Quantum Mechanics Qualification, March 6, 2011.

You must provide the details or reasonings to justify your answers.

1. (10+5+5%) **Three noninteracting electrons** are put in an 1-D infinite potential well, $V(x) = 0$ for $0 \leq x \leq L$ and $V(x) = \infty$ elsewhere. (1) Write down the spatial and spinal wave function of one of the ground states. (2) What is the total angular momentum of this state? (3)What is the energy and the degeneracy of the ground state?

2. (5+10%) The symbol $|j, m\rangle$ denotes the simultaneous eigenstate of $\hat{J}^2$ and $\hat{J}_z$ with eigenvalues $j(j+1)\hbar^2$ and $m\hbar$ respectively. Given a prepared state

$$|\psi\rangle = \alpha|j, m+1\rangle + \beta|j, m-1\rangle$$

where α and β are complex numbers and $|\alpha|^2 + |\beta|^2 = 1$, calculate $\langle \hat{J}_x \rangle$ and $\langle \hat{J}_y^2 \rangle$.

3. (10+10%) The Hamiltonian of a system is given by

$$H = \frac{17}{8}a^\dagger a - \frac{15}{16}\left(a^2 + (a^\dagger)^2\right) + \frac{41}{16}$$

where

$$a = \frac{1}{\sqrt{2}}(q + ip)\,, \; a^\dagger = \frac{1}{\sqrt{2}}(q - ip)\,, \text{ and } [q, p] = i.$$

(1) What are the eigen-energies of this system? (2) What is the ground state wave function? Remember to fix the normalization.

4. (10+10%) Consider a central scattering potential $V(r) = V_0/r$ for $r < a$ and $V(r) = 0$ elsewhere. Where V_0 is a constant. (1) Calculate the Born approximation scattering amplitude. (2) Use the result to evaluate the total scattering cross section in the limit of very low incident energy ($ka \ll 1$).

5. (10+5%) A system is governed by the Hamiltonian H_0. Denote the eigenstates by $|n\rangle$ with energy E_n, $n = 1, 2, 3, \cdots$, i.e., $H_0|n\rangle = E_n|n\rangle$. We assume there is no degeneracy. At $t = 0$, a potential $V(t)$ is turned on whose matrix elements $\langle n|V(t)|m\rangle$ vanish except that $\langle 2|V(t)|3\rangle = V_0 e^{+i\omega t}$ and $\langle 3|V(t)|2\rangle = V_0 e^{-i\omega t}$, where V_0 is real.
(1) Assume that at $t = 0$ the system is at state $|2\rangle$, using time-dependent perturbation theory to first order in V_0, evaluate the probability of finding the state in $|2\rangle$ at time $t > 0$. (2)Discuss the physics when the perturbation is in resonant ($\omega = E_3 - E_2$).

6. (3+7%) (1)Explain the physics of hyperfine splitting. (2)Given that the hyperfine splitting in the hydrogen atom produces 21 cm radiation, find the corresponding wavelength (in cm) for the deuterium atom (consists of one proton and one neutron) with the nucleus spin-1, $g_p = 2 \times (2.79)$, and $g_n = 2 \times (-1.91)$.