Quantum Mechanics

Problem 1 25% Answer the following questions *briefly*

(a) What is the most important consequence for a system being translational invariance?

(b) Write down the Dirac equation and indicate what kind of particles it is intended to describe.

(c) Consider a two-level system characterized by the Hamiltonian

$$\mathbf{H} = \Delta \left[|1><1| - 2|2><2| + 2|1><2| + 2|2><1| \right],$$

where Δ is a real number with the dimension of energy. $|1>$ and $|2>$ are normalized and are also orthogonal to each other. If one performs ideal measurements on the energy of this system, what possible values he would get?

(d) Consider 10 identical bosons in one dimension. Let x_i and p_i $(i = 1, 2, 3, ...10)$ be the corresponding position and momentum operators for 10 particles. Which of the following operator(s) is(are) observable(s)?

$|p_1 - p_{10}| + |p_2 - p_9| + |p_3 - p_8| + |p_4 - p_7| + |p_5 - p_6|$, $x_1 + x_2 + ...x_{10}$, $p_2^2 + p_4^2 + p_6^2 + p_8^2 + p_{10}^2$, $\sum_{i<j} \frac{e^2}{4\pi\epsilon_0(x_i - x_j)}$.

(e) Suppose that in the Aharonov-Bohm (AB) effect experiment, the particle sources are point particles with energy E and charge q. If the interference pattern is shifted by Δx in comparison to the pattern without the magnetic field, find the magnetic flux that is behind the double-slit (assume that the seperation between two slits is d and the distance from the slits to the screen is L).

Problem 2 Consider two particles of the same mass m in one dimension. Two particles are connected by a spring with spring constant k.

(a) 15% Suppose that the total momentum of two particles be p, find all possible total energies for the following cases: (i) two particles are non-identical (ii) two particles are identical fermions (iii) two particles are identical bosons.

(b) 8% Suppose that in addition to being connected by a spring, two particles carry charges of q and $-q$ and move along x axis. A uniform electric field E is applied along $+x$ direction. If two particles are in an energy eigenstate, find the minimum of root-mean-square relative distance (i.e., minimum of $\sqrt{\langle (x_1 - x_2)^2 \rangle}$) between two particles.

Problem 3 20% Consider the coupling of three spin-1/2 particles. Let $|\alpha\beta\gamma\rangle$ denote the state when the first particle in ithe state $|\alpha\rangle$, the 2nd in $|\beta\rangle$ and the 3rd in $|\gamma\rangle$, where α, β, and γ are either + (spin up) or − (spin down).

(a) Construct all states with total angular momentum $J = 3/2$.

(b) By setting coefficients associated with states starting with + (i.e., $|+\beta\gamma\rangle$) being positive, construct all states with $J = 1/2$. (Hint: add two spin-1/2 partciles first, and then include the 3rd particle. You may find the following formula useful: $J_\pm |j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle$)

Problem 4 15% A particle of mass m is described by the wave function

$$\psi_E (r, \theta, \phi) = A \exp(-r/a_0),$$

where r, θ and ϕ are spherical coordinates and a_0 is a positive constant. Assuming that ψ_E is an energy eigenstate in a central potential $V(r)$. and $V(r) \to 0$ as $r \to \infty$, find the energy E of this state, $V(r)$, and all other possible energies for this particle.

Problem 5 Consider a one-dimensional harmonic oscillator with mass m and natural frequence ω. The oscillator is in its ground state $|0\rangle$ at $t = -\infty$. A time-dependent potential is switched on as follows

$$V(x, t) \;=\; \frac{V_0 x}{1 + (t/\tau)^2},$$

where τ is a positive constant and V_0 is also a constant.

(a) 10% Find the probability that the oscillator is in the state $|n\rangle$ $(n \neq 0)$ at $t = \infty$ to the order of V_0^2.

(b) 7% Suppose that for $t \geq 0$, $V(x, t) = V_0 x$. Find the wavefunction ψ at $t = 0$ to $O(V_0)$ in the adiabatic limit $\tau \to \infty$. Find a static Hamiltonian H such that to $O(V_0)$, ψ is an eigenstate to H.