Quantum Mechanics

Useful information

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2+\beta x} = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/4\alpha}$$

Problem 1 20% Answer the following questions *briefly*

(a) 5% What are generators of rotation?

(b) 5% Find Hermitian conjugates of the operators: $x\frac{\partial}{\partial x}$ and $4^{|\alpha\rangle\langle\beta|}$.

(c) 5% Consider 6 identical Fermions in one dimension. Let x_i and p_i $(i = 1, 2, 3, ...6)$ be the corresponding position and momentum operators for 6 particles. Which of the following operator(s) is(are) observable(s)?

$p_1 + p_2 + ...p_{10}$, $x_2^2 + x_4^2 + x_6^2$, $\sum_{i<j} \frac{e^2}{4\pi\epsilon_0(x_i - x_j)}$.

(d) 5% Let $\hat{\mathbf{P}}$ and $\hat{\mathbf{L}}$ be the three dimensional momentum and orbital angular momentum operators. Consider the operator

$$\exp\left(\frac{i\hat{\mathrm{J}}_x\phi}{\hbar}\right)\hat{\mathrm{P}}_y\hat{\mathrm{L}}_z\exp\left(-\frac{i\hat{\mathrm{J}}_x\phi}{\hbar}\right).$$

Here $\hat{\mathbf{J}}$ is the total angular momentum operator. Express the above operators in terms of $\hat{\mathrm{P}}_y$, $\hat{\mathrm{P}}_z$, $\hat{\mathrm{L}}_y$, $\hat{\mathrm{L}}_z$, and ϕ.

Problem 2 22% Consider a particle at one dimension. Let the mass of the particle be m and its position is described by x.

(a) 9% If at $t = 0$, the particle is found precisely at $x = 0$ and the particle is other free for $t > 0$, find the wavefunction $\psi(x, t)$ of the particle at a later time t.

(b) Suppose that at $t = 0$, the particle is not so precisely at $x = 0$ but is located at $x = 0$ with the wavefunction being given by $\psi(x, 0) = (\pi a^2)^{-1/4}\exp(-x^2/2a^2)$, where $a > 0$. The particle is otherwise free for $t > 0$.

(i) 6% Find the position operator in the Heisenberg picture $\hat{\mathrm{x}}_H(t)$ (for $t > 0$) in terms of the operators $\hat{\mathrm{x}}$ and $\hat{\mathrm{p}}$ defined in the Schrodinger's picture. Calculate the commutator $[\hat{\mathrm{x}}_H(t), \hat{\mathrm{x}}_H(t')]$.

(ii) 7% Using results from (i), calculate $\Delta x(t)$ for $t > 0$.

Problem 3 31% Suppose that the Hamiltonian of a particle is given by

$$H = 5a^\dagger a + \beta a^2 + \beta(a^\dagger)^2 + 3$$

where after appropriate choice of units ($\hbar = 1$ and etc), $[x, p] = i$ and a and $a^\dagger$ are given by

$$a = \frac{1}{\sqrt{2}}(x + ip) \text{ and } a^\dagger = \frac{1}{\sqrt{2}}(x - ip).$$

(a) 9% When $\beta = 2$, if one performs ideal measurements on the energy of this particle, what possible values he would get?

(b) 6% Find the normalized ground state wavefunction $\tilde{\phi}_0(x)$ for the case $\beta = 2$.

(c) 8% Suppose that in addition to H with $\beta = 2$, a perturbed potential $V = \alpha x^4$ is applied to the particle. Find the energy shifts for all energy levels to the order of α.

(d) 8% Suppose that β is switched on as follows: $\beta(t) = \frac{\beta_0}{1+(t/\tau)^2}$, where τ is a positive constant. At $t = -\infty$, the particle is in its ground state. (i) Find the probability that the particle makes transition to an excited state at $t = \infty$ to the order of β_0^2.

(ii) For $\beta_0 = 2$ and to $O(\beta_0)$ in the wavefunction, find the probability of finding the particle in the ground state of $\beta = 2$ (i.e., the ground state found in (b)) at $t = \infty$. Note that literally, the propability is greater than one due to trucation but do not worry about it.

Problem 4 (a) 4% Find $\mathbf{L} \cdot \mathbf{S} \left(Y_1^{-1}(\theta, \phi) + 2Y_1^0(\theta, \phi)\right) |+\rangle$ in terms of $Y_l^m(\theta, \phi)$, $|+\rangle$, and $|-\rangle$, where $\mathbf{L}$ are the orbital angular momentum operator and $\mathbf{S}$ are the spin operators.

(b) 4% A beam of unploarized spin-1/2 particles, moving along the y-axis, is incident on two collinear Stern-Gerlach apparatuses, the first with $\mathbf{B}$ along the z axis and the second with $\mathbf{B}$ along the z' axis, which lies in the x-z plane at an angle $\pi/3$ relative to the x axis. Both apparatuses transmit only the uppermost beams, what fraction of particles leaving the first will leave the last?

(c) 4% Consider a system of two particles with spins $s_1 = 3/2$ and $s_2 = 1/2$. By considering the addition of two spins, find all possible eigenvalues to square of the difference of spin operators $(\mathbf{S}_1 - \mathbf{S}_2)^2$.

Problem 5 10% Consider the mutual elastic scattering of two spin-1/2 fermions. The Hamiltonian for this system is $\hat{\mathrm{H}} = \frac{\hat{\mathrm{p}}_1^2}{2m} + \frac{\hat{\mathrm{p}}_2^2}{2m} + \mathrm{V}(\mathbf{r}_1 - \mathbf{r}_2)$, where $\mathrm{V}(\mathbf{r}_1 - \mathbf{r}_2) = g\delta(\mathbf{r}_1 - \mathbf{r}_2)$. In the lab frame, the scattering is set up in the way that one fermion is initially at rest, while the other one is incident with a momentum $\hbar\mathbf{k}$. Both fermions are not polarized. Use the Born approximation to calculate the differential cross section $\sigma(\theta, \phi)$ in the center of mass (CM) frame. What is the differential cross section in the lab frame? What would be the differential cross section in the CM frame if these particles are identical fermions?