Quantum Mechanics Qualification
Spring, 2015.

1. (7%) In what situation it is correct to directly solve the equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right)\psi(\vec{r}) = E\,\psi(\vec{r})\,?$$

   Give a counter example.

2. (8%) Give two physical examples that one of the associated Hilbert spaces is finite dimensional and the other one is $\infty-$dimensional.

3. (10%) Use some of the spherical harmonics of $l = 2$ to construct a normalized wave function which is an eigen-state of $\hat{L}_x$.

4. (7+8%) (a) In a 3-dim Hilbert space, construct two physical operators A and B and they satisfy: (1) A, B share one and only one common eigenvalue, also (2) $[A, B] \neq 0$. (b) Use your A, B to verify the generalized uncertainty principle.

5. (10%) Work out all the Clebsch-Gordan coefficients of $\frac{1}{2} \times \frac{1}{2}$.

6. (5+5+5+5%) An electron is trapped in a 2-dimensional infinite potential well in a rectangular area $a \leq x \leq a+L$ and $b \leq y \leq b+2L$. (a) Write down the corresponding Schrodinger wave function with proper normalization. (b) What is the probability of finding the ground state electron in the rectangular region $a \leq x \leq a + L/3$ and $b \leq y \leq b + L$?
   If 10 more electrons are filled in and we assume there is no interaction among the electrons,
   (c) what is the ground state energy of the 11-electron system? (d) What's the minimal energy to excite the system?

7. (7+5+8%) Consider a perturbation $H' = \alpha x^4$ to the harmonic oscillator problem.
   (a) Show that the first-order correction to the unperturbed eigen-energies are

$$E_n^{(1)} = \frac{3\hbar^2\alpha}{4m\omega^2}[1 + 2n + 2n^2]$$

   (b) No matter how small α is, the perturbation expansion will break down for some large enough n. Why?
   (c) If we make the perturbation time-dependent, $H' = \alpha x^4 e^{-t^2/\tau^2}$, between $t = -\infty$ and $t = +\infty$. What is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t = +\infty$?

   You might find the following identity useful.

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}\,\exp\frac{b^2}{4a}$$

8. (10%) Consider the S-wave neutron-neutron scattering where the interaction potential is approximated by $V(r) = V_0 \vec{S}_1 \cdot \vec{S}_2 e^{-r/a}$, where $\vec{S}_1$ and $\vec{S}_2$ are the spin vector operators of the two neutrons, and $V_0 > 0$. Find the differential cross section in the first Born approximation.