Quantum Mechanics Qualification, Sep. 19, 2010.

You must provide the details or reasonings to justify your answers.

1. (9 +6%)[**Addition of angular momenta**]
   Work out the Clebsch-Gordan(CG) coefficients of
   (a) $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$
   (b) $1 \otimes 1 = 2 \oplus 1 \oplus 0$

2. (15%)[**time-dependant perturbation**]
   At $t = -\infty$ a particle is in its ground state in an 1D harmonic oscillator potential. At $t = 0$, a perturbation $V(x,t) = V_0 \hat{x} e^{-t/\tau}$ is turned on. Calculate to first order the probability that at $t = \infty$ the system will have made a transition to its first, second, and third excited states.

3. (6+8+6%)[**Identical Particles**]
   (a)What are Fermions, and what are Bosons? How do they differ from each other?
   (b) Write down the spatial and spinal wave function of the first excited state for two noninteracting electrons in an infinite potential well, $V(x) = 0$ for $0 \le x \le L$ and $V(x) = \infty$ elsewhere.
   (c) Similarly, write down the spatial wave function of the first excited state for two noninteracting spin-0 particles in the same potential well.

4. (10+10%)[**Quantum measurements**]
   (a) Measurement of an electron's spin along the z-axis(S_z) using a Stern-Gerlach apparatus gives the eigenvalue $-\hbar/2$. What is the probability that a subsequent measurement of the spin in the direction $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ yields $+\hbar/2$?
   (b) Measurement of an electron's spin along the axis $\hat{n}$ gives the eigenvalue $+\hbar/2$. What is the probability that a subsequent measurement of the spin along the x-axis yields $+\hbar/2$?

5. (5+10%)[**electron in a constant B-field**]
   An electron moves in a uniform magnetic field which is pointing to the z-direction ($\vec{B} = B\hat{z}$).
   (a) Evaluate
   $$\left[\hat{\Pi}_x, \hat{\Pi}_y\right],$$
   where $\hat{\Pi}_x = \hat{p}_x - e\hat{A}_x/c$, $\hat{\Pi}_y = \hat{p}_y - e\hat{A}_y/c$.
   (b) By comparing the Hamiltonian and the commutation relation obtained in (a) with those of the 1D simple harmonic oscillator problem, show that the energy eigenvalues are:
   $$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \left(\frac{|eB|\hbar}{mc}\right)\left(n + \frac{1}{2}\right)$$

where $\hbar k$ is the continuous eigenvalue of the $\hat{p}_z$ operator and n is a positive integer (including zero).

6. (10+5%) A particle whose wave function is given by

$$\psi(\vec{r}) = \left(\frac{1}{\sqrt{3}} Y_{11}(\theta,\phi) - \frac{\sqrt{7}}{\sqrt{6}} Y_{1,-1}(\theta,\phi) + \frac{1}{\sqrt{2}} Y_{10}(\theta,\phi) \right) f(r)$$

where $f(r)$ is a normalized radial function, $\int_0^\infty r^2 f^2(r) dr = 1$.
(a) What are the expectation values of $\hat{L}^2$, $\hat{L}_z$, and $\hat{L}_x^2$?
(b) What is the expectation value of an operator $\hat{V}(\theta) = 2\cos^2\theta$ in this case?

You may find the following information useful:

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta \,, \; Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$