Universal dynamics of quantum spin decoherence in a nuclear spin bath

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We systematically investigate the universal spin decoherence dynamics of a localized electron in an arbitrary nuclear spin bath, which can even be far away from equilibrium due to the weak nuclear-lattice interaction. We show that the electron spin relaxation dynamics (as well as spin pure dephasing and Hahn echo decay) can *always* have a universal behavior as long as the initial state is composed of a sufficiently large amount of spin eigenstates. For a given system, the pattern of the universal dynamics depends on the complicated initial condition only via a *single* parameter, which measures the amount of phase coherence between different spin eigenstates in the initial state. Our results apply even when the number of the involved nuclei is not large and, therefore, provide a solid foundation in the comparison of the theoretical/numerical results to the experimental measurement. As an example, we also show the numerical results for systems of noninteracting spin bath in a zero magnetic field regime and discuss the features of universal decoherent dynamics.

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I. INTRODUCTION

Localized spins in solid-state systems are one of the most promising candidates for realizing the quantum computation due to its long coherence time[1] and the possible scalability.[2] Recently, quantum control of single localized spin becomes experimentally feasible[3] but the nuclear spin bath induced decoherence, which is the dominant decoherent mechanism in the low temperature regime, is still hindering further developments. To study the effects of the nuclear spin bath, both analytical approaches[4–6] and numerical simulations[7,8] are developed for different parameter regimes. The effect of dipolar interaction between the nuclear spins are also studied in Refs. 9. However, an uncontested conclusion about the spin decoherence dynamics and its relation to the experimental measurement is still unavailable even though the deleterious effects of nuclear spin have been verified in recent experiments.[10,11]

From the experimental side, the most crucial limitation results from the fact that the initial nuclear spin configuration is very little known nor controllable. This is a highly nontrivial problem because even if a thermalized spin bath is assumed in the beginning (as done in most theoretical work[5,6,9]), any quantum measurement or manipulation of electron spin can just destroy the equilibrium and lead to a highly nonequilibrium nuclear spin dynamics. The coherent time of the nuclear spin bath is known to be extremely long (can exceed 1 s in a GaAs quantum well[12] and 25 s in a ^{29}Si isotope[13]) and, therefore, it is very questionable if the nuclear spin bath could be well thermalized for the next quantum measurement/manipulation in a short time during the quantum computation process. In order to have a meaningful comparison between the theoretical results and the experimental measurement, the first and the most important question one should ask is if there could be any universal dynamics in such a system, which is insensitive to the details of the initial nuclear spin configuration.

From the theoretical side, answering the above question is also very difficult because the spin dynamics of one configuration can be very different from the other[7] even though their initial configurations are similar. Moreover, in a typical quantum dot system, the number of nuclei can be very huge ($N \sim 10^{3-5}$) and, hence, it is also a significant challenge for an ordinary numerical simulation to explore such huge phase space. These challenges are fundamentally important to the understanding of the spin decoherence mechanism and to its future application in quantum computation. However, to the best of our knowledge, there has been no systematic study in the literature on this important issue.

In this paper, we address this issue by rigorously proving the existence of a generic and universal electron spin decoherent dynamics in an arbitrary nuclear spin bath. By "universal dynamics," we mean an electron spin evolution that is of zero standard deviation over different initial conditions in the whole phase space. More precisely, we show that (1) the universality of spin decoherence *always* exists if only the initial state is composed of a sufficiently large amount of spin eigenstates and, (2) for a given system, such universal dynamics depends on the initial configuration only through a *single* parameter, which measures the amount of phase coherence between the spin eigenstates of the initial wave function. (3) The universality is ensured by the large amount of phase space rather than the large value of the nuclear number, N, and, therefore, the numerical simulation for a small size system (say $N \sim 10-20$) can still be good enough to compare to a realistic system with much more nuclei.[14] Finally, (4) the universality of spin dynamics applies to the decoherence of the diagonal part (S_z), as well as the off-diagonal part (S_x) of the electron spin, no matter if it is in a free induction decay (FID) or in a Hahn echo decay. Therefore, our results resolve the fundamental problems in the comparison of a theoretical calculation and an experimental measurement, and provide a new direction for the future study of the spin decoherence. We also study the spin dynamics for systems of different electron and/or nuclear spins and find that the spin dynamics is mainly determined by the geometric structure of the system density of states and is therefore insensitive to the magnitude of the nuclear spin.

This paper is organized as fellows: In Sec. II, we describe the system Hamiltonian and the initial wave functions in our study. In Sec. III, we show the universal dynamics of elec-

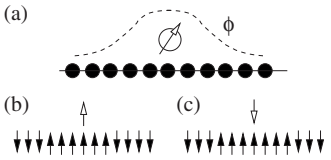

FIG. 1. (a) Schematic pictures for electron spin (white circle) coupled to a nuclear spin bath (black dots). The electron is assumed to be described by the orbital envelope wave function, ϕ, and interacts with the nuclear spins (located at $\vec{r}_i$) via a hyperfine interaction $A_i=A_0|\phi(\vec{r}_i)|^2$, where A_0 is the coupling strength. (b) and (c) are two spin eigenstates with maximum/minimum energies in zero magnetic field and $J_z=0$ case.

tron spin relaxation. We study the universal dynamics by using both numerical and analytical methods. In Sec. IV, we discuss the microscopic origin of the universality. In Sec. V, we generalize our consideration to other spin systems. We conclude in Sec. VI.

II. SPIN EIGENSTATES AND PHASE SPACE

A general spin decoherence due to nuclear spin bath is described by the following Hamiltonian:

$$\hat{H} = \hat{\mathbf{S}} \cdot \sum_{i}^{N} A_i \hat{\mathbf{I}}_i + \hat{H}_{n-n} + \hat{H}_Z, \tag{1}$$

where $\hat{\mathbf{S}}$ and $\hat{\mathbf{I}}_i$ are the dimensionless spin operators ($\hbar \equiv 1$) of the localized electron and the nucleus at lattice site i, respectively. A_i is the hyperfine coupling strength, depending on the wave function profile of the localized electron [Fig. 1(a)], and we use $A_i=A_0e^{-(3i/N)^2}$ for our numerical calculation with N being the number of nuclei. We note that increasing N will not change the shape of the electron wave function but just to reduce the average distance between the nuclei. Therefore, as we will show below that, the effect of a larger number of nuclei is just to reduce the standard deviation of the spin dynamics instead of its average value. Here, $\hat{H}_{n-n}$ and $\hat{H}_Z$ are the interaction between the nuclear spin and the Zeeman term due to an external magnetic field, respectively. Even in the simplest case where both $\hat{H}_Z$ and $\hat{H}_{n-n}$ are zero or neglected, the resulting dynamics due to the electron–nuclear coupling is only still quite complex because it involves a huge amount of eigenstates in the Hilbert space. In order to have a meaningful comparison between the theoretical (numerical) results and the experimental observation, the first question one should ask is if there could be any universal dynamics in such a spin system, which is insensitive to a general initial condition of the system and, therefore, can be observed and is repeatable in a realistic experiment. After all, it is very difficult to control and/or manipulate the spin configuration of nuclei in solid-state systems. This question, to the best of our knowledge, has not been answered or even not addressed yet in the literature.

It is convenient to use spin eigenstate, $|\mathbf{S}\rangle_e \otimes |j\rangle_n \equiv |\mathbf{S}\rangle_e \otimes |\{I_{1,z}, I_{2,z}, \cdots, I_{N,z}\}_j\rangle_n$, as the basis of calculation, where $|\mathbf{S}\rangle_e$ is electron spin eigenstate along a certain direction (which will be specified below) and $\hat{I}_{i,z}$ is the nuclear spin eigenvalue along the direction of magnetic field ($\hat{z}$) at the ith site. For simplicity, in this paper, we assume the electron spin is initially polarized only along the z or x axis and, therefore, a general initial wave function can be written to be $|\psi_0\rangle_{x,z} = |S_{x,z}\rangle_e \otimes \Sigma_{j=1}^{M_\Omega} a_j |j\rangle_n$, where $a_j=r_je^{i\varphi_j}$ is the coefficient of the jth spin eigenstate with phase φ_j and amplitude r_j. Here, $|S_z\rangle_e \equiv |+\rangle_e$ and $|S_x\rangle_e \equiv \frac{1}{\sqrt{2}}(|+\rangle_e + |-\rangle_e)$, and M_Ω is the size of the Hilbert space of the nuclear bath. Using the above expression, we consider the following three spin dynamics, which are related to the spin relaxation, spin pure dephasing, and Hahn echo decay, respectively. The first two can be expressed as

$$\langle S_{z,x}(t)\rangle \equiv {}_{z,x}\langle\psi_0|\hat{S}_{z,x}(t)|\psi_0\rangle_{z,x} = \sum_{j=1}^{M_\Omega} r_j^2 S_{j,j}^{z,x}(t) + \sum_{j\neq l}^{M_\Omega} r_j r_l e^{-i(\varphi_j-\varphi_l)} S_{j,l}^{z,x}(t),$$

where $S_{j,l}^{z,x}(t) \equiv {}_n\langle j|\otimes {}_e\langle S_{z,x}|\hat{S}_{z,x}(t)|S_{z,x}\rangle_e \otimes |l\rangle_n$ is the matrix element. $\langle S_z(t)\rangle$ can be very different from $\langle S_x(t)\rangle$ if the nuclear spin is polarized by an external magnetic field or with a finite total angular momentum in a certain direction. Similarly, the Hahn echo decay is given by $\rho_{+-}^H(\tau) \equiv {}_e\langle +|\hat{\rho}_H(\tau)|-\rangle_e$, where the Hahn echo density matrix

$$\hat{\rho}_H(\tau) \equiv \mathrm{Tr}_n\{U(\tau)|\psi_0\rangle_{xx}\langle\psi_0|U(\tau)^\dagger\} = \sum_{j\ n}\langle j|U(\tau)|\psi_0\rangle_{xx}\langle\psi_0|U(\tau)^\dagger|j\rangle_n$$

and $U(\tau) \equiv e^{-iH\tau}\sigma_x e^{-iH\tau}$.[9,15] The characteristic time scale T_2 of pure dephasing is related to the single spin FID while Hahn echo decay[15] is usually used to extract single spin behavior from an ensemble measurement.

III. UNIVERSAL DYNAMICS

In this section, we show that the universality of spin decoherence always exists if only the initial state is composed of a sufficiently large amount of spin eigenstates and, for a given system, such universal dynamics depends on the initial configuration only through a single parameter, which measures the amount of phase coherence between the spin eigenstates of the initial wave function. We first show the numerical results for spin relaxation, spin pure dephasing, and Hahn echo decay, respectively. Then we rigorously give the proof of the universality.

A. Numerical study

In order to explore the spin dynamics from different initial conditions in the whole phase space, in this paper, we allow both the amplitude, $\{r_j\}$, and the phase, $\{\varphi_j\}$, to be independent variables, and be randomly chosen according to the distribution functions $\mathcal{P}_r(r_j)$ and $\mathcal{P}_\varphi(\varphi_j)$, respectively. The ensemble-averaged spin dynamics for $\langle S_z(t)\rangle$ becomes $[\langle S_z(t)\rangle] \equiv \frac{[\langle S_z(t)\rangle]_{r,\varphi}}{[\langle S_z(0)\rangle]_{r,\varphi}}$, where $[f(r)]_r \equiv \int_0^1 \mathcal{P}_r(r) f(r) dr$ denotes the average of a function $f(r)$ and, similarly, $[f(\varphi)]_\varphi \equiv \int_0^{2\pi} \mathcal{P}_\varphi(\varphi) f(\varphi) d\varphi$. $[\langle S_z(0)\rangle]_{r,\varphi}$ in the denominator is for normalization. At the same time, the associated normalized standard deviation (NSD) is defined as follows:

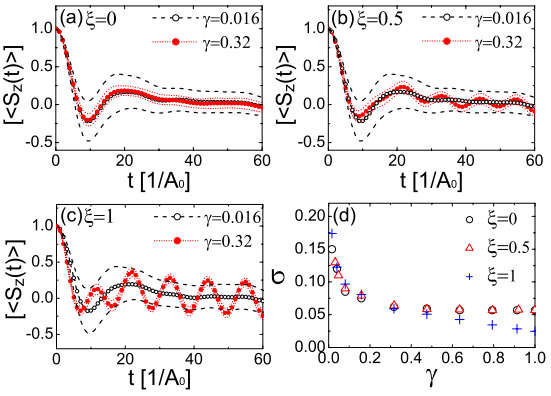

FIG. 2. (Color online) [(a)–(c)] Averaged electron spin relaxation (symboled lines) for $\xi=0$, 0.5, and 1, respectively. Dashed and dotted lines are the uncertain range $\{[\langle S(t)\rangle] \pm \sigma(t)\}$ for $\gamma=0.016$ and 0.32, respectively. (d) Time averaged NSD $[\sigma \equiv \lim_{T\to\infty} \int_0^T \sigma(t)dt/T]$ vs γ. $N=11$ in all figures.

$$\sigma(t) \equiv \sqrt{\frac{[\langle \hat{S}_z(t)\rangle^2]_{r,\varphi} - [\langle \hat{S}_z(t)\rangle]^2_{r,\varphi}}{[\langle S_z(0)\rangle]^2_{r,\varphi}}}. \tag{2}$$

Similar definition of averaged dynamics, as well as the NSD for $\langle S_x(t)\rangle$ and $\rho^H_{+-}(t)$, can be obtained easily. We note that, if the NSD of the averaged spin dynamics goes to zero in the limit of infinite phase space, the averaged dynamics is also "the most probable" dynamics with an almost zero probability in the other time-evolution behavior. As a result, we can define it as a universal dynamics of the given system, which is independent (in the probability sense) of the details of the initial nuclear spin configuration. On the other hand, the system has no universal dynamics if the NSD is of the order of one since the average value could not represent the characteristic dynamics of a general initial condition.

Before analytically studying the universality of spin dynamics in a general system, it is more instructive to show some numerical results of the simplest system without the magnetic field and nuclear spin interaction ($\hat{H}_Z=\hat{H}_{n-n}=0$). We will first present the result for $\langle S_z(t)\rangle$ then the results for $\langle S_x(t)\rangle$ and $\rho^H_{+-}(t)$. For the convenience of later discussion, we restrict the calculation inside a subspace Γ where the total angular momentum $J_z=S_z+\Sigma_{i=1}^N I_{i,z}$ is zero and choose $\mathcal{P}_r(r)=\gamma+(1-\gamma)\delta(r)$ and $\mathcal{P}_\varphi(\varphi)=(1-\xi)/2\pi+\xi\delta(\varphi)$. Here, $\gamma\in[0,1]$ can be understood as the probability to have a nonzero contribution in the subspace Γ while $\xi\in[0,1]$ is the probability to have a phase coherence at a given value (set to be zero). They satisfy the normalization condition: $\int_0^1 \mathcal{P}_r(r)dr=\int_0^{2\pi} P_\varphi(\varphi)d\varphi=1$ for all ξ and γ.

In Figs. 2(a)–2(c), we show the averaged electron spin $(S=\frac{1}{2})$ relaxation $[\langle S_z(t)\rangle]$ in a noninteracting spin bath $(I=\frac{1}{2})$ with zero magnetic field for $\xi=0$, 0.5, and 1 with two different values of γ. We observe that when γ is small $(\gamma=0.016)$, meaning only a few spin eigenstates are involved in $|\psi_0\rangle_z$, the NSDs are very large, i.e., no universal dynamics. This explains why in the literature, different initial states can result in very different time-evolution patterns.[7] When γ becomes larger $(\gamma=0.32)$, the NSD decreases in all figures [see also Fig. 2(d)] while the averaged dynamics also begins to show different behaviors for different values of ξ: two-decay curves for $\xi\to 0$ and a single mode oscillation for $\xi\to 1$. In fact, as we will show later, the NSD always decreases to zero even when γ is finite as long as the size of the phase space becomes large enough. These results indicate that universal dynamics can always be expected if the initial state is composed of a sufficiently large portion of spin eigenstates in the phase space, simply due to the strong quantum interference effects. We could also show that the two-decay time scale of Fig. 2(a) is due to the structure of the system density of states and will discuss that in more details in the latter section.

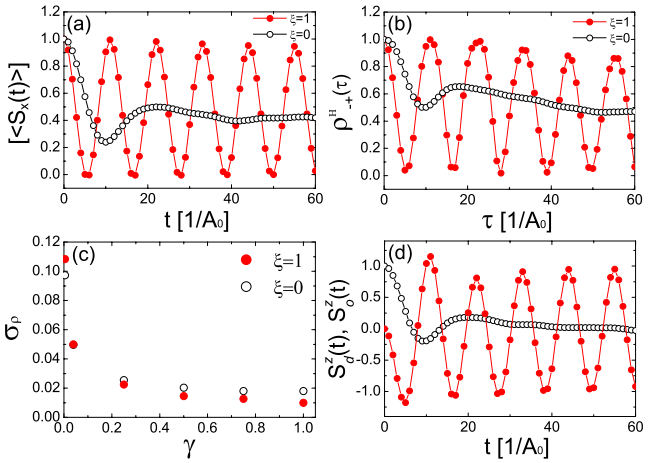

FIG. 3. (Color online) (a) $[\langle S_x(t)\rangle]$ and (b) $\rho^H_{+-}(\tau)$ as a function of time t and τ, respectively (see text). Lines with filled and open symbols are for $\xi=1$ and $\xi=0$. (c) The NSD at $\tau=10$ for the Hahn echo decay. (d) $S^z_d(t)$ (open circle) and $S^z_o(t)$ (filled circle) for the spin relaxation dynamics in Eq. (3).

In Figs. 3(a) and 3(b), we respectively plot pure dephasing $[\langle S_x(t)\rangle]$ and Hahn echo decay $[\rho^H_{+-}(\tau)]$ as a function of time for fully coherent ($\xi=1$, filled circle) and fully incoherent ($\xi=0$, open circle) choices of the initial condition. To simplify the numerical calculation, we choose an initial wave function in a subspace of $\Sigma_{i=1}^N I_{I,z}=0$ for the pure dephasing $[\langle S_x(t)\rangle$, see Fig. 3(a)], and $J_z=0$ [same as $\langle S_z(t)\rangle$] for the Hahn echo decay [$\rho^H_{+-}(\tau)$, see Fig. 3(b)]. The electron for the former case is initially polarized in the x direction so that the average total angular momentum in z direction, $\langle S_z(t)\rangle$, is still zero. As a result, the dynamics of dephasing, $\langle S_x(t)\rangle$, is different from the relaxation, $\langle S_z(t)\rangle$ due to the different choices of subspace where the initial wave function is defined. We believe such convection is justified and will not affect any of our conclusion, because here we just used this numerical results as an example to understand the general properties of the universal dynamics. Full numerical results for any realistic situation will need a much larger phase space and much longer time. Within this subspace, different initial wave functions still result in different dynamics (not shown here). However, when the initial wave function is composed of sufficiently large amounts of eigenstates in the subspace, we again find a universal dynamics with almost zero NSD. In Figs. 3(a) and 3(b), we show the results for $\gamma=1$ for pure dephasing and Hahn echo decay in the two subspace described above. In Fig. 3(c), the NSD of the Hahn

echo decay is plotted as a function of γ at $\tau=10$. From these results, we find similar single mode oscillation for $\xi=1$ while a two-decay curves for $\xi=0$ in all the three dynamics $[\langle S_z(t)\rangle, \langle S_x(t)\rangle$, and $\rho_{+-}^{H}(\tau)]$.

B. Analytical study

To analytically study the universal spin dynamics, we have to do the ensemble-average first so that

$$[\langle S_z(t)\rangle] = S_d(t) + \frac{[r]_r^2}{[r^2]_r}|[e^{-i\varphi}]_\varphi|^2 S_o(t), \tag{3}$$

where we have used $[\langle S_z(0)\rangle]_{r,\varphi}=[r^2]_r\Sigma_j\langle j|\hat{S}_z|j\rangle=[r^2]_r M_\Gamma$ in the normalization; $S_d(t)\equiv M_\Gamma^{-1}\Sigma_j S_{j,j}(t)$ and $S_o(t)\equiv M_\Gamma^{-1}\Sigma_{j_1\neq j_2}S_{j_1,j_2}(t)$ are the diagonal and off-diagonal matrix elements of the electron spin. We note that Eq. (3) indicates that the averaged spin dynamics depends on the initial condition only via a *single* parameter, $\beta\equiv([r]_r^2/[r^2]_r)|[e^{-i\varphi}]_\varphi|^2$, which depends on the phase distribution function, $\mathcal{P}_\varphi$, much more significantly than on the amplitude distribution function, $\mathcal{P}_r$, since $[r]_r^2\sim[r^2]_r$ for the usual function of $\mathcal{P}_r$ and $r\geq 0$. We note that although the experimental preparation of a coherent nuclear spin bath (i.e., finite value of β) is not easy at the present stage, it has been realized how to control the coherent electron spin dynamics via interaction with a *single* nuclear spin in a diamond.[11] Therefore, at least in a small quantum dot system, a coherent preparation and control of a few nuclear spins can still be realized. In Fig. 3(d), we show the time evolution of both $S_d(t)$ and $S_o(t)$ of the spin matrix element. Not surprisingly, they are of very different properties: the diagonal part, $S_d(t)$, shows a clear two-decay process with a fast decay in a short time and a slow decay in a long time. However, the off-diagonal part, $S_o(t)$, does not decay at all and shows a single mode oscillation. It is easy to see that the numerical results shown in Figs. 2(a)–2(c) can be obtained as a superposition of $S_d(t)$ and $S_o(t)$, just as suggested by Eq. (3). The numerical comparison between these two approaches (ensemble average before and after the time evolution) agree excellently well (not shown here), showing that only a single parameter, β, is necessary to reproduce all the ensemble-averaged spin relaxation dynamics.

In order to examine if the ensemble-averaged results of Eq. (3) is a universal dynamics, we need to calculate the fluctuation [NSD, Eq. (2)] of this average. For simplicity, we first study the case in the completely random phase limit, say $[e^{i\varphi}]_\varphi=0$. We then have $[\langle\hat{S}_z(t)\rangle]_{r,\varphi}^2=M_\Gamma^2[r^2]^2S_d(t)^2$ according to Eq. (3). After some algebra, we can derive:

$$[\langle\hat{S}_z(t)\rangle^2]_{r,\varphi}-[\langle\hat{S}_z(t)\rangle]_{r,\varphi}^2 = ([r^4]_r-2[r^2]_r^2)\sum_j S_{j,j}(t)^2+[r^2]_r^2\sum_{j,l}S_{j,l}S_{l,j}$$

. Since we are interested in the upper bound of the NSD, we may use

$$\begin{aligned}\sum_{j,l}S_{j,l}(t)S_{l,j}(t) &= \sum_j\langle j|\hat{S}_z(t)\hat{P}_\Gamma\hat{S}_z(t)|j\rangle\\ &\leq \sum_j\langle j|\hat{S}_z(t)^2|j\rangle\\ &\leq S(S+1)\sum_j\langle j|j\rangle\\ &= S(S+1)M_\Gamma,\end{aligned}$$

where $\hat{P}_\Gamma\equiv\Sigma_l|l\rangle|l\rangle$ is to project a state onto the subspace Γ with electron spin eigenvalue $S_z=S$. Here, we have used the fact that for any state, the expectation value of $\hat{S}_z(t)^2$ must be equal to or smaller than the expectation value of the total electron spin $\mathbf{S}^2$, which is, however, a conserved quantity of our system [see Eq. (1)]. Similarly, we also have $\Sigma_j S_{j,j}(t)^2\leq\Sigma_{i,j}|S_{i,j}(t)|^2\leq S(S+1)M_\Gamma$ and, therefore, $[\langle\hat{S}_z(t)\rangle^2]_{r,\varphi}-[\langle\hat{S}_z(t)\rangle]_{r,\varphi}^2\sim\mathcal{O}(M_\Gamma)$. In other words, after being divided by $[\langle S_z(0)\rangle]_{r,\varphi}^2\propto M_\Gamma^2$, we find $\sigma(t)\propto M_\Gamma^{-1/2}$ and, therefore, goes to zero in the limit of $N\gg 1$ or $M_\Gamma\to\infty$.

We can also apply a similar method to study the NSD of a phase coherent initial state, i.e. $[e^{i\varphi}]_\varphi\neq 0$. After some algebra, the expansion of $[\langle\hat{S}_z(t)\rangle^2]_{r,\varphi}-[\langle\hat{S}_z(t)\rangle]_{r,\varphi}^2$ will have two additional summations (besides those of the two shown above) with nonuniversal prefactors: first, we have

$$\begin{aligned}\sum_{i,j,l}S_{i,l}(t)S_{l,j}(t) &= \langle V|\hat{S}_z(t)\hat{P}_\Gamma\hat{S}_z(t)|V\rangle\\ &\leq \langle V|\hat{S}_z(t)\hat{S}_z(t)|V\rangle\\ &\leq S(S+1)\langle V|V\rangle\\ &= S(S+1)\sum_{i,j}\langle i|j\rangle = S(S+1)M_\Gamma,\end{aligned}$$

where we define $|V\rangle\equiv\Sigma_j|j\rangle$ as an auxiliary state. Secondly,

$$\begin{aligned}\sum_{l,j}S_{l,l}(t)S_{l,j}(t) &\leq \sqrt{\sum_l S_{l,l}(t)^2\sum_l|\sum_j S_{l,j}(t)|^2}\\ &\leq M_\Gamma^{1/2}\sqrt{\langle V|\hat{S}_z(t)\hat{P}_\Gamma\hat{S}_z(t)|V\rangle}\\ &\leq M_\Gamma^{1/2}\sqrt{\langle V|\hat{S}_z(t)^2|V\rangle}\leq\sqrt{S(S+1)}M_\Gamma,\end{aligned}$$

where we have used the fact that the inner product of two vectors must be equal to or smaller than the product of their length. Therefore, after being normalized by the initial spin average, we find $\sigma(t)\propto M_\Gamma^{-1}$ and becomes zero in a large system size just like for the complete random phase case ($\xi=0$). From the above results, we conclude that no matter how much phase coherence there are between the spin eigenstates of the initial wave function, the spin relaxation dynamics can always be universal (with zero NSD) in the limit of an infinitely large phase space ($M_\Gamma\gg 1$). Similar derivation for the dynamics of pure dephasing $[\langle S_x(t)\rangle]$ and Hahn echo decay $[\rho_{+-}^{H}(\tau)]$ can be obtained straightforwardly.

IV. MICROSCOPIC ORIGIN OF UNIVERSALITY

After concluding the universality of the most general spin relaxation system [Eq. (1)], in the rest of this paper, we return to a less general case, i.e., the zero magnetic field and noninteracting spin bath ($\hat{H}_Z=\hat{H}_{n-n}=0$), to study the micro-

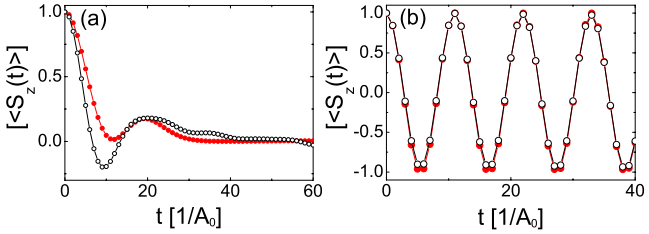

FIG. 4. (Color online) (a) Comparison between the calculated $\langle S_z(t)\rangle$ (open circle) with $\xi=0$ and $\gamma=1$ and $\langle S_z(t)\rangle_{\mathrm{DOS}}$ (filled circle) obtained from the density of states only. (b) The comparison between the calculated $\langle S_z(t)\rangle$ with $\xi=1$ and $\gamma=1$, and the result obtained by including $|E_{\mathrm{max/min}}\rangle$ only (see text).

scopic origin of the universal spin relaxation curves shown in Figs. 2(a)–2(c). We first rewrite $\langle \hat{S}_z(t)\rangle$ in terms of the energy eigenstates, $|E\rangle$:

$$\langle \hat{S}_s(t)\rangle = \int dE\mathcal{D}(E)\int dE'\mathcal{D}(E')C_E C^*_{E'} \times \langle E|\hat{S}_z|E'\rangle e^{-i(E-E')t}, \tag{4}$$

where $\mathcal{D}(E)$ is the density of states (DOS) of the system and $C_E \equiv \langle \psi_0|E\rangle$. According to our numerical calculation, we observe that the matrix element, $\langle E|\hat{S}_z|E'\rangle$, varies almost randomly and is structureless for different energies, and, therefore, it cannot be the origin of the smooth universal dynamics we calculated above. Since we have proved the existence of a universal electron spin dynamics and now we are interested in the simplest possible explanation for the features of such dynamics, we can first neglect such structureless random matrix element for simplicity. As we will see later, it turns out that this simplification does bring a very useful understanding of the universal spin dynamics.

For the case when the initial wave function, $|\psi_0\rangle$, is totally randomly distributed in the phase space Γ, i.e., $\gamma=1$ and $\xi=0$, we can further assume that $\langle \psi_0|E\rangle$ is also independent of the energy E in the above equation. As a result, Eq. (4) can be approximated by $\langle S_z(t)\rangle_{\mathrm{DOS}} \equiv |\int dE\mathcal{D}(E)e^{-iEt}|^2$, which is just the power spectrum of the density of states. In Fig. 4(a), we show the full numerical result of $\langle S_z(t)\rangle$ for $\xi=0$, compared to $\langle S_z(t)\rangle_{\mathrm{DOS}}$ given above. One can see that the latter can qualitatively reproduce all the important structure of the full numerical results. This agreement helps us to conclude that the decay time of $\langle S_z(t)\rangle$ is mainly determined by the width of the DOS peaks [see Fig. 5(a)] while the time scale of the second peak of $\langle S_z(t)\rangle$ is given by the energy separation between the two peaks in DOS.

As for the single mode oscillation shown in Fig. 2(c) for the full spin coherent initial state ($\xi=1$), we can apply a similar study but notice that the coefficient C_E is not a constant for all the eigenstate energy any longer. Our numerical calculation shows that $|C_E|$ is very small ($\propto M_\Gamma^{-1/2}$) for all energy except near $E=E_{\mathrm{max/min}}$, where $|E_{\mathrm{max/min}}\rangle$ is the eigenstates of the top (maximum) and bottom (minimum) of the eigenenergy band. This is because $|E_{\mathrm{max/min}}\rangle$ has a large overlap with some particular spin eigenstate [as shown in Figs. 1(a) and 1(b)] and the overlapping coefficients are not cancelled out due to the same phase in a full spin coherent initial state ($\xi=1$). In Fig. 4(b), we compare the numerical result of the universal dynamics ($\gamma=1$) of a full coherent initial state ($\xi=1$) and the result calculated by using $|E_{\mathrm{max/min}}\rangle$ only (with proper normalization). We find the agreement is excellent, predicting the same oscillation frequency and even the same phase. The agreement justifies the approximations used in the derivations after Eq. (4) and also shows that the universal behavior of the spin relaxation dynamics can be simply explained by the structure of the density of states and the two special spin configurations, as shown in Figs. 1(a) and 1(b). As for the results with $0<\xi<1$, it can be also explained well by a linear combination of the above two results, as suggested by Eq. (3).

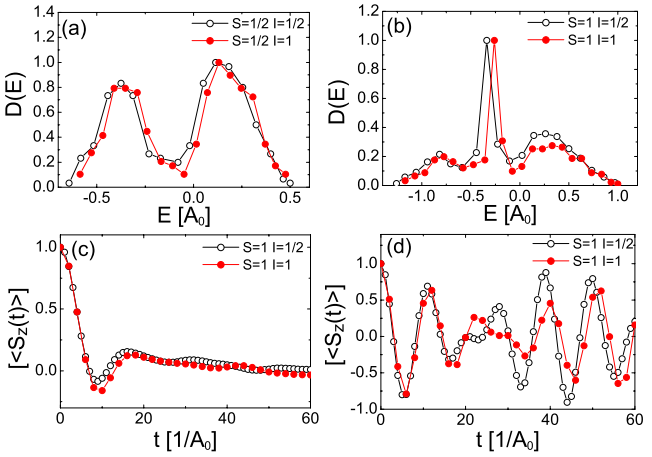

FIG. 5. (Color online) (a) and (b) are the density of states of the systems with different electron/nuclear spins. Note that in both figures, the energy axis for $I=1$ cases (filled circle) have been rescaled by a factor 1/2 in order to fit the same scale as the $I=1/2$ case. (c) and (d) are ensemble-averaged spin relaxation curves for $S=1$ and $\gamma=1$ case with $\xi=0$ and $\xi=1$, respectively. For comparison, results of different nuclear spins are shown together after rescaling the time axis (see above).

V. RESULTS FOR DIFFERENT SPINS

After systematically investigating the spin relaxation dynamics for a spin-half electron inside a spin-half nuclear spin bath, here we further extend the study of universality to systems of different electron/nuclear spins. In Figs. 5(a) and 5(b), we show the density of states for $(S,I)=(\frac{1}{2},\frac{1}{2})$, $(S,I)=(\frac{1}{2},1)$, $(S,I)=(1,\frac{1}{2})$, and $(S,I)=(1,1)$ in different curves. For the convenience of comparison, we rescale the energy scale in each plot and normalize the height of DOS by the total size of phase space Γ. Surprisingly, we find that the DOS structure is almost the same for different nuclear spins I as long as the electron spin S is the same. This reflects the fact that the total Hilbert space of the nuclear spin bath has been large enough due to the number of nuclei so that the spin degrees of freedom does play very little role in the structure of the energy spectrum. Analyzing the energy eigenstate configuration, we find the spin configuration near the degeneracy regime (position of the peaks) are polarized,

either parallel or anti-parallel, to the electron spin. Similar observation also applies to the triple peak structure in Fig. 5(b) for $S=1$: In Figs. 5(c) and 5(d), we show the spin relaxation curves for $S=1$ with the spin phase random ($\xi=0$) and the spin phase coherent ($\xi=1$) initial wave functions, respectively, after properly rescaling the horizontal axis. One can see that the results in Fig. 5(c) are very similar to the spin-half case [Fig. 2(a)] while it shows a beating oscillation for a coherent initial wave function for Fig. 5(d). The rescaled time evolution for $I=\frac{1}{2}$ and $I=1$ are very similar except for a small phase twist. We then conclude that the spin relaxation dynamics is insensitive to the nuclear spin degrees of freedom, which is consistent with our earlier statement that the universal spin dynamics is independent of the nuclear spin configuration. Our results for $S=I=1$ can also be applied to the study of spin dynamics in the mixtures of spinful cold atoms in an all-optical trap, where the localized "electron" and the "nuclei" can be prepared easily by using an optical lattice with the proper wavelength difference. The advantage of the cold atom system is that the initial spin configuration can be prepared easily and the coupling strength, A_0, can be tuned via optical Feshbach resonance and/or other method.

VI. CONCLUSIONS

In this paper, we rigorously prove that the electron spin decoherence due to the nuclear spin bath can always be universal if only coupled by a sufficiently large amount of spin eigenstates. There are several features about the universal dynamics that we want to emphasize: First, in the derivation above, we do not rely on any particular form of the distribution function ($\mathcal{P}_r$ and $\mathcal{P}_\varphi$), hence, the universality of spin dynamics is independent of the nuclear spin configuration. However, if the initial state is composed of only finite numbers of spin eigenstates (as done in the literature), our derivation will fail since $[r^2]_r \to 0$ in the denominator of $\sigma(t)$, i.e., no universal dynamics can be expected. Second, the universality does not rely on any particular Hamiltonian so our conclusion also applies to systems that include nuclear spin interaction, finite magnetic field, or any other more complicated system. Different system Hamiltonians just bring different averaged results of spin dynamics but the huge phase space (*not* necessarily the huge nuclear number) can *always* ensure it to be the most probable one regardless of the details in the initial condition. Such important results lead to another conclusion that a numerical simulation of a much smaller system (say, $N \sim 10-20$) can still have large enough phase space $[M_\Omega=(2S+1)\times(2I+1)^N \sim 10^{3\sim 6}]$ and, hence, gives similar results, as given by macroscopic number of nuclei.[14] The excellent agreement between our results of small size calculation [Fig. 2(a) with $N=11$] and a meanfield type calculation of a much larger system (for example, Fig. 4 of Ref. 8 with $N=2000$) ensures the existence of such size-independent universal dynamics. Our results, therefore, make it possible to have a realistic comparison between the theoretical calculation and the experimental data, leaving only a single unknown parameter β as a fitting parameter. (For example, in the Fig. 4 of Ref. 8, $\beta=0$ is expected due to the thermalized initial bath.) Finally, our derivation relies on the fact that the electron spin is a conserved quantity with an upper-bounded eigenvalue (not scaled with system size). This may explain why spin eigenstate can be a special basis for studying universal physics and restricts a naive application of our results to the relaxation dynamics of other physical quantities.

It is also worthy to note that the universal dynamics may bear a close relationship with the quantum central limit theorem (QCLT). QCLT has been used to study the quantum state estimation without using a large ensemble,[16] and to explain why quantum and classical random walks possess different behaviors.[17] It is natural to conjecture that the existence of the universal dynamics and the reason why a small size system can already capture the behavior of the macroscopic system, can be understood in the context of QCLT. For example, in Ref. 16, it was pointed out that a quantum state estimation with small error using a small size ensemble is possible. This is clearly resembling our work, where a small size system can capture the universal dynamics of a macroscopic system. However, such a connection is not at all transparent in the details of the theory due to the different language and models of interest. Further study about such an interesting connection shall be very important for future investigation but, unfortunately, is beyond the scope of this paper.

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