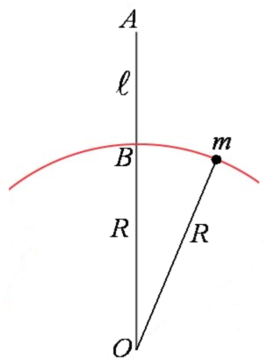
Qualifying Exam for PhD Candidates (February, 2010)

Classical Mechanics

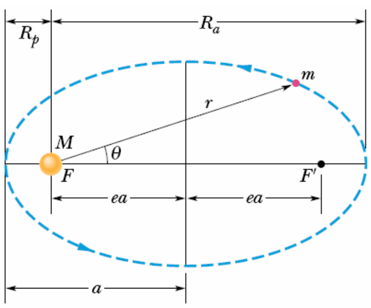
1. (a) What is the principle of least action or Hamilton's principle? (3%) (b) Derive the Euler-Lagrange equations from the Hamilton's principle. (4%) (c) Show that the Lagrange equation is equivalent to Newton's equation of motion if the Lagrangian is given by $L=\sum_i^N \frac{1}{2} m_i v_i^2 - U(\vec{r}_1,\vec{r}_2,...,\vec{r}_N)$ for a closed system of $N$ particles with a potential energy $U(\vec{r}_1,\vec{r}_2,...,\vec{r}_N)$. (3%) (d) Despite of the fact that the Lagrangian formulism is equivalent to Newton's law of motion, are there any advantages of Lagrangian formulism? If so, please name some. (3%) (e) What is the Hamiltonian of a system if its Lagrangian is given by $L(q_1,q_2,...,\dot{q}_1,\dot{q}_2,...,t)$ ?(4%) (f) Derive the Hamilton's equations and show that $\frac{dH}{dt}=\frac{\partial H}{\partial t}$, i.e. $H$ is a constant of motion if it does not depend on time explicitly. (3%)

2. A particle of mass $m$ is confined to move without friction on a circle of radius $R$. The particle is also subjected to a central force $\vec{F}(\vec{r})=-k\vec{r}$ with the center fixed at position $A$ shown in the attached diagram. Clearly, the equilibrium position of particle is at point $B$ on the circle. (a) What is the convenient generalized coordinate to describe the particle's motion? (3%) (b) Write down its Lagrangian. (3%) (c) What is its generalized momentum? (3%) (d) Write down its Hamiltonian. (3%) (e) Using action-angle variable technique, or whatever method you prefer, to obtain the small-amplitude period of the particle about its equilibrium position. (8%)




3. (a) A particle of mass $m$ moves in a circular orbit under the influence of a central potential $-km/r^n$, where $k$ is a positive constant. For small perturbation to a circular orbit, what is the range of $n$ for the orbit to be stable? (10%)

(b) A satellite of mass $m$ moves in a circular orbit of radius $R$ around the Earth of mass $M$. It receives an impulse of $\vec{P}$ parallel to its instantaneous velocity $\vec{v}$. For the satellite to be in a bounded orbit after the impulse, what is the maximum value of $P$? For $P$ less than this maximum value, find the period and eccentricity of the satellite after the pulse. (10%)



4. Bungee Jumping (高空彈跳)

Bungee Jumping probably originated from the indigenous people of the Penecoste Islands in the South Pacific. The local people have jumped off bamboo towers with vines tied to their legs for thousands of years, as part of the ceremony for young boys becoming brave adults. England's Oxford Dangerous Sports Club started the modern version of the sport on April 1, 1978 from the Clifton Suspension Bridge in Bristol, England. In 1988 the Kockleman Brothers in the US and A.J. Hackett in New Zealand started commercial jumping for the public. Today the bungee jumping has gradually become a popular sport worldwide. There are two main types of bungee jumping cords, sheathed and all-rubber.

(高空彈跳起源於南太平洋中的小島 Pentecoste Islands，島上的原住民將樹藤綁在腳踝上，自竹子搭成的高塔上向下躍下，以示成年男子的膽識，已有數千年的歷史。這項原住民的活動，在 1978 年經由英格蘭的“危險運動俱樂部”改良，並在英格蘭 Bristol 的 Clifton 吊橋上示範。而將高空彈跳商業化，則始於 1988 年美國的 Kockleman 兄弟及紐西蘭的 A.J. Hackett。至今高空彈跳已逐漸普及全球。現在使用的彈跳粗繩主要有兩種，一種是以橡膠為內心，外加套鞘構成，一種是全橡膠製成。)

For simplicity of this problem, we consider a homogeneous rubber rope and assume that the rope satisfies the following equation when it is stretched.

$$\frac{F}{A} = Y\frac{\ell - \ell_0}{\ell_0}$$

where $A$ and $\ell$ are the cross sectional area and the length of the rope under the tensile force $F$, $Y = 1.70 \times 10^6\, Pa$ is the material's Young's modulus. Furthermore we assume that the rubber density $\rho = 1.25 \times 10^3\, kg \cdot m^{-3}$ remains unchanged when it is stretched, thus we have $A\ell = A_0 \ell_0$, where $A_0 = 1.00 \times 10^{-3}\, m^2$ and $\ell_0 = 20.0 m$ are the natural cross sectional area and the length of the rope.

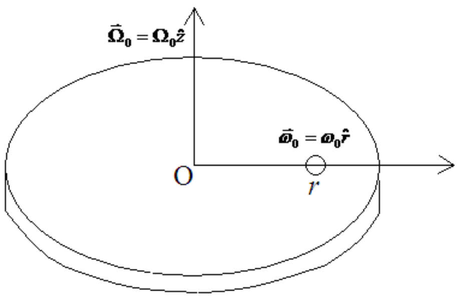
(a) When the rope is placed on a horizontal plane, derive the relationship between $F$ and $\ell$ in terms of known quantities. If $\Delta\ell \equiv \ell - \ell_0 << \ell_0$, then what is the spring constant of the rope? (4%)

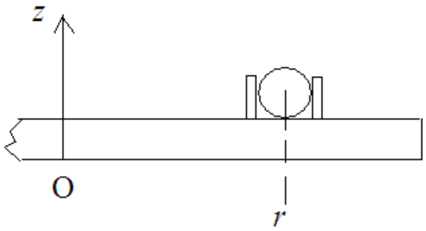
(b) When the rope is in vertical position with one end of the rope fixed at a certain height and the other end attached with an object of mass $m = 50.0kg$, what is the length $\ell_1$ of rope if both the rope and the object are in static equilibrium? (4%)

(c) If the object is executing small-amplitude oscillations about its equilibrium position, what is period of the oscillations? (6%)

(d) A person of 50.0 *kg* weight jumps from the platform with the rope securely attached. Assume that the entire rope is placed on the platform before jumping and the friction between the rope and the platform can be ignored. What is the maximum extension of the rope if the persons jump down with negligible initial speed and negligible air resistance during his fall? (6%)

5. Rolling ball on a turntable. A uniform solid ball of radius *a* and mass *m* is placed gently on a horizontal turntable which is rotating at a fixed angular velocity $\vec{\Omega}_0 = \Omega_0 \hat{z}$. The position of the ball is at a distance *r* from the center of the turntable and, initially, it has zero center-of-mass velocity but a spin of $\vec{\omega}_0 = \omega_0 \hat{r}$, as shown in the diagram below. We further assume that there two smooth concentric circular walls at *r*–*a* and *r*+*a* to restrict the motion of the ball in the circular track of radius *r*, as shown in the cross section view below. There is no friction between the wall and the ball. The static and dynamic friction between the ball and the turntable are $\mu_k$ and $\mu_s$ respectively.




(a) Due to friction, the motion of the ball will become pure rotation on the turntable. When this happens, what are its spin $\omega \hat{r}$ and orbital angular velocity $\Omega \hat{z}$ in the laboratory reference frame. (10%)

(b) After the ball establishes pure rotation, the retaining walls are instantly removed vertically. Will the ball's motion remains the same? Explain. (10%)