Quantum Mechanics Qualification
Spring, 2014.

1. (5% each)Briefly answer the following questions:

   - For a given state $|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|2\rangle + \frac{i}{\sqrt{3}}|3\rangle$, where $|1\rangle, |2\rangle, |3\rangle$ are three orthonormal basis. What is the probability of finding $|\psi\rangle$ in $|3\rangle$?
   - Continue from the previous question, if there is another state $|\phi\rangle = |1\rangle + \frac{i}{\sqrt{2}}|2\rangle - \sqrt{3}|3\rangle$, $\langle 2|\psi\rangle\langle\phi|1\rangle$ =?
   - In a spherically symmetrical potential, a state is described by a normalized wave function $\psi(r,\theta,\phi) = R(r) \times [\sqrt{2/7}Y_4^2 - i\sqrt{1/7}Y_4^0 + i\sqrt{4/7}Y_4^{-1}]$. What is $\langle\psi|L_x^2 + L_y^2|\psi\rangle$?
   - What is the Hermitian conjugate of the operator $x\frac{\partial}{\partial x}$?
   - What are the Clebsch-Gordan coefficients?
   - What is Path Integrals?
   - What is spontaneous emission?

2. (15%) Consider two particles of the same mass m in one dimension and they are connected by a spring with spring constant k. Suppose that the total momentum of the system is p, find all possible total energies for the following cases: (i) two particles are different (ii) two particles are identical fermions (iii) two particles are identical bosons.

3. (8%+7%) A point particle of mass m is subject to the following central potential

$$V(r) = \begin{cases} \infty, & r < a \\ 0, & a < r < a+b \\ V_0(>0), & a+b < r < a+b+\triangle \\ 0, & a+b+\triangle < r\,. \end{cases}$$

   (a) In the limit of $V_0 \to \infty$, find the corresponding eigen-energies and the normalized wavefunctions for $l = 0$ bound state. Recall that the Laplacian operator is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

   in the spherical coordinate. (b) Now consider the case that V_0 is finite but much larger than the ground state energy obtained in (a). Use the WKB approximation for tunneling to estimate the escaping rate (= probability per unit time) of the particle in the lowest bound state.

4. (10%) A point particle of mass m and incident energy E is scattering off the potential $V(\vec{r}) = ge^{-r^2/R^2}$. Calculate the first Born approximation to the differential and total cross sections.

5. (10%+5%+10%) Consider a perturbation $H' = \alpha x^4$ to the harmonic oscillator problem.

(a) Show that the first-order correction to the unperturbed eigen-energies are

$$E_n^{(1)} = \frac{3\hbar^2\alpha}{4m\omega^2}[1 + 2n + 2n^2]$$

(b) No matter how small α is, the perturbation expansion will break down for some large enough n. Why?

(c) If we make the perturbation time-dependent, $H' = \alpha x^4 e^{-t^2/\tau^2}$, between $t = -\infty$ and $t = +\infty$. What is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t = +\infty$?

You might find the following identity useful.

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}\exp\frac{b^2}{4a}$$