Ph.D. Qualify Exam **Classical Electrodynamics** Spring 2014

1. (20 %) [Maxwell equations and EM waves]
   (a) Write down the differential form of the four Maxwell equations.
   (b) Derive the wave equations in free space for the electric field from Maxwell equations. You may need this identity $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$.
   (c) Show that the electric field and magnetic field are perpendicular to each other in an EM wave.
   (d) A laser beam has an intensity of 1W/m^2. Find the peak value of its electric field and magnetic field. ($\mu_0 = 4\pi \times 10^{-7}$ N/A^2)

2. (20 %) Explain the following items:
   (a) Linear and circular polarization of light.
   (b) Phase velocity and group velocity
   (c) Multipole expansion of electric potential.
   (d) Rayleigh scattering.
   (e) Liénard-Wiechert potential.
   (f) Kramers-Kronig relations.

3. (20 %) [Green function]

   To solve an electrostatic problem, $\nabla^2 \Phi(\vec{r}) = -\rho(\vec{r})/\varepsilon_0$, for arbitrary charge distribution $\rho(\vec{r})$, one may use the Green function approach. The Green function for 3-dimensional Poisson equation satisfies:

   $$\nabla^2 G(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \text{ ----- } (1)$$

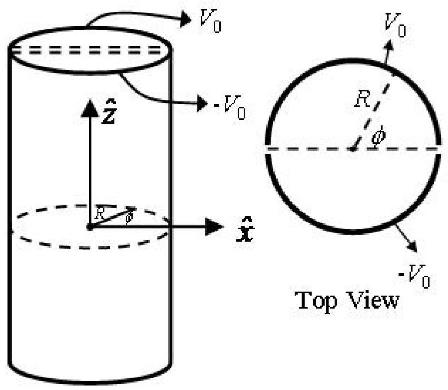
   (a) Brief explain how we can obtain $\Phi(\vec{r})$ by solving $G(\vec{r}, \vec{r}\,')$.
   (b) To solve $G(\vec{r} - \vec{r}\,')$, first find the Fourier transform of the Green function, i.e. find $\tilde{G}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int G(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d^3\vec{r}$, by taking the Fourier transform of equation (1).
   (c) Find the Green function $G(\vec{r} - \vec{r}\,')$ by making an inverse transform of $\tilde{G}(\vec{k})$, i.e. find $G(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int \tilde{G}(\vec{k}) e^{i\vec{k}\cdot\vec{r}} d^3\vec{k}$. Show this is exactly the form of Coulomb potential. (You may need this identity $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.)
   (d) For a three-dimensional Helmholtz equation, the Green's function satisfies:

   $$\left(\nabla^2 - \lambda^2\right) G(\vec{r} - \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \text{ ------ } (2)$$

   where λ is a constant. Repeat the same procedure as in (b) and (c), find $G(\vec{r} - \vec{r}\,')$ and obtain the so-called Yukawa potential. (You may need this identity $\int_0^\infty \frac{x^2}{x^2 + a^2} \frac{\sin x}{x} dx = \frac{\pi}{2} e^{-a}$.)

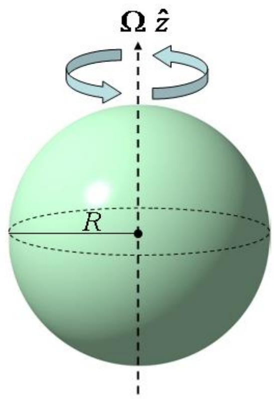
4. (20 %) [Electrostatics]

A very long hollow conducting tube with radius R is cut into halves through its center. The upper half is kept at potential V_0 and the lower half is kept at $-V_0$. Find the electric potential inside and outside the tube. The general solution of Laplace equation in cylindrical coordinate with z-symmetry is $\Phi(r,\phi)=\sum_{n=1}^{\infty}(A_n r^n + B_n r^{-n})(C_n \cos n\phi + D_n \sin n\phi)$. You may need this identity: $\int_0^{\pi} \sin n\phi \cdot \sin m\phi \cdot d\phi = \begin{cases} 0, & if\ n \neq m \\ \frac{\pi}{2}, & if\ n = m \end{cases}$.



5. (20 %) [Magnetostatics]

An insulating solid sphere of radius R has a spherically symmetric charge distribution $\rho(r)$. The sphere is rotating about an axis through its center at angular frequency Ω.



(a) Give an expression for the electric potential function $\Phi(\vec{r})$ at any point $\vec{r}$ in terms of an integral involving $\rho(r)$. The potential is zero at infinity.

(b) Give an expression for the current density $\vec{J}(\vec{r})$ in terms of the quantities given at any point inside the sphere.

(c) Give an expression for the magnetic field $\vec{B}(\vec{r})$ at any point $\vec{r}$ in terms of an integral involving the current density you find in (b).

(d) Find a simple expression for the ratio $\frac{|B(0)|}{\Phi(0)}$ in terms of only Ω and some constants.

(e) What is the direction of $\vec{B}(0)$ if $\rho(r)$ is positive everywhere?

You may need this identity: $\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\cdot\vec{B}-(\vec{A}\cdot\vec{B})\cdot\vec{C}$