# Qualifying Examination [Classical Mechanics]

9/28/2013

**1. (20%)** A pointlike mass $m$ is undergoing a three-dimensional projectile motion in a uniform gravitational field $g$. Consider that the air resistance is negligible.

- (a) Write the Hamiltonian function for this problem $H = H(p_i, q_i)$, choosing as generalized coordinates the Cartesian coordinates $(x, y, z)$ with the $z$ axis pointing in the vertical direction.
- (b) Show that these equations lead to the known equations of motion for Projectile motion and that the Hamiltonian function corresponds to the Total Mechanical energy of the particle.
- (c) Once the particle reaches the highest point in its trajectory, a retarding force proportional to the velocity of the particle starts acting. Assume the proportionality constant to be known and given by $k$. Calculate the vertical velocity as a function of time for the descending particle, and find the terminal velocity.

**2. (20%)** A rigid body consists of six particles, each of mass $m$, fixed to the ends of three light rods of length $2a$, $2b$, and $2c$ respectively, the rods being held mutually perpendicular to one another at their midpoints.

- (a) Write down the inertia tensor for the system in the coordinate axes defined by the rods;
- (b) Find angular momentum and the kinetic energy of the system when it is rotating with an angular velocity $\omega$ about an axis passing through the origin and the point $(a, b, c)$.

**3. (20%)** Many features of the orbits around a nonrotating black hole can be well approximated within the Newtonian equations of motion via a pseudo-Newtonian gravitational potential for $r > r_*$

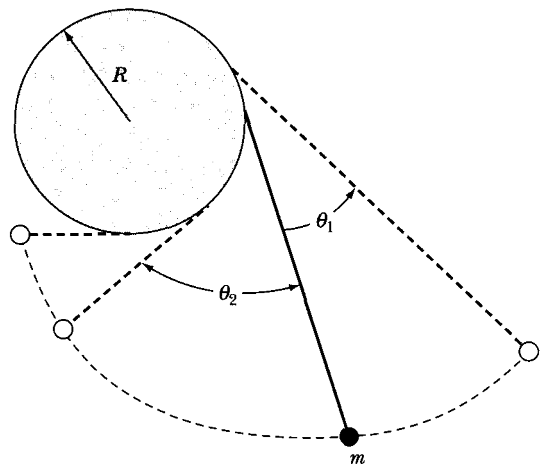
$$\Psi = -\frac{GM}{(r - r_*)}.$$

Consider only the region outside the black hole "horizon" $r_* = 2GM/c^2$, with $M$ the mass of the "black hole" and $m\,(\ll M)$ the mass of the orbiting test particle.

- (a) Calculate the total energy and obtain the effective potential $\Psi_e(\tilde{l}, r)$ defined via the conserved total energy
$$E = m\left[\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \Psi_e\right],$$
where the conserved angular momentum $l = m\tilde{l}$. Sketch the (dimensionless) function $\Psi_e/c^2$, for a few values of $\alpha \equiv c\tilde{l}/(GM)$ in the range 0-5, using the dimensionless coordinate $x = r/r_* = rc^2/(2GM)$.
- (b) Over what range of $r$ do circular orbits exits? For such a circular orbit, what is the corresponding angular velocity $\Omega(r)$?
- (c) Over what range of $r$ are the circular orbits unstable?
- (d) For what range of $\alpha$ will a particle dropped nearly from rest very far away ($E = 0$) be swallowed by the black hole?

**4. (20%)** A pendulum is constructed by attaching a mass $m$ to an extensionless string of length $l$. The fixed end of the string is connected to a fixed vertical disk of radius $R\,(R < l/\pi)$, as shown in the figure below. The motion of the pendulum occurs in the plane of the figure and the string connection to the disk is at a place such that in the oscillations of interest here the string is always tangent to the disk.

- (a) Find the Lagrangian and the equation of the motion for the pendulum.
- (b) Expand $\theta$ about some angle $\theta_0$ and assume that $\epsilon \equiv \theta - \theta_0$ is small. Calculate the period of the small oscillations.
- (c) In the small oscillations regimes find the line about which the angular motion extends equally in either direction (i.e. $\theta_1 = \theta_2$).




**5. (20%)** Use following information to answer these questions: The rest-mass energy of an electron and an charged pion are 500 keV and 140 MeV respectively. The average lifetime of a charged pion at rest is $2.60 \times 10^{-8}$.

- (a) Consider a liner accelerator in which electrons are accelerated to 50 GeV per electron. The accelerator is 3 km long. The electrons are accelerated in "bunches" that is 1 cm long as measured in the laboratory frame of reference. Calculate the apparent length of the accelerator and a bunch of electrons as measured in the reference frame of the 50-GeV electrons.
- (b) Suppose an electron with total energy of 0.5 GeV collides head-on with a positron with total energy of 0.5 GeV. The electron and position annihilate and create a pair of back-to-back charged pions:
$$e^+ + e^- \to \pi^+ + \pi^-.$$
What is the average lifetime of these pions, as measured in the laboratory reference frame?
- (c) Suppose an electron with total energy of 0.5 GeV collides head-on with a positron with total energy of 0.5 GeV. The electron and position annihilate and create a pair of back-to-back neutral pions:
$$e^+ + e^- \to \pi^0 + \pi^0.$$
Assume that the mass of the neutral pion is the same as that of the charged pion. By far the most common decay mode of the $\pi^0$ is to two photons: $\pi^0 \to \gamma + \gamma$. What is the minimum possible photon energy and the maximum possible photon energy measured in the laboratory frame of reference?