Quantum Mechanics Qualification
Fall, 2015.

1. (5%) In what situation it is correct to directly solve the following equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right)\psi(\vec{r}) = E\,\psi(\vec{r})\,?$$

   Give a counter example.

2. (10%) Use some of the spherical harmonics of $Y_{l=1}^{m}(\theta,\phi)$, to construct your own normalized wave function which is an eigen-state of $\hat{L}_y$.

3. (7+8%) (a) In a 3-dim Hilbert space, construct two physical operators A and B (as two 3×3 matrices) and they satisfy: (1) A, B share one and only one common eigenvalue, also (2) $[A, B] \neq 0$. (b) Use your A, B to verify the generalized uncertainty principle.

4. (7+5+8%) Consider a perturbation $H' = \alpha x^4$ to the harmonic oscillator problem.
   (a) Work out the first-order correction to the eigen-energies of state $|n^{(0)}\rangle$. (hint: $a = (ip + m\omega x)/\sqrt{2\hbar m\omega}$ where ω is the natural frequency.)
   (b) No matter how small α is, the perturbation expansion will break down for some large enough n. Why?
   (c) If we make the perturbation time-dependent, $H' = \alpha x^4 e^{-t^2/\tau^2}$, between $t = -\infty$ and $t = +\infty$. What is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t = +\infty$?

   You might find the following identity useful.

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}\,\exp\frac{b^2}{4a}$$

5. (10%) Consider a central scattering potential $V(r) = V_0$ for $r < a$ and $V(r) = 0$ elsewhere, where V_0 is a constant. Use the Born approximation to evaluate the total scattering cross section in the limit of ($ka \ll 1$), where k is the momentum of the incident particle.

6. (8+8%) (a) Write down the spatial and spinal wave function of the first excited state for two noninteracting electrons in an infinite potential well, $V(x) = 0$ for $-L \leq x \leq L$ and $V(x) = \infty$ elsewhere.
   (c) Similarly, write down the spatial wave function of the first excited state for two noninteracting spin-0 particles in the same potential well.

7. (7+7%) (a) Give an explicit example of two states in the Hydrogen atom with all quantum numbers specified and the $E1$ transition between the two states is allowed.
   (b) As in (a), provide an example where $E1$ transition is NOT allowed.

8. (10%) For a point particle, show that $\frac{\partial}{\partial x}$ is NOT a Hermitian operator in the position representation. (Hint: consider a general matrix element $\int \psi_1^*(x)\frac{\partial}{\partial x}\psi_2(x)dx$.)