Quantum Mechanics Qualification
Feb. 24, 2013.

1. Answer the following questions briefly(5+5+5+5+5%)
   (a) What are the generators of spatial translations?
   (b) What are the Hermitian conjugates of operators $x^2\frac{d}{dx}$ and $e^{-i\,|\alpha\rangle\langle\beta|}$?
   (c) For a wavefunction $\psi = \frac{1}{\sqrt{3}}Y_2^{-1}(\theta,\phi) - \frac{\sqrt{2}}{\sqrt{3}}Y_3^{-1}(\theta,\phi)$, find $\langle\psi|L_+|\psi\rangle$ and $L_z\psi$.
   (d) A recent hot news was a bright meteor (shooting star) flying across the sky over Chelyabinsk, Russia. What is the physics and quantum processes behind the flare?
   (e) What is Aharonov-Bohm effect?

2. (10%) A system consists of two particles with spins $s_1 = 3/2$ and $s_2 = 1/2$. Find all possible eigenvalues of the operator $(\mathbf{s}_1 - \mathbf{s}_2)^2$.

3. (15%) Consider two particles of the same mass m in one dimension and they are connected by a spring with spring constant k. Suppose that the total momentum of the system is p, find all possible total energies for the following cases: (i) two particles are different (ii) two particles are identical fermions (iii) two particles are identical bosons.

4. (10%) Consider a one-dimensional harmonic oscillator with mass m and natural frequency ω. The oscillator is in its ground state $|0\rangle$ at $t = -\infty$. A time dependent potential is switched on as follows
$$V(x,t) = \frac{V_0 x}{1 + (t/\tau)^2}$$
where τ is a positive constant and V_0 is a constant as well.
Find the probability that the oscillator is in the state $|n\rangle (n \neq 0)$ at $t = \infty$ to order of V_0^2.

5. (10%) Consider a central scattering potential $V(r) = V_0$ for $r < a$ and $V(r) = 0$ elsewhere, where V_0 is a constant. Use the Born approximation to evaluate the total scattering cross section in the limit of $(ka \ll 1)$, where k is the momentum of the incident particle.

6. (15%) A perfectly elastic steel ball of diameter 1 cm is dropped vertically from a height 10 cm above an identical ball which is anchored to the ground. Taking into account the uncertainty principle, estimate the maximum number of bounces that can occur. (The density of iron is around $\sim 8g/cm^3$.)

7. (15%) A point particle of mass m is subject to the following central potential
$$V(r) = \begin{cases} 0, & 3a < r < 5a \\ \infty, & \text{elsewhere}\,. \end{cases}$$
Find the corresponding eigen-energy and the normalized wavefunction for $l = 0$. Recall that the Laplacian operator is given by
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$
in the spherical coordinate.