbb{P}_{B+} \mathbb{P}_{B+} | \psi \rangle} \right)$$

$$= \frac{\langle \psi | \mathbb{P}_{B+} \mathbb{P}_{A+} \mathbb{P}_{B+} | \psi \rangle}{\langle \psi | \psi \rangle}$$

$$\mathbb{P}_{A+} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbb{P}_{B+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

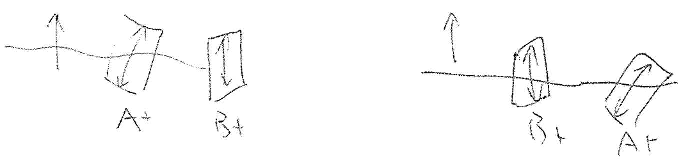
$$\mathbb{P}_{A+} \mathbb{P}_{B+} \mathbb{P}_{A+} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbb{P}_{B+} \mathbb{P}_{A+} \mathbb{P}_{B+} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{Prob}(A_+ \to B_+) = \frac{(a^* \; b^*)\frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}}{(a^* \; b^*)\begin{pmatrix} a \\ b \end{pmatrix}} = \frac{\frac{1}{4}(a+b)(a^*+b^*)}{|a|^2+|b|^2}$$

$$\text{Prob}(B_+ \to A_+) = \frac{(a^* \; b^*)\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}}{(a^* \; b^*)\begin{pmatrix} a \\ b \end{pmatrix}} = \frac{\frac{1}{2}|a|^2}{|a|^2+|b|^2}$$

对 $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $\text{Prob}(A_+ \to B_+) = \frac{1}{4}$, $\text{Prob}(B_+ \to A_+) = \frac{1}{2}$

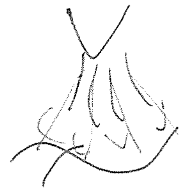


$\xrightarrow{}$ (×50%) $\xrightarrow{}$ (×50%) $= \frac{1}{4}$ $\qquad$ $\xrightarrow{}$ 100% $\xrightarrow{}$ (×50%) $= \frac{1}{2}$

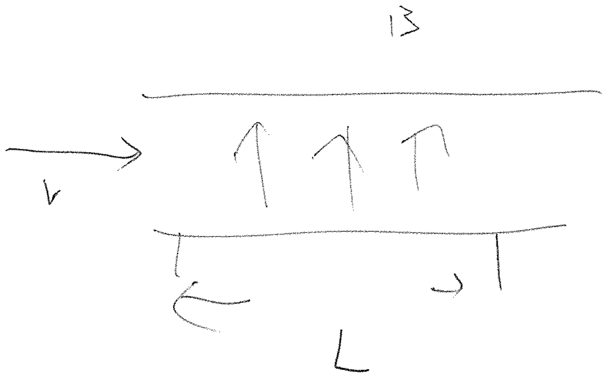
$$\mathbb{P}_{A+} \cdot \mathbb{P}_{A-} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$\mathbb{P}_{A+} \cdot \mathbb{P}_{B\pm} \neq 0$. similarly $\mathbb{P}_{B+} \mathbb{P}_{B-} = 0$

$$E = -\vec{\mu}\cdot\vec{B} \qquad \vec{F} = -\vec{\nabla} V$$



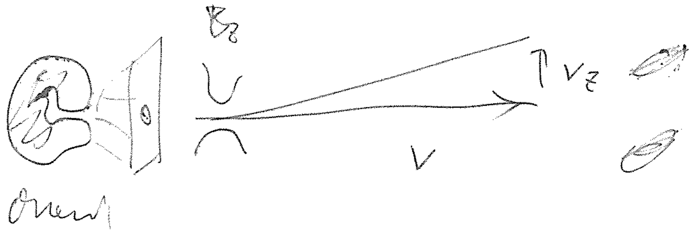
$$\vec{F}_z = +\vec{\mu}\frac{dB}{dz} = m\ddot{z}$$



$$\Delta t = \frac{L}{v}$$

$$\ddot{z} = \frac{\mu_z}{m}\frac{dB}{dz}$$

$$v_z = \frac{\mu_z}{m}\frac{dB}{dz}\frac{L}{v}$$



Stern - Gerlach exp

1. $\mathcal{E}$
2. $|\psi\rangle$ is a ray
3. Hermitian, complete A
4. $\hat{A} \to a_n \; (a(\lambda))$
5. $\text{Prob}(A \to a_n \, (a(\lambda))) = \dfrac{\langle\psi|\mathbb{P}_a|\psi\rangle}{\langle\psi|\psi\rangle}$
6. $|\psi\rangle \to \mathbb{P}_a|\psi\rangle$.

Ag : $Z = 47$

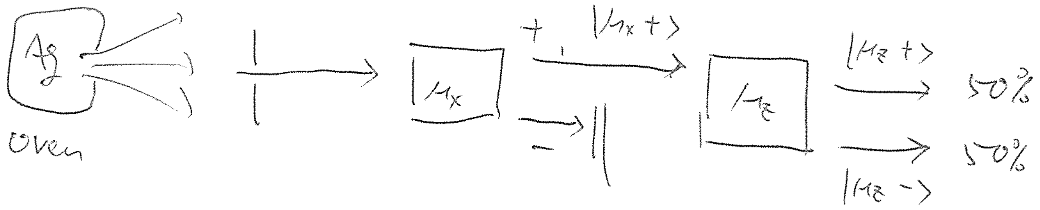
$(Kr)\, 4d^{10}\, \underline{5s}$

One outter electron

magnetic moment $\pm\mu_0$

$\mu_0 = \dfrac{e\hbar}{2m_e c}$, Bohr magneton.

- We only talk about magnetic moment, not spin at this moment



- $\hat{\mu}_x$ has 2 possible outcomes, by rule 3.4, $\hat{\mu}_x$ has eigenvalues: $\pm\mu_0$ (also true for $\hat{\mu}_y$, $\hat{\mu}_z$)
- $\mathcal{E}$ must be 2-dim, at least; it can be spanned by the eigenkets of $\hat{\mu}_x$ with eigenvalues $\pm\mu_0$.
- Since it's only 2-dim, it can <u>also</u> be spanned by the eigenkets of $\hat{\mu}_y$ & $\hat{\mu}_z$.
- ~~Any~~ eigenkets of any of $\hat{\mu}_{x,y,z}$ must be expressible as linear combinations of the eigenkets of any other operator.
- Denote the normalized eigenvector (upto a phase) as

$$|\mu_x, \pm\rangle, \quad |\mu_y, \pm\rangle, \quad |\mu_z, \pm\rangle$$

and

$$\langle\mu_x, +|\mu_x, -\rangle = \langle\mu_y, +|\mu_y, -\rangle = \langle\mu_z, +|\mu_z, -\rangle = 0$$

- by rule-2, the state of Ag at various stages ~ some state vector.

by rule-6, after $\hat{\mu}_x$ with $+\mu_0$, the state is $|\mu_x, +\rangle$

- The state entering the 2nd magnet is a linear combination of $|\mu_z, \pm\rangle$.

$$\Rightarrow \quad |\mu_x +\rangle = C_+ |\mu_z +\rangle + C_- |\mu_z -\rangle$$

and $\quad C_\pm = \langle \mu_z, \pm | \mu_x, +\rangle$

- by rule-5,

$$\begin{aligned} \text{Prob}(\hat{\mu}_z = +\mu_0) &= \langle \mu_x + | \Pi_{z+} | \mu_x +\rangle \\ &= \langle \mu_x + | \left(|\mu_z, +\rangle\langle \mu_z, +| \right) |\mu_x +\rangle \\ &= \left| \langle \mu_z + | \mu_x +\rangle \right|^2 = |C_+|^2 = \frac{1}{2} \end{aligned}$$

$$\Rightarrow \quad |\mu_x +\rangle = \frac{1}{\sqrt{2}} \left(|\mu_z, +\rangle + e^{i\alpha} |\mu_z, -\rangle \right)$$

with an overall phase to be absorbed by the kets.

- similarly,

$$|\mu_x, -\rangle = \frac{1}{\sqrt{2}} \left(|\mu_z, +\rangle + e^{i\beta} |\mu_z, -\rangle \right)$$

- But $\quad \langle \mu_x, + | \mu_x, -\rangle = \frac{1}{2} \left(1 + e^{i(\beta - \alpha)} \right) = 0$

$$\Rightarrow \quad \boxed{e^{i\beta} = -e^{i\alpha}}$$

$$\Rightarrow \quad |\mu_x, \pm\rangle = \frac{1}{\sqrt{2}} \left(|\mu_z +\rangle \pm e^{i\alpha} |\mu_z, -\rangle \right)$$

- similarly,

$$|\mu_y, \pm\rangle = \frac{1}{\sqrt{2}}\left(|\mu_z, +\rangle \pm e^{i\theta} |\mu_z, -\rangle \right)$$

θ is another phase

- The operator can be constructed by their projection operator and eigenvalues:

$$\hat{\mu}_z = +\mu_0 \mathbb{P}_{z+} + (-\mu_0) \cdot \mathbb{P}_{z-}$$

$$= +\mu_0 |\mu_z, +\rangle\langle\mu_z, +| - \mu_0 |\mu_z, -\rangle\langle\mu_z, -|$$

reminding: $\hat{A} \simeq \begin{pmatrix} \langle 1|A|1\rangle & \langle 1|A|2\rangle & \cdots \\ \langle 2|A|1\rangle & \langle 2|A|2\rangle & \\ \vdots & & \ddots \end{pmatrix}$ in basis: $|1\rangle, |2\rangle, |3\rangle$ ...

~~This is the simple case for eigenbasis~~

~~for~~ the case of eigenbasis, $\hat{A}$ is diagonal.

$$\hat{A} |\psi\rangle = \hat{A} (c_1 |1\rangle + c_2 |2\rangle + \cdots) \qquad c_n = \langle n|\psi\rangle$$

$$= (c_1 \hat{A} |1\rangle + c_2 \hat{A} |2\rangle + \cdots)$$

$$= (c_1 a_1 |1\rangle + c_2 a_2 |2\rangle + \cdots)$$

The operation of $\hat{A} \simeq \sum_n a_n |n\rangle\langle n| = \sum a_n \mathbb{P}_n$

and $\hat{\mu}_x = +\mu_0 |\mu_x +\rangle\langle\mu_x +| - \mu_0 |\mu_x -\rangle\langle\mu_x -|$

$$= \mu_0 \Big[\frac{1}{\sqrt{2}} (|\mu_z +\rangle + e^{i\alpha} |\mu_z -\rangle) \frac{1}{\sqrt{2}} (\langle\mu_z +| + e^{-i\alpha} \langle\mu_z, -|)$$

$$- \frac{1}{\sqrt{2}} (|\mu_z +\rangle - e^{i\alpha} |\mu_z -\rangle) \frac{1}{\sqrt{2}} (\langle\mu_z +| - e^{-i\alpha} \langle\mu_z, -|) \Big]$$

$$\Rightarrow \hat{\mu}_x = \frac{\mu_0}{2}\Big\{\Big(|\mu_z+\rangle\langle\mu_z+| + |\mu_z-\rangle\langle\mu_z-| + e^{i\alpha}|\mu_z-\rangle\langle\mu_z+| + e^{-i\alpha}|\mu_z+\rangle\langle\mu_z-|\Big)$$

$$-\Big(|\mu_z+\rangle\langle\mu_z+| + |\mu_z-\rangle\langle\mu_z-| - e^{i\alpha}|\mu_z-\rangle\langle\mu_z+| - e^{-i\alpha}|\mu_z+\rangle\langle\mu_z-|\Big)\Big\}$$

$$= \mu_0\Big[e^{i\alpha}|\mu_z-\rangle\langle\mu_z+| + e^{-i\alpha}|\mu_z+\rangle\langle\mu_z-|\Big]$$

(you can check that $\hat{\mu}_x = (\hat{\mu}_x)^\dagger$)

Similarly

$$\hat{\mu}_y = \mu_0\Big[e^{i\theta}|\mu_z-\rangle\langle\mu_z+| + e^{-i\theta}|\mu_z+\rangle\langle\mu_z-|\Big]$$

Now, we do $\hat{\mu}_x$ first $\Rightarrow \hat{\mu}_y$,

still 50% $\hat{\mu}_y = +\mu_0$, and 50% $\hat{\mu}_y = -\mu_0$

therefore $\frac{1}{2} = |\langle\mu_x+|\mu_{y,}\pm\rangle|^2$

$$= \frac{1}{4}\Big|\big(\langle\mu_z+| + e^{-i\alpha}\langle\mu_z-|\big)\big(|\mu_z+\rangle \pm e^{i\theta}|\mu_z-\rangle\big)\Big|^2$$

$$= \frac{1}{4}\Big|1 \pm e^{i(\theta-\alpha)}\Big|^2 = \frac{1}{4}\Big|1 \pm \cos(\theta-\alpha) \pm i\sin(\theta-\alpha)\Big|^2$$

$$= \frac{1}{4}\Big((1\pm\cos(\theta-\alpha))^2 + \sin^2(\theta-\alpha)\Big)$$

$$= \frac{1}{2}\big(1 \pm \cos(\theta-\alpha)\big) \quad \Rightarrow \quad \theta-\alpha = \pm\frac{\pi}{2}$$

or $\quad e^{i\theta} = \underline{\underline{\pm i}}\, e^{i\alpha}$

Now it's convection.

Traditionally, $\hat{\mu}_x$ is choosen to be real.

In the $\hat{\mu}_z$ basis.

$$\hat{\mu}_x = \mu_0 \left[|\mu_z +\rangle\langle \mu_z -| + |\mu_z -\rangle\langle \mu_z +| \right]$$

$$\hat{\mu}_y = \mu_0 \left[-i |\mu_z +\rangle\langle \mu_z -| + i |\mu_z -\rangle\langle \mu_z +| \right]$$

$$\hat{\mu}_z = \mu_0 \left[+ |\mu_z +\rangle\langle \mu_z +| - |\mu_z -\rangle\langle \mu_z -| \right]$$

In the matrix representation

$$|\mu_z \pm\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{\mu}_x \sim \mu_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\mu}_y \sim \mu_0 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\mu}_z \sim \mu_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli matrices are rediscovered by solely using the rules of QM and the experimental results!.