Quantum Mechanics Qualification, Sep. 18, 2011.

You must provide the details or reasonings to justify your answers.

1. (8+7%) The Hamiltonian of a system is given by

$$H = \frac{41}{9} a^\dagger a + \frac{20}{9}\left(a^2 + (a^\dagger)^2\right) + \frac{25}{9}$$

where

$$a = \frac{1}{\sqrt{2}}(q + ip)\,, a^\dagger = \frac{1}{\sqrt{2}}(q - ip)\,, \text{ and } [q, p] = i.$$

(1) What are the eigen-energies of this system? (2) What is the ground state wave function? Remember to fix the normalization.

2. (5+5%) A particle whose wave function is given by

$$\psi(\vec{r}) = \left(\frac{\sqrt{2}}{\sqrt{3}} Y_{11}(\theta,\phi) - \frac{\sqrt{11}}{\sqrt{6}} Y_{1,-1}(\theta,\phi) + \frac{1}{\sqrt{2}} Y_{10}(\theta,\phi)\right) f(r)$$

where $f(r)$ is a normalized radial function, $\int_0^\infty r^2 f^2(r)dr = 1$.
(a) What are the expectation values of $\hat{L}^2$ and $\hat{L}_z$?
(b) What is the expectation value of an operator $\hat{V}(\theta) = 3\cos^2\theta$ in this case?

You may find the following information useful:

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta\,,\; Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

3. (8+8%) (a) Write down the spatial and spinal wave function of the first excited state for two noninteracting electrons in an infinite potential well, $V(x) = 0$ for $0 \le x \le L$ and $V(x) = \infty$ elsewhere.
(b) Similarly, write down the spatial wave function of the first excited state for two noninteracting spin-0 particles in the same potential well.

4. (10%) An astronaut in his spacesuit (total mass of 200kg) is stranded 3m from the spaceship. He hands and shoot a 10 Megawatt laser directly away from the ship and leaves it shinning. About how long will it take him to move 3m and reach safety? He started from rest relative to the ship.

5. (5+5+10+5+5%) An electron is trapped in a 2-dimensional infinite potential well in a rectangular area $a \le x \le a + L$ and $b \le y \le b + 2L$. (a) Write down the corresponding Schrodinger wave function with proper normalization. (b) What is the probability of finding the ground state electron in the rectangular region

$a \leq x \leq a + L/3$ and $b \leq y \leq b + L$?
(c) If a perturbation

$$\Delta V(x, y, t) = \lambda \cos \frac{\pi(y-b)}{2L} \cos \frac{3\hbar\pi^2 t}{8mL^2}$$

is applied to the system, discuss the rate of transition from the ground state to the excited states.

If 10 more electrons are filled in and we assume there is no interaction among the electrons,
(d) what is the ground state energy of the 11-electron system? (e) What's the minimal photon energy to excite the system?

6. (9%) The Clebsch-Gordan coefficients are coefficients those are used for adding two angular momenta (j_1 and j_2) to get the final angular momentum (j), and be represented as $\langle j_1, m_1; j_2, m_2 | j\, m\rangle$. It is known that $\langle 3/2, 3/2; 1/2, -1/2 | 2\, 1\rangle = 1/2$. Find $\langle 3/2, -1/2; 1/2, -1/2 | 2\, -1\rangle$, $\langle 3/2, 1/2; 1/2, 1/2 | 2\, 1\rangle$, and $\langle 3/2, 3/2; 1/2, -1/2 | 1\, 1\rangle$.

7. (10%) Calculate the Born approximation to the differential and total cross sections for scattering a particle of mass m off the δ-function potential $V(\vec{r}) = g\delta^3(\vec{r})$.