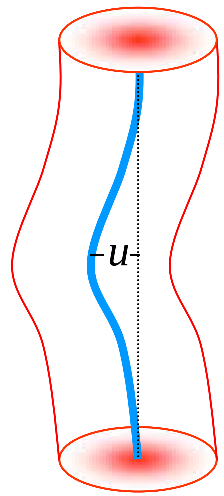
The path integral theory of thermal fluctuations of the flux lines in London appr. and melting of the vortex lattice.

1. The small tilt approximation

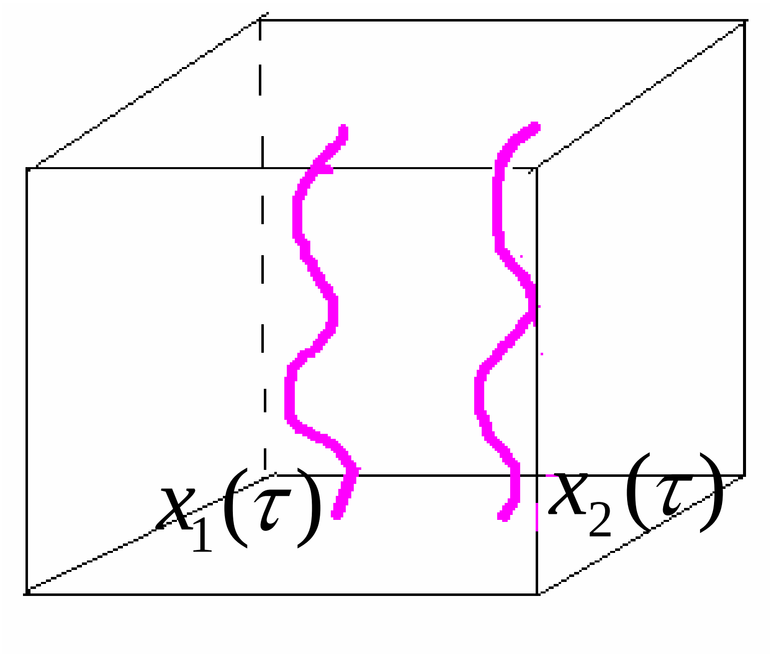
Abrikosov vortices in this regime are approximated by infinitely thin elastic lines with line energy



$$\varepsilon = \frac{\Phi_0^2}{2\pi^2\lambda^2} = \frac{H_{c1}\Phi_0}{4\pi}$$

Interaction is assumed mainly magnetic, thereby pair - wise (superposition principle). Force per unit length:

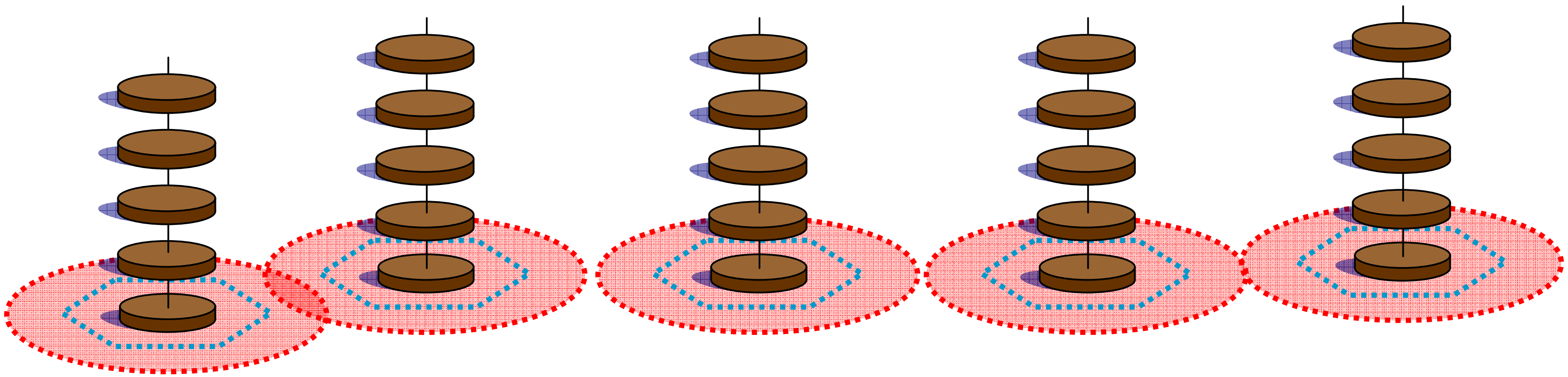
$$Force \equiv -\frac{d}{dr}\Delta F = \begin{cases} \varepsilon \dfrac{1}{r}, & \xi << r << \lambda \\[2ex] \dfrac{\varepsilon}{2}\left(\dfrac{\pi}{2\lambda r}\right)^{1/2} e^{-r/\lambda}, & r >> \lambda \end{cases}$$



Parallel vortices repel, anti-parallel attract, however the picture is more complicated than that: the force between curved vortices is of the vector-vector type

$$E_{\text{int}} = \frac{\varepsilon_0}{2} \iint_{\tau_1,\tau_2} \left[\frac{d\hat{x}_1 \cdot d\hat{x}_2}{|x_1 - x_2|} e^{-|x_1 - x_2|/\lambda} - \frac{|d\hat{x}_1| \cdot |d\hat{x}_2|}{|x_1 - x_2|} e^{-|x_1 - x_2|/\xi} \right]$$

Thermal fluctuations



Thermodynamics of lines is described (in the ergodic or thermal equilibrium situationnns) by a statistical sum

$$Z = \int \prod_a Dx_a(z) \exp \frac{1}{T} \left[\sum_a \varepsilon \oint_\tau dl(\tau) + \frac{1}{2} \sum_{a \neq b} E_{\text{int}}(x_a, x_b) \right]$$

This statistical sum is the "line representation of "dual" relativistic scalar QED (Higgs model) in 3D

$$F_{dual}[\chi^*, \chi] =$$

$$\chi^* D^2 \chi + v\left(\chi^* \chi\right) + \frac{\lambda}{2}\left(\chi^* \overleftrightarrow{D} \chi\right)^2 + \frac{B^2}{8\pi}$$

Kovner, B.R. IJMPA7, 7419 (1992), MPLA8, 1343 (1993); Sudbo et al (1999)

The small tilt approximation

This theory is too complicated. Generally all possible configurations including vortex loops contribute.

However when magnetic field is not too weak, vortices are nearly aligned with magnetic field. In this case loops, overhangs… can be ignored and one develops a small tilt approximation.

$$\frac{dl(\tau \equiv z)}{dz} = \sqrt{\left(\frac{dx(z)}{dz}\right)^2 + \left(\frac{dy(z)}{dz}\right)^2 + 1}$$

$$\approx 1 + \frac{1}{2}\left[\left(\frac{dx(z)}{dz}\right)^2 + \left(\frac{dy(z)}{dz}\right)^2\right] = 1 + \frac{1}{2}\left(\dot{r}\right)^2$$

We can regards z as "time" and then the approximation is just a nonrelativistic approximation of the dual theory. Interaction also simplifies and becomes the "instantaneous" Abrikosov repulsion:

$$E_{int} = \frac{\varepsilon}{2}\int dz \;\; K_0\left(\frac{r_1(z) - r_2(z)}{\lambda}\right)$$

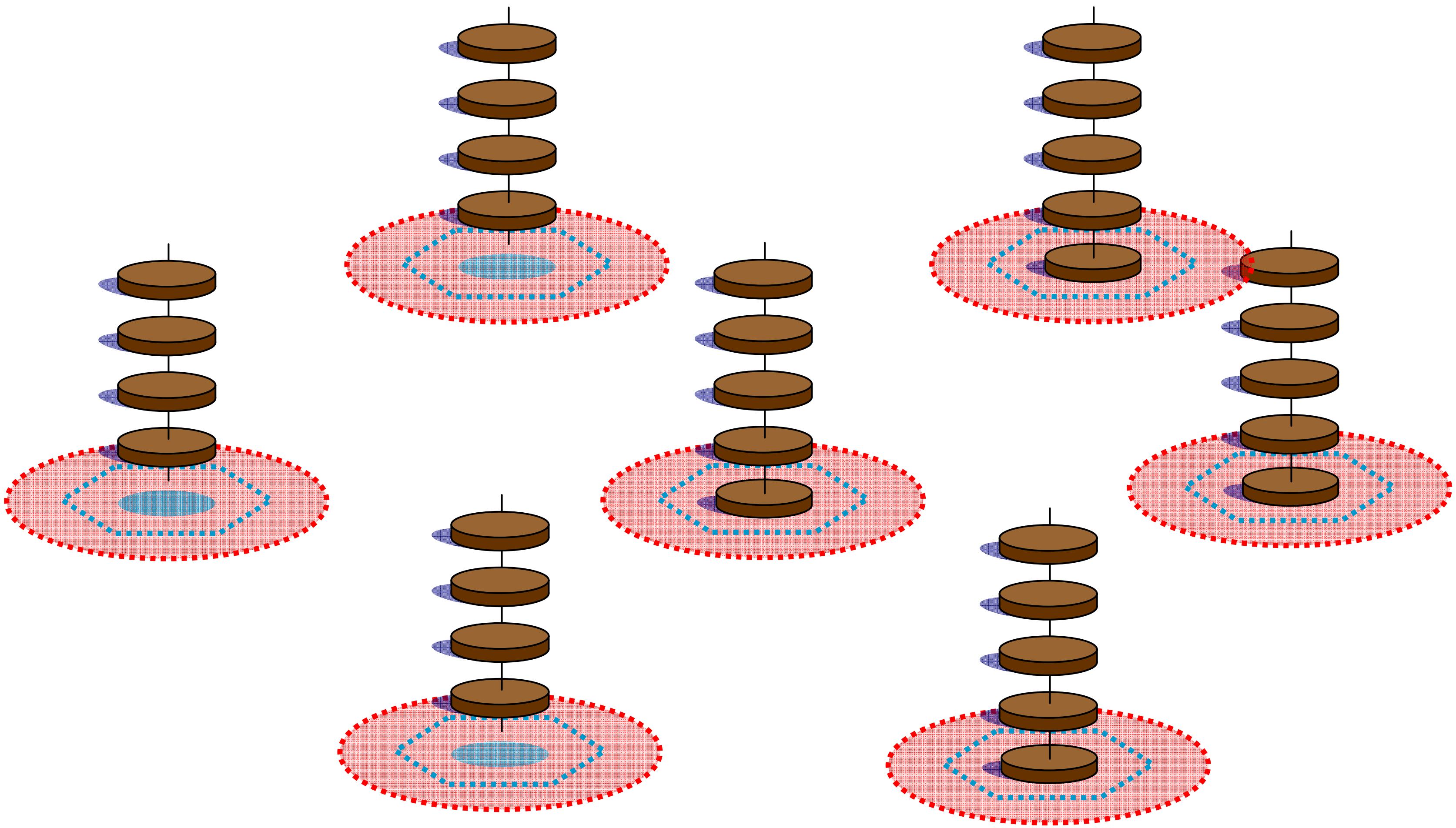
Within this approximation the statistical sum becomes identical to the (Euclidean version of, namely, no complex "I") the path integral of the quantum of many -body problem of interacting bosons:

$$Z = \int \prod_a Dx_a(z) \exp\left\{ -\frac{1}{T} \int dz \left[\sum_a \frac{\varepsilon}{2} \left(\dot{r} \right)^2 + \frac{1}{2} \sum_{a \neq b} V(x_a - x_b) \right] \right\}$$

With T analogous to $\hbar$ and $\mathfrak{m}$ to mass of a "particle" and z playing a role of Euclidean time.

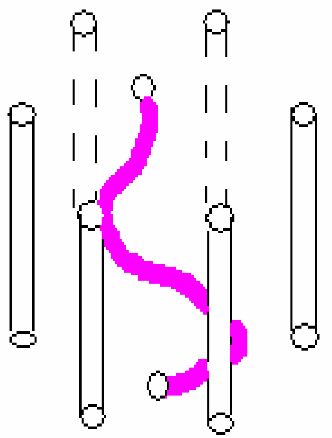
Nelson, PRL (1988)

Thermal fluctuations in vortex lattice



2. The QM analogy: mean displacement in vortex lattice

A standard way of calculating displacement due to thermal fluctuations or disorder in lattice is via elasticity theory



Here I will employ the less rigorous but simpler Einstein approximation: a vortex is considered in a field of all the other vortices as if they are not vibrating: the cage model.

This will be enough to qualitatively map the melting line via Lindemann criterium

The cage model

To further simplify the calculations only harmonic part of the cage potential will be used.

$$V_{cage}(r) = \frac{1}{2} \sum_{\substack{all\ \ the\ \ other \\ vortices}} V(r_a - r) = \frac{1}{2} k\ \ r^2$$

$$k = \frac{1}{2} \sum_a \Delta V(r) \Big|_{r_a}$$

For very small fields $B < H_{c1}$ **distances are large,**

a>●🗁 **and the interaction is exponentially weak**

$$V(r) = \varepsilon \exp\left[-r/\lambda\right]$$

Therefore only 6 nearest neighbours at distance

$a \approx \sqrt{\Phi_0 / B}$ **are significant:**

$$k_{weak}(B) \approx \frac{\varepsilon}{\lambda^2} \exp\left[-\frac{a(B)}{\lambda}\right] = \frac{\varepsilon}{\lambda^2} \exp\left[-\frac{\sqrt{\Phi_0 / B}}{\lambda}\right]$$

For larger inductions $B > H_{c1}$ **the interaction is logarithmic**

$$V(r) = -\varepsilon \log\left[r / \xi\right]$$

and more neighbours should be summed up, resulting in

$$k_{strong}(B) \approx \frac{\varepsilon}{a(B)^2} = \frac{\varepsilon}{\Phi_0} B$$

The QM analogy for calculation of averages

Now we consider the thermal motion of a single vortex in harmonic potential: quantum mechanics of harmonic oscillator.

Within the quantum mechanical analogy thermal average of a physical quantity is equivalent to quantum mechanical VEV

$$\langle O \rangle_{thermal} = Z^{-1} \int Dr(z) O(r) \exp \frac{1}{T} \int_z \left[\frac{\varepsilon}{2} \left(\dot{r}(z) \right)^2 + \frac{1}{2} V(r) \right]$$

$$= \langle vac \,|\, \hat{o} \,|\, vac \rangle$$

Therefore for square of displacement r^2 in harmonic oscillator ground state one obtains using the well known ground state of harmonic oscillator:

$$\langle r^2 \rangle = \int d^2 r \;\; r^2 \left|\psi(r)\right|^2 =$$

$$\int d^2 r \;\; r^2 \left(\frac{k\varepsilon}{\pi T^2}\right)^{1/4} \exp\left[-\sqrt{\frac{k\varepsilon}{T^2}} r^2\right] = \frac{T}{\sqrt{k\varepsilon}}$$

3. Lindemann crirerium for melting of the vortex lattice

A phenomenological model states that when the displacement reaches

$$\left\langle (\Delta r)^2 \right\rangle_{\substack{thermal \\ disorder}} = c_L^2 a^2$$

the lattice becomes unstable (spinodal) and slightly before that will melt into a homogeneous state (liquid=gas for purely repelling interactions). Typical values of the Lindemann constant range between

$$c_L \sim 0.1 - 0.3$$

Using the displacement calculated with QM, the Lindemann criterium takes a form:

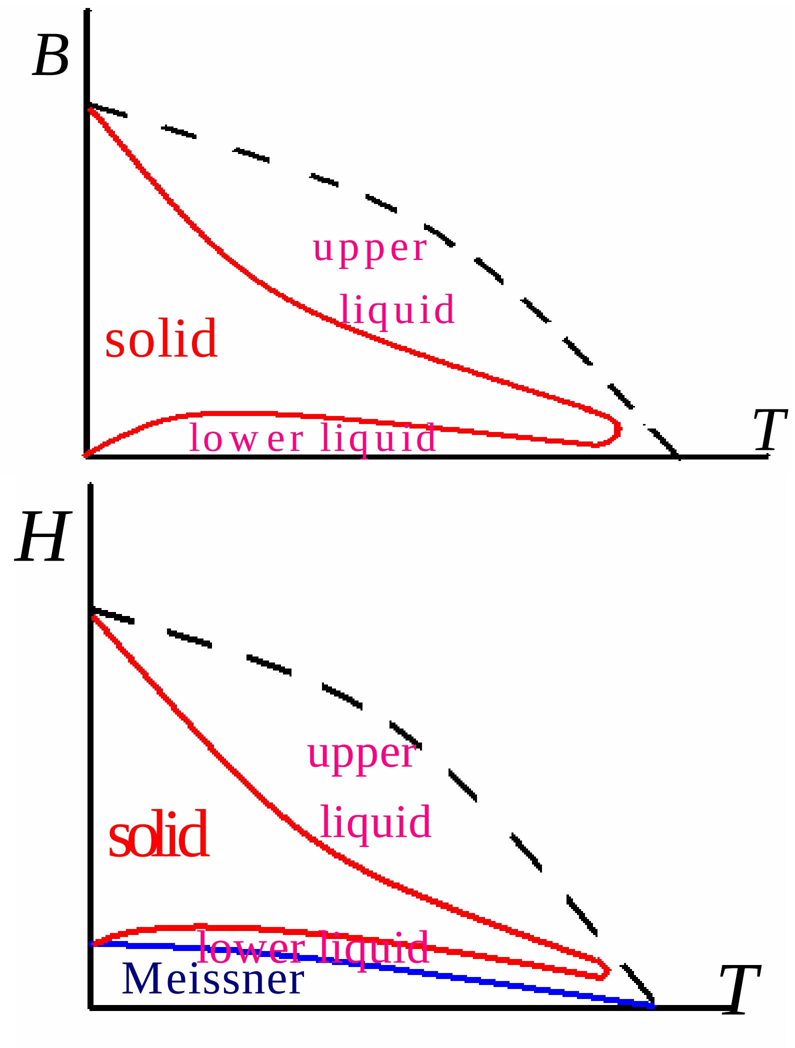
$$\frac{T}{\sqrt{k\varepsilon}} = c_L^2 a^2 \Rightarrow T = c_L^2 a(B)^2 \sqrt{k(B)\varepsilon}$$

For very weak fields one obtains a logarithmic melting line separating the "random walk" lower liquid or gas from solid

$$T = c_L^2 \frac{\Phi_0}{B_m} \frac{\varepsilon}{\lambda} \exp\left[-\sqrt{\frac{\Phi_0}{B_m} \frac{1}{2\lambda}} \right] \Rightarrow B_m \propto \frac{1}{(\log T)^2}$$

This is similar to melting of atomic solids in a sense that density of crystal is higher than that of liquid

Phase diagram including thermal fluctuations



The melting line has a characteristic (Nelson's nose) shape. Note that the Meissner phase is well separated from the crystal.

For larger fields one gets a negative power

$$T = c_L^2 \varepsilon a(B) = c_L^2 \varepsilon \sqrt{\frac{\Phi_0}{B_m}} \Rightarrow B_m = \frac{c_L^4 \varepsilon^2 \Phi_0}{T^2}$$

This segment is like melting of ice: density of liquid is larger than that of the crystal.

Near H_{c2} **one has to go beyond London approximation**

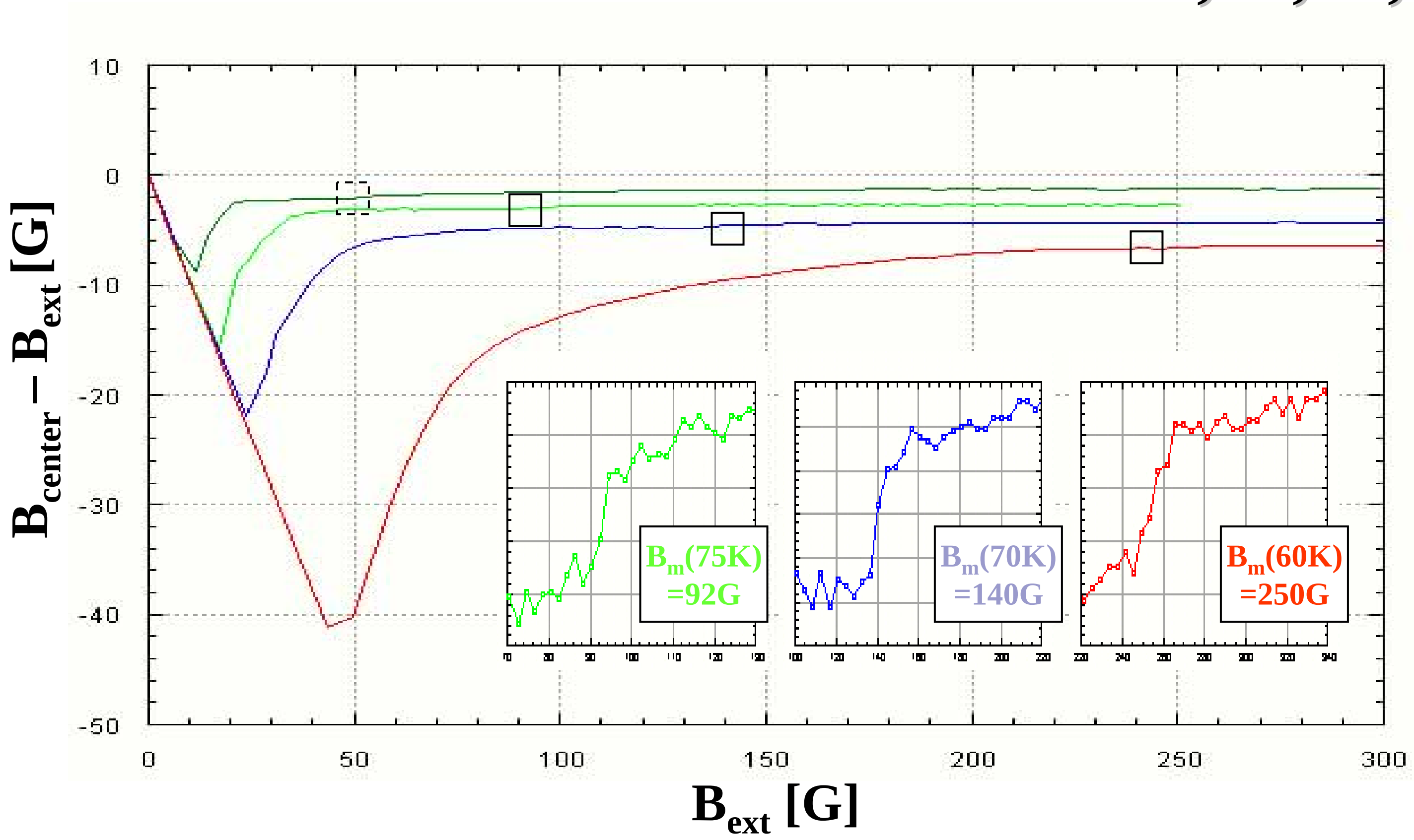
The Lindemann criterium is a one phase instability model and provides an estimate of spinodal only. It does not allow to determine what kind of transition occurs (first, second order or KT).

To find the melting line, one has to calculate free energies of both phases. This has not been achieved yet in London approximation.

Experiments

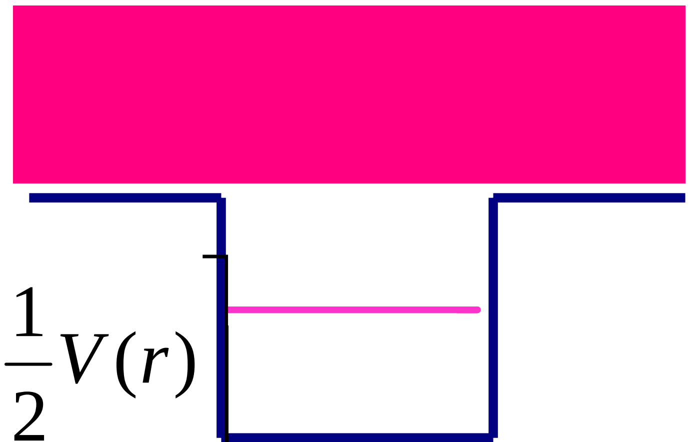
Show the transition is first order

T = 60, 70, 75, 80 K



4. Thermal depinning

A columnar pin can be modeled well by a finite well, say cylindrical well of width U and size R.



The binding free energy is

$$\exp\left\{U(T)L_z/T\right\} = Z^{-1}\int Dr(z)\exp\frac{1}{T}\int_z\left[\frac{\varepsilon}{2}\left(\dot{r}(z)\right)^2+\frac{1}{2}V(r)\right]$$

$$Z=\int Dr(z)\exp\frac{1}{T}\int_z\left[\frac{\varepsilon}{2}\left(\dot{r}(z)\right)^2\right]$$

It is dominated by the ground state. The ground state energy of a particle in a well is found by solving an algebraic equation

$$U(T) \approx E_0 = U - c\frac{T^2}{2\varepsilon R^2}$$

Where c is a zero of a Bessel function. Then at certain temperature

$$T_{depin} \sim R\sqrt{\varepsilon U}$$

bound states disappear and vortex gets "liberated" by thermal fluctuations. In dynamics this temperature is close to the crossover from the "flux creep" to flux flow.

The theory can be extended to include defect surfaces (1D well), while Lindemann criterium can be applied also to the columnar disorder case.

$c_L \approx 0.1 - 0.3$

Conclusion

1. **Type II superconductors in magnetic field provide a convenient laboratory for studying multisoliton physics and beyond.**
2. **Convenient environment for experiments creates mant theoretical questions, sometimes answers.**