Qualification examination (spring, 2010)

- Quantum Mechanics -

註：總分 110 分，滿分 100 分

1. (Propagator, 20%)

Given an arbitrary wavefunction $\psi(x)$ at $t=0$, the propagator $U(x',0;x,t)$ can lead us to its form at any later time t under the influence of potential $V(x)$:

$$\psi(x,t)=\int_{-\infty}^{\infty}U(x',0;x,t)\psi(x')dx'. \qquad (1)$$

(a) Please find the $U(x',0;x,t)$ for a free particle.

(b) Setting $x'=x$ and integrating $U(x,0;x,t)$ over x, we would end up with $\int_{-\infty}^{\infty}U(x,0;x,t)dx=\sum_n e^{-iE_n t}$ where E_n are the eigenvalues. Insert the identity $\int_0^{\infty}\delta(E-E_n)dE=1$ inside the summation and we have:

$$\int_{-\infty}^{\infty}U(x,0;x,t)dx=\int_0^{\infty}e^{-iEt}D(E)dE \qquad (2)$$

where $D(E)=\sum_n \delta(E-E_n)$ plays the role of density of state. Equation (2) is very useful because it tells us that, without any knowledge of the potential, we can still solve for the eigenenergies by inverse-Laplace-transforming $\int_{-\infty}^{\infty}U(x,0;x,t)dx$ with respect to the imaginary time it. Please practice this property to derive the eigenvalues for the harmonic oscillator from

$$U(x',0;x,t)=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}}\exp\left\{\frac{im\omega}{2\hbar\sin\omega t}\left[\left(x'^2+x^2\right)\cos\omega t-2x'x\right]\right\}.$$

(You may need to use the relation: $\sum_{n=-\infty}^{\infty}e^{ink}=2\pi\sum_{\ell=-\infty}^{\infty}\delta(k-2\ell\pi)$)

2. (Delta-function potential, 20 points)

Find the ground state energy for attractive delta-function potentials

(a) $V(x)=-\alpha\cdot\delta(x)$

(b) $V(\vec r)=-\alpha\cdot\delta(\vec r)$ in two dimensions

(c) $V(x)=-\alpha\cdot\delta(x)-\alpha\cdot\delta(x-L)$. Draw a schematic plot to show how this energy changes with L, but there is no need to solve for its exact value except at the asymptotic cases of $L=0,\infty$.

3. (Spin, 20 points)

An electron of spin $\left|S_x=\frac{1}{2}\right\rangle$ was injected with velocity $(v,0,0)$ into $x\geq 0$ regime where there is a magnetic field $(0,0,B)$ (while $B=0$ elsewhere). What is its spin state after being bent by the field and eventually exiting the regime?

4. (Perturbation theory and variational method, 20%)

Consider a perturbation $H'=\alpha x^4$ to the oscillator problem.

(a) Show that $E_n^1=\dfrac{3\hbar^2\alpha}{4m^2\omega^2}\left[1+2n+2n^2\right]$.

(b) Argue that no matter how small α is, the perturbation expansion will break down for some large enough n. What is the physical reason?

(c) If we are careful with how the perturbation is turned on; let's say, as $H'=\alpha x^4 e^{-t^2/\tau^2}$ between $t=-\infty$ and $t=\infty$, what is the probability that the oscillator originally in the ground state ends up in the state $|n\rangle$ at $t=\infty$?

5. (Addition of angular momenta, 10 points)

Write down the spatial and spinal parts of the first-excited eigenfunctions for two noninteracting electrons in an infinite potential well, $V(x)=\begin{cases}0, & 0\leq x\leq L\\ \infty, & elsewhere\end{cases}$.

6. (Harmonic trap in 2D, 20 points)

For a two-dimensional harmonic potential, we can assume separation of variables in the polar coordinates and write the eigenfunction as $\psi(r,\theta)=\phi(r)e^{in\theta}$ where n are integers. The 2D Schrodinger equation is then simplified to 1D:

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}+\frac{n^2\hbar^2}{2mr^2}+\frac{K}{2}r^2\right]\phi(r)=E\cdot\phi(r)$$

Expand the solution in polynomials and plug it into the equation before matching the coefficients. Show that both the eigenvalues and degeneracy match what we expect from the Cartesian coordinates; namely, $E=(n_1+n_2+1)\hbar\sqrt{\dfrac{K}{m}}$ where $n_{1,2}$ are nonnegative integers.