Qualifying examination – Statistical Mechanics

Fall, 2012

1. (main concepts in statistical mechanics, 30 points)
   (a) What are the fundamental assumptions of equilibrium statistical mechanics?
   (b) Distinguish the following energies: Helmholtz $(F \equiv U - TS)$ free energy, grand potential $(-PV)$, Gibbs $(G \equiv U - TS + PV)$ thermodynamic potential, and enthalpy $(H \equiv U + PV)$.
   (c) Gibbs' paradox and how one resolves the unphysical puzzle.
   (d) Bose-Einstein condensation.
   (e) Determine the critical mass of a neutron star (called the Chandrasekhar limit) beyond which the gravitational pull can overcome the quantum pressure and cause further shrinkage of the dead star into a white dwarf.

2. (entropy, 20 points)
   What is the entropy of the following three-dimensional systems consisting of N atoms with atomic weight m and at temperature T?
   (a) A crystal. At very low temperature, the interaction between these atoms can be neglected. Assume the ground state of each atom is doubly degenerate.
   (b) An ideal classical gas.
   (c) An ideal Fermi gas with T much less than the Fermi temperature.

3. (specific heat, 30 points)
   Calculate the specific heat of the following systems at low temperature.
   (a) A classical molecular gas consisting of identical atoms with atomic weight m. Assume that the gas is dilute so that intermolecular interactions can be neglected. To consider rotations, use I to denote the moment of inertia of each molecule. In the mean time, approximate the bonding between atoms by a spring constant K to describe the vibrating motion. (10 points)
   (b) A solid in the Einstein model; e.g., N number of independent oscillators with intrinsic harmonic frequency ω. (5 points)
   (c) A one-dimensional solid in the Debye model; e.g., N atoms with atomic weight m are connected in series by springs with spring constant K. For simplicity, adopt the periodic boundary condition; that is, arrange these atoms in a circle with one extra spring connecting the first and last atoms. (15 points)

4. (one-dimensional Ising model, 20 points)

The Hamiltonian of one-dimensional Ising model is

$$H = -J\sum_{i=1}^{N-1} S_i S_{i+1} - B\sum_{i=1}^{N} S_i$$

where $S_i = \pm 1$ and B is the external magnetic field. When the coupling constant J is positive/negative, it favors parallel/antiparallel alignments for the spins.

(a) Please show that the magnetization, $M = \sum_{i=1}^{N} \langle S_i \rangle$ where $\langle\ \rangle$ denotes the statistical average, is always zero in the absence of *B*. In other words, there is no phase transition into a magnetic state at any temperature *T*.

(b) Find the magnetic susceptibility, $\chi \equiv \frac{dM}{dB}$, when *B* is much less than both *J* and $k_B T$.