Qualification examination (fall, 2011)

- Classical Mechanics -

1. (Calculus of variations, 20%)
When minimizing a function, we differentiate it with respect to its free parameter. But if it is an integral that we want to minimize, the variable becomes a function and the equivalent action of differentiation is the variation. For classical actions $S \equiv \int_{t_i}^{t_f} L(q,\dot{q})dt$, the starting and end points $q(t_i), q(t_f)$ are fixed, and so only variation and no differentiation is needed. Let me now test you on a problem that requires both actions; namely, determine the shape and contact angle of a liquid drop on a solid surface. Use $\varepsilon_{1,2}, \varepsilon_{2,3}, \varepsilon_{1,3}$ to denote the amount of energies that can be lowered for each unit area between air/liquid, liquid/desk, and air/desk. Make sure you include the surface tension T and the constraint that the liquid volume needs to be conserved.
   (a) Variate the drop shape over the total potential energy and write down the Euler-Lagrange equation of motion that determines the drop shape. No need to solve this differential equation.
   (b) Differentiate with respect to the drop height and obtain a simple relation for the contact angle, which is often ascribed to the balance of forces.

2. (Hamiltonian equations of motion, Goldstein's Exercise 8.14, 20 points)
The Lagrangian for a system can be written as
$$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y}^2 - k\sqrt{x^2+y^2},$$
where $a, b, c, f, g,$ and k are constants. What is the Hamiltonian? What quantities are conserved?

3. (Canonical transformation, Goldstein's Exercise 9.24, 20 points)
   (a) Show that the transformation $Q = p + iaq,\ P = \frac{p - iaq}{2ia}$ is canonical and find a generating function.
   (b) Use the transformation to solve the linear harmonic oscillator problem.

4. (2009 考古題, 加點變形, 20%)
A uniform ladder leans against a smooth vertical wall. The initial angle between the floor and ladder is θ_0. If the floor is also smooth, the ladder will slide down.

(a) Find the angle between the floor and the ladder when the ladder loses contact with the vertical wall.

(b) Find the shortest distance between the top of the ladder and the foot of the wall when the ladder lies totally flat on the floor.

5. (2008 考古題, 加點變形, 20 points)

A point particle of mass m and initial speed v_0 is scattered by another originally static point particle of mass M. The perpendicular distance between the center of force and the incident velocity is d (called the impact parameter). Assuming that the force is repulsive and the magnitude of this force is $\frac{mc}{r^2}$ where c is a positive constant and r is the distance between these two particles.

(a) Calculate the minimum value of r.

(b) Calculate the differential and total cross sections.

6. (2002 考古題，加分題，10 分)

A raindrop falls through a cloud collecting mass as it falls. Assume that the drop remains spherical and that the mass of the raindrop increases at a rate which is proportional to the cross-sectional area of the drop multiplied b the speed of fall. If the drop starts from rest when it is infinitely small, what is the acceleration?