The Thermodynamic Properties of Bose-Einstein Condensation

玻色-愛因斯坦凝聚態的統計特性

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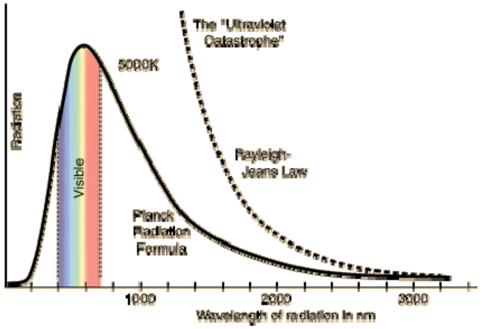
2009_1201 於清華物理系

Outline :

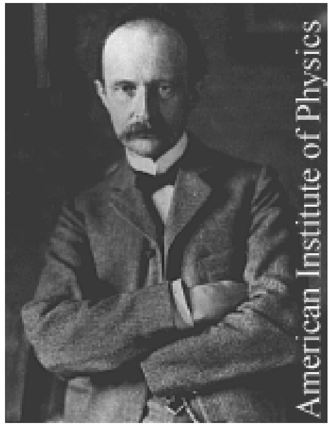
- Bose-Einstein Condensate 簡介
- Thermodynamic Limit
- Weakly Interacting
- Phase Transition

(1901) Black Body Radiation



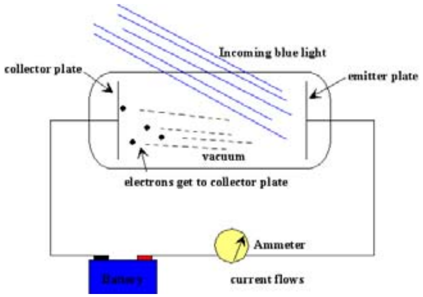
$$\langle E \rangle = \frac{h\nu}{e^{h\nu/kT} - 1}$$

$$E_{quanta} = nh\nu, n = 0, 1, 2, 3, \cdots$$

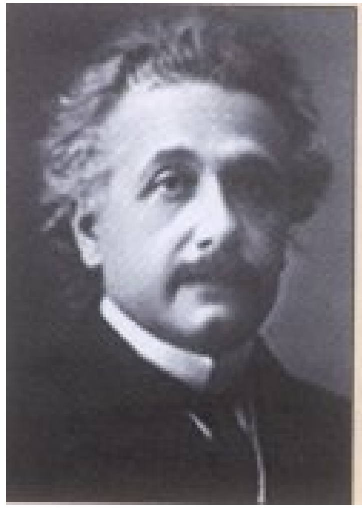


Max Planck (1858-1947)

(1905)The Photoelectric Effect

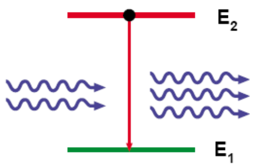


$$E_{photon} = h\nu$$



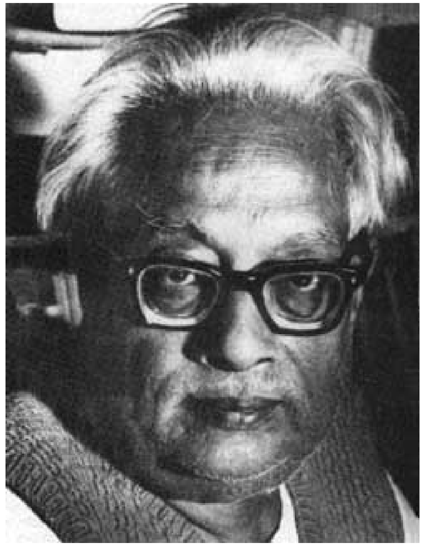
Albert Einstein(1879~1955)

(1917) stimulated emission



$$\langle W_T(\omega) \rangle = \frac{A_{21}}{\left(\frac{N_1}{N_2}\right) B_{12} - B_{21}} = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1}$$

The Bose-Einstein Distribution



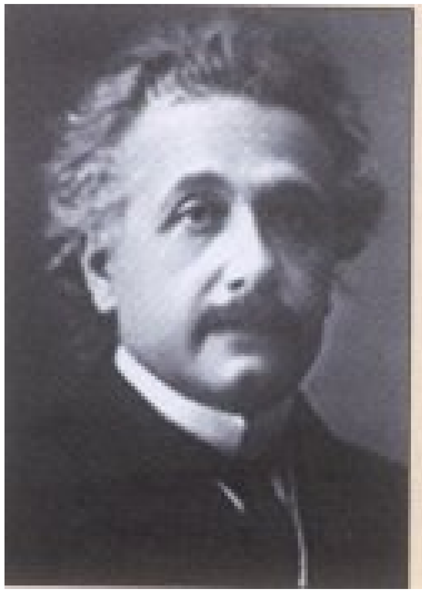
Satyendra N. Bose, (1894~1974)

* Identical particle *

For photon (Bose,1924) : $$f(E) = \frac{1}{e^{E/kT} - 1}$$

For atoms (Einstein,1925) : $$f(E) = \frac{1}{e^{(E-\mu)/kT} - 1}$$

μ is a Lagrangian multiplier for the conservation of particle number



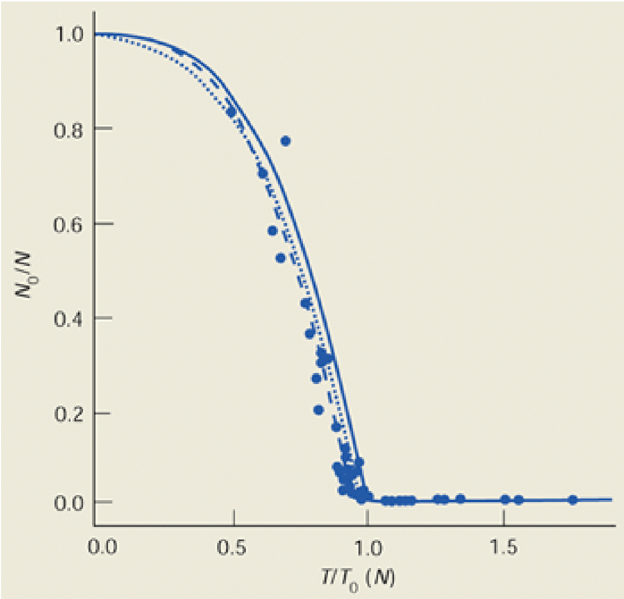
Albert Einstein(1879~1955)

Einstein considered *N* non-interacting bosonic and non-relativistic particles in a cubic box of volume L^3 with periodic boundary conditions.
In the *thermodynamic limit*, defined as

$$N, L \to \infty \quad \text{with} \quad N/L^3 = \rho = \text{constant},$$

$$\frac{N}{V} = \frac{g_{3/2}(z)}{\lambda^3} \quad , \quad z = e^{\beta\mu}, \quad g_\eta(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^\eta}, \quad \lambda = \sqrt{\frac{2\pi\hbar^2}{mk_B T}}$$

Bose–Einstein condensate



a phase transition occurs at a temperature T_c defined by:

$$\rho\lambda_{dB}^{\ 3} = \zeta\left(\frac{3}{2}\right) \cong 2.612, \quad \zeta(\eta) = \sum_{j=1}^{\infty} \frac{1}{j^{\eta}}$$

$$T > T_c \qquad \frac{N_0}{N} \to 0$$

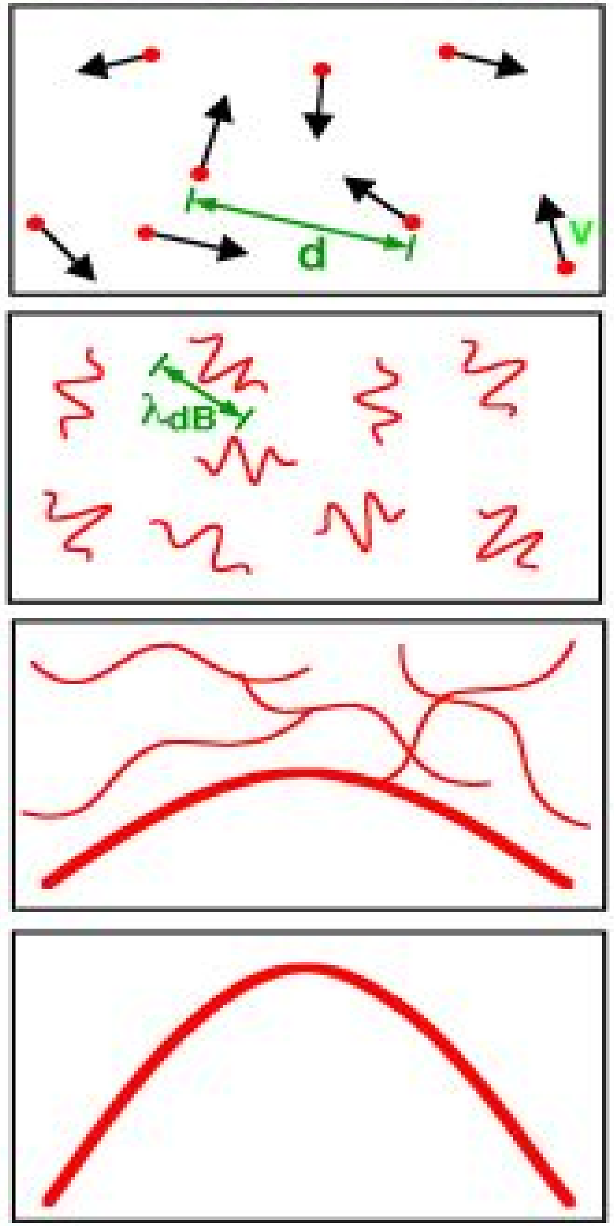
$$T < T_c \qquad \frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

For *T* < *T*$_c$ *the system has formed a Bose–Einstein condensate in p = 0.*

The number *N*$_0$ *of particles in the condensate is on the order of N, that is* macroscopic.

As we will see, the *macroscopic population of a single quantum state* *is the key feature of a Bose–Einstein condensate, and gives rise to* interesting properties, *e.g. coherence (as for the laser).*

What is Bose-Einstein condensation (BEC)?



High Temperature T:

thermal velocity v

density d^{-3}

"Billiard balls"

Low Temperature T:

De Broglie wavelength

$\lambda_{dB}=h/mv \propto T^{-1/2}$

"Wave packets"

$T=T_{crit}$: Bose-Einstein Condensation

$\lambda_{dB} \approx d$

"Matter wave overlap"

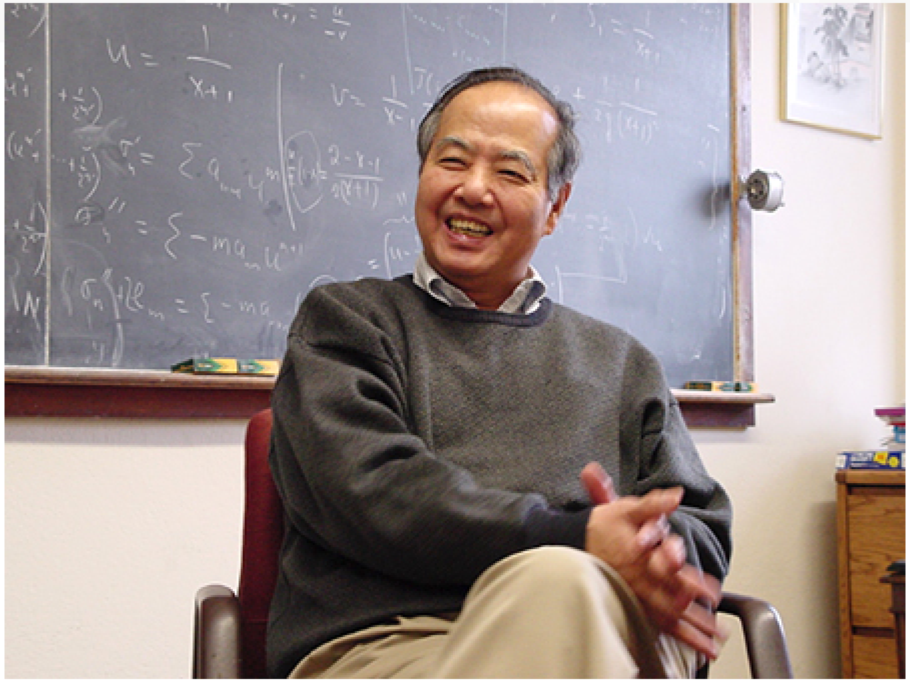
T=0: Pure Bose condensate

"Giant matter wave"

Wolfgang Ketterle

迄今在地球上找到的超流液體只有He^4 II 和He^3 II，人們一定會問，H^1 是玻色子，何以不能發生玻-愛凝聚現象呢？這是由於氫原子之間的相互作用太強，以致遠在凝聚條件達到以前，它已經成了固體，完全失去自由玻色子的特性了，否則一定會存在第三種超流液體。

李政道 **1979** 統計力學
p.57 凡異出版社

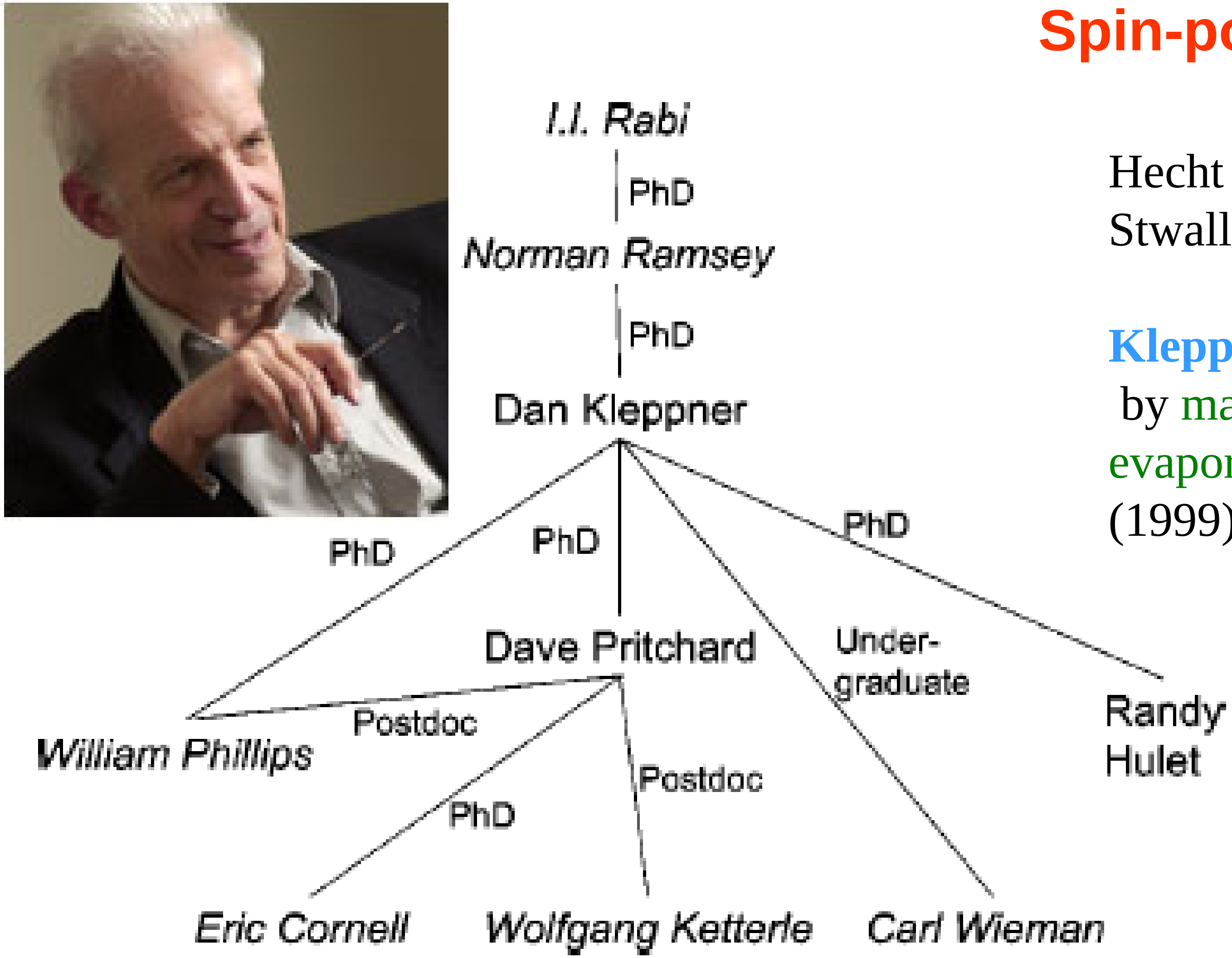


李政道**(1926~)**

(1957)

Family tree of atomic physicists.

People with names in italics are Nobel laureates.



Spin-polarized Hydrogen

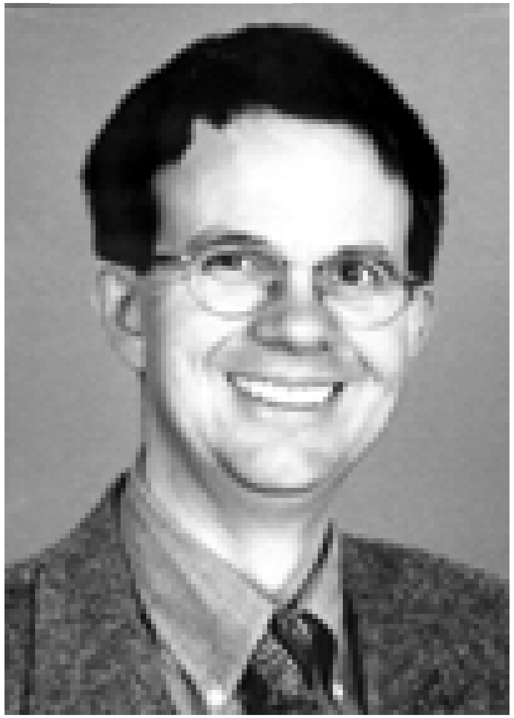
Hecht (1959)
Stwalley and Nosanow (1976)

Kleppner at MIT, since 1985,
by magnetic trapping and
evaporative cooling.
(1999)Hydrogen BEC

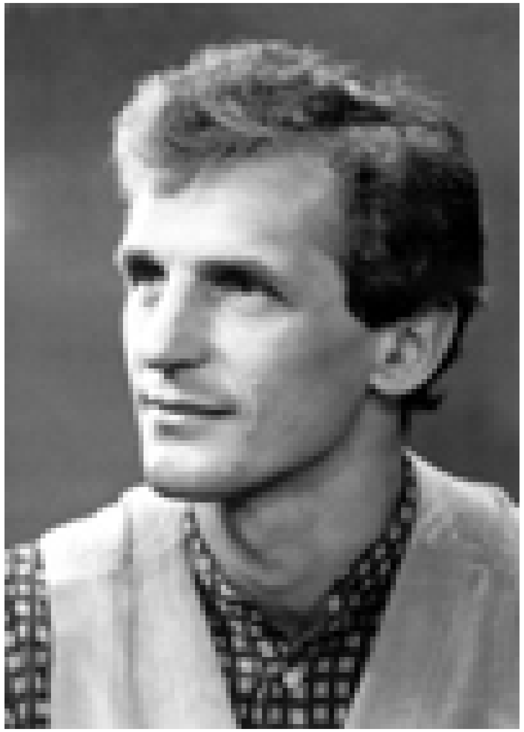
Daniel Kleppner
awarded Wolf
Prize in Physics

Rev. Mod. Phys., Vol. 74,p.1131 , 2002

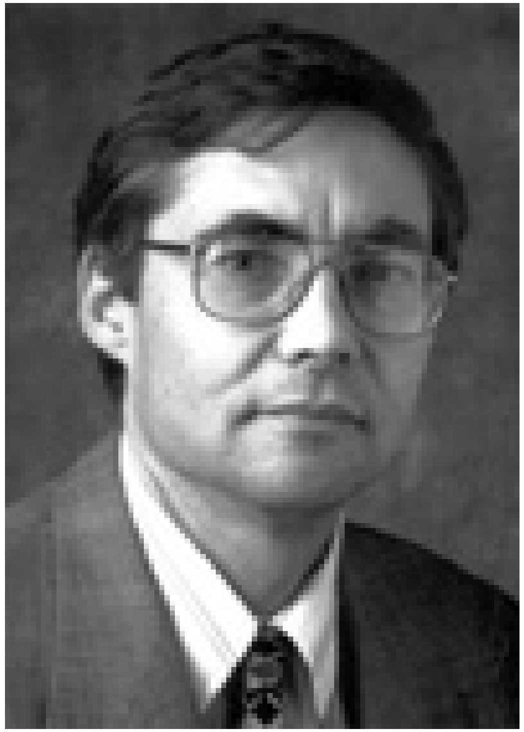
The Nobel Prize in Physics 2001



Eric A. Cornell



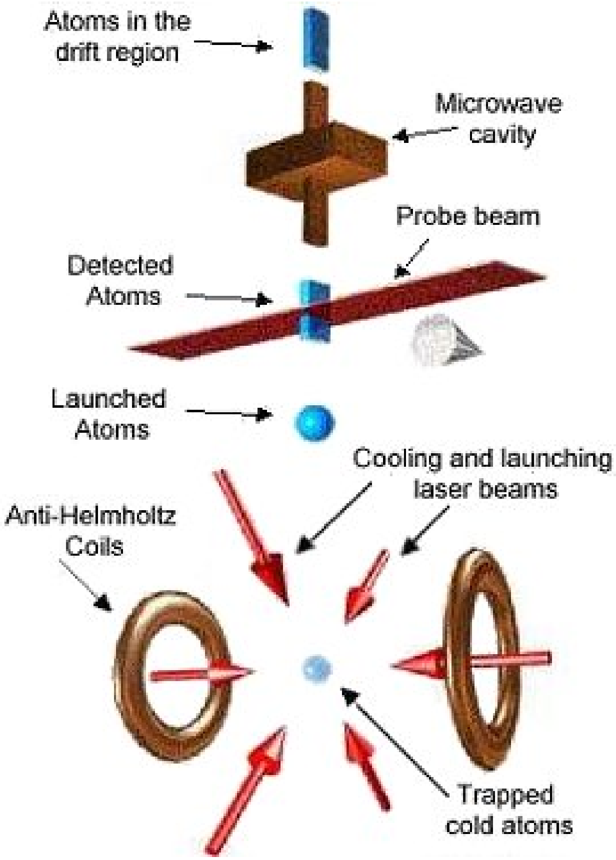
Wolfgang Ketterle



Carl E. Wieman

"for the achievement of **Bose-Einstein condensation** in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates"

Magneto-optical Trap



Laser Cooling

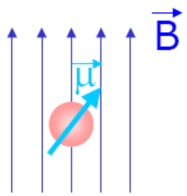
The Nobel Prize in Physics 1997

"for development of methods to cool and trap atoms with laser light"

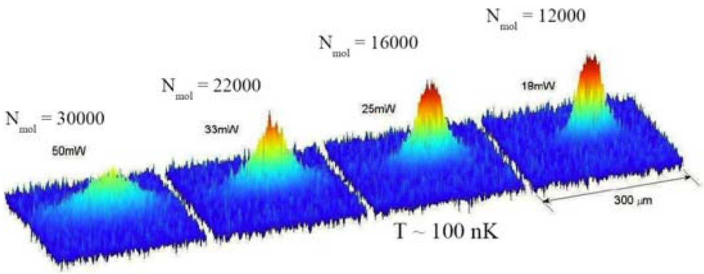
Steven Chu	Claude Cohen-Tannoudji	William D. Phillips

Magnetic Field Gradient

Zeeman energy:



$$E_Z = -\vec{\mu}\cdot\vec{B}$$
$$= g_f m_F \mu_B B$$



Why interesting?

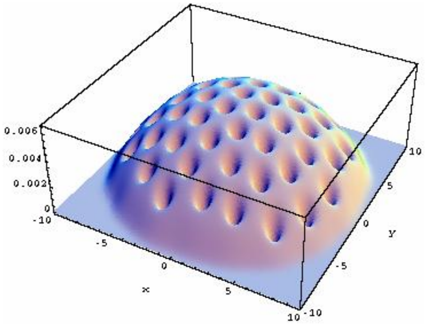
I. Simple systems for the theory

An important theoretical frame for Bose–Einstein condensation in interacting systems was developed in the 50's by Beliaev, Bogoliubov, Gross,Pitaevskii in the context of superfluid helium.

$$\mu\,\Phi = \left(-\frac{\hbar^2 \nabla^2}{2m} + V(r) \right) \Phi + g\,|\Phi|^2\,\Phi, \quad g = \frac{4\pi\hbar^2}{m} a_0$$

Gross, E. P., 1961, Nuovo Cimento 20, 454.
Pitaevskii, L. P., 1961, Zh. Eksp. Teor. Fiz. 40, 646 [Sov. Phys. JETP 13, 451 (1961)].

This theory however is supposed to work better if applied to Bose condensed gases where the interactions are much weaker.

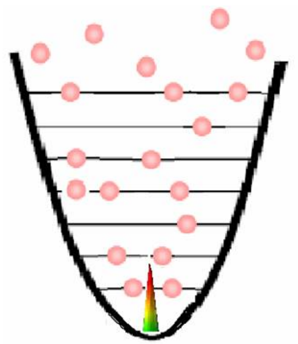


II. New features : (respect to superfluid He 4)

Spatial inhomogeneity

Finite size effects

Tunability



$$V = \frac{1}{2}\omega_x{}^2 x^2 + \frac{1}{2}\omega_y{}^2 y^2 + \frac{1}{2}\omega_z{}^2 z^2$$

Grand Canonical Ensemble for non-interacting particles

I. Energy Level $E_i \ \psi_i = \hat{H} \ \psi_i$

II. Grand Partition Function

$$D = \sum_{N=0}^{\infty} z^N Q_N, \quad z = e^{\beta\mu}, Q_N = \int \frac{d^{3N} p d^{3N} q}{N! h^{3N}} e^{-\beta H(p,q)} = \sum_n e^{-\beta E_n} = \mathrm{Tr}\rho$$

III. Thermodynamic Functions

$$\frac{PV}{k_B T} = \ln D, \quad \langle N \rangle = z \frac{\partial}{\partial z} \ln D, \quad U = -\frac{\partial}{\partial \beta} \ln D$$

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V, \quad S = \int_0^T dT \frac{C_V}{T},$$

$$A(N,V,T) = U - TS, \quad G(N,P,T) = A + PV$$

Thermodynamic limit

Statistical Mechanics Kerson Huang 2nd **p.289**

$$\frac{N}{V} = \frac{1}{\lambda^3} g_{3/2}(z) + \frac{\langle n_0 \rangle}{V} + \left(\frac{\langle n_1 \rangle}{V} + \frac{\langle n_2 \rangle}{V} + \cdots \right)$$

where, in the parentheses, there appear any finite number of terms. Every term in the parentheses, however, approaches zero as $V \to \infty$. For example,

$$\frac{\langle n_1 \rangle}{V} = \frac{1}{V} \frac{1}{z^{-1} e^{\beta \epsilon_1} - 1} \le \frac{1}{V} \frac{1}{e^{\beta \epsilon_1} - 1}$$

where

$$2m\epsilon_1 = (2\pi\hbar)^2 \frac{l_1}{V^{2/3}}$$

$$l_1 = \text{sum of the squares of three integers not all zero}$$

Hence

$$\frac{\langle n_1 \rangle}{V} \le \frac{1}{V} \frac{2m\beta V^{2/3}}{(2\pi\hbar)^2 \beta^2 l_1} \xrightarrow[V \to \infty]{} 0 \tag{12.53}$$

This shows that (12.41) is valid.

$$\frac{1}{v} = \frac{1}{\lambda^3} g_{3/2}(z) + \frac{1}{V} \frac{z}{1-z} \tag{12.41}$$

Grand Canonical Ensemble for non-interacting Bose gas

Free Gas, no trap

$$\frac{P}{k_B T} = \frac{1}{\lambda^3} g_{5/2}(z) - \frac{1}{V}\ln(1-z)$$

$$\frac{N}{V} = \frac{1}{\lambda^3} g_{3/2}(z) - \frac{1}{V}\frac{z}{1-z}$$

Exact to $\left(\frac{\Delta E}{k_B T}\right)$

Euler's Theorem for homogenous functions

$$dU = TdS - PdV + \mu dN, \quad U = TS - PV + \mu N,$$

$$A = U - TS = -PV + \mu N, \quad G = A + PV = \mu N$$

Trapped Gas

$$\ln D = \left(\frac{E_{char}}{k_B T}\right)^{\eta} g_{\eta+1}(z) - \frac{1}{V}\ln(1-z)$$

$$N = \left(\frac{E_{char}}{k_B T}\right)^{\eta} g_{\eta}(z) - \frac{1}{V}\frac{z}{1-z}$$

Expansion of Bose-Einstein function F. London

$$g_{1/2}(e^{-\alpha}) = \left(\frac{\pi}{\alpha}\right)^{1/2} - 1.460 + (0.208)\alpha - (0.0128)\alpha^2 + O(\alpha^3)$$

$$g_{3/2}(e^{-\alpha}) = 2.612 - 2(\pi\alpha)^{1/2} + (1.460)\alpha - (0.104)\alpha^2 + O(\alpha^3)$$

$$g_{5/2}(e^{-\alpha}) = 1.342 - (2.612)\alpha + \frac{4}{3}\left(\pi\alpha^3\right)^{1/2} - (0.730)\alpha^2 + O(\alpha^3)$$

Chebyshev Polynomial Expansion

If $\eta \neq$ integer, and $-\pi < \alpha < 0$, then $g_{\eta+1}(e^{\alpha}) = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!}\zeta(\eta+1-j) - \frac{\pi(-\alpha)^{\eta}}{\Gamma(\eta+1)\sin(\eta\pi)}$

If $\eta \neq$ integer, and $0 < \alpha < \pi$, then

$$g_{\eta+1}(e^{\alpha}) = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!}\zeta(\eta+1-j) - \frac{\pi(\alpha)^{\eta}}{\Gamma(\eta+1)\tan(\eta\pi)}$$

If η = integer, and $-2\pi < \alpha < 2\pi$ then

$$g_{\eta+1}(e^{\alpha}) = \sum_{j=0\neq\eta}^{\infty} \frac{\alpha^j}{j!}\zeta(\eta+1-j) - \frac{(\alpha)^{\eta}}{\Gamma(\eta+1)}\left[\ln|\alpha| - \Psi(\eta+1) + \Psi(1)\right]$$

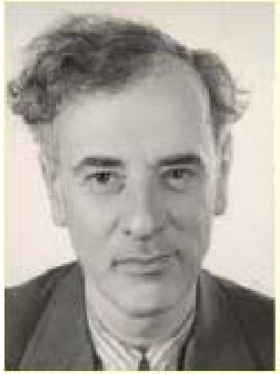
where $\Psi(\eta) = \frac{d}{d\eta}\ln\Gamma(\eta)$

Note on the Bose-Einstein Integral Functions (John E. Robinson,1951)

Two-fluid model in BEC

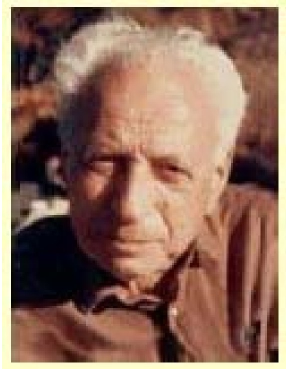
A. Griffin, Phys. Rev. B, Vol 53, p.9341 (1996)

Tisza-Landau two-fluid hydrodynamics (1938~1941)



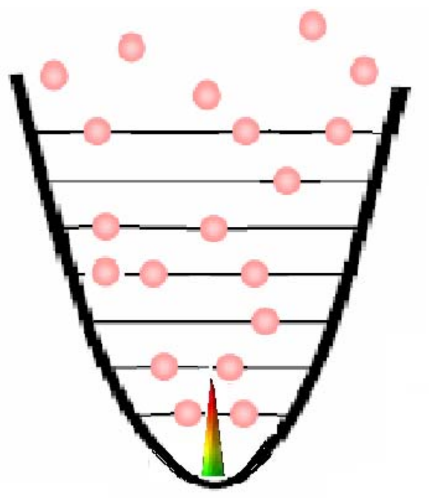
Lev Landau (1908~1968)

Superfluid **: component of liquid which is associated with macroscopic occupation (BEC) of a single particle state . Carries zero entropy, flows without dissipation with irrotational velocity.**



Laszlo Tisza (July 7, 1907 – April 15, 2009

Normal fluid **: comprised of incoherent thermal excitations , behaves like any fluid at finite temperatures in local thermodynamic equilibrium. This requires strong collisions.**



The condensate part

$$\left(-\frac{\hbar^2\nabla^2}{2m}+U_{\text{ex}}(\mathbf{r})-\mu\right)\Phi(\mathbf{r})+g[n_c(\mathbf{r})+2\tilde{n}(\mathbf{r})]\Phi(\mathbf{r})=0,$$

and the thermal part

$$i\frac{\partial\tilde{\psi}(\mathbf{r},t)}{\partial t}=\left(-\frac{\hbar^2\nabla^2}{2m}+U_{\text{ex}}(\mathbf{r})-\mu\right)\tilde{\psi}(\mathbf{r},t)+2gn(\mathbf{r})\tilde{\psi}(\mathbf{r},t),$$

Diverge ?

Projected Gross-Pitaevskii equation

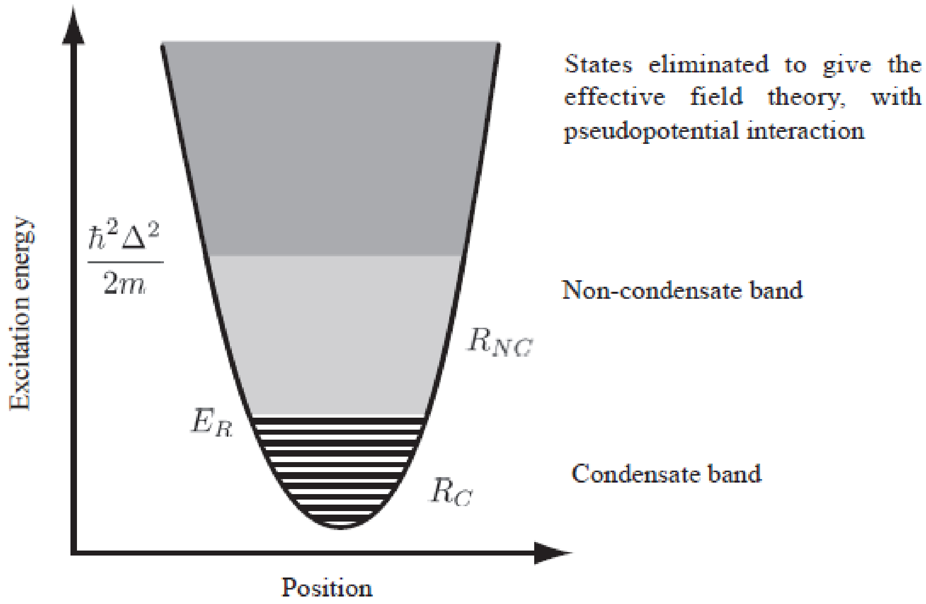
Gardiner-Zoller Quantum Kinetic Theory

Condensate band and non-condensate band

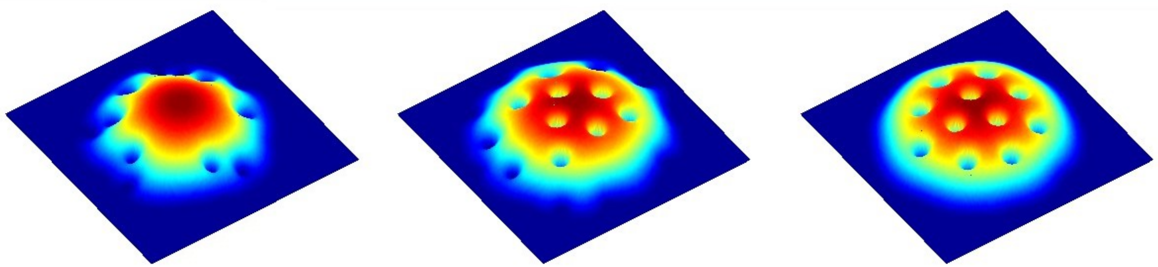
Projectors :

$$\psi_{NC}(\mathbf{x}) = \int d^3\mathbf{x}'\, \mathcal{P}_{NC}(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}') \equiv \mathcal{P}_{NC}\{\psi(\mathbf{x})\},$$

$$\phi(\mathbf{x}) = \int d^3\mathbf{x}'\, \mathcal{P}_C(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}') \equiv \mathcal{P}_C\{\psi(\mathbf{x})\}.$$



$$i\hbar\frac{\partial\Psi(\tilde{\mathbf{x}})}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{trap}}(\tilde{\mathbf{x}})\right)\Psi(\tilde{\mathbf{x}}) + P\{U_0|\Psi(\tilde{\mathbf{x}})|^2\Psi(\tilde{\mathbf{x}})\},$$



In weakly interacting,

Mean Field Theorem 是否足以決定相變溫度 ?

PHYSICAL REVIEW A ,R4633 (1996)

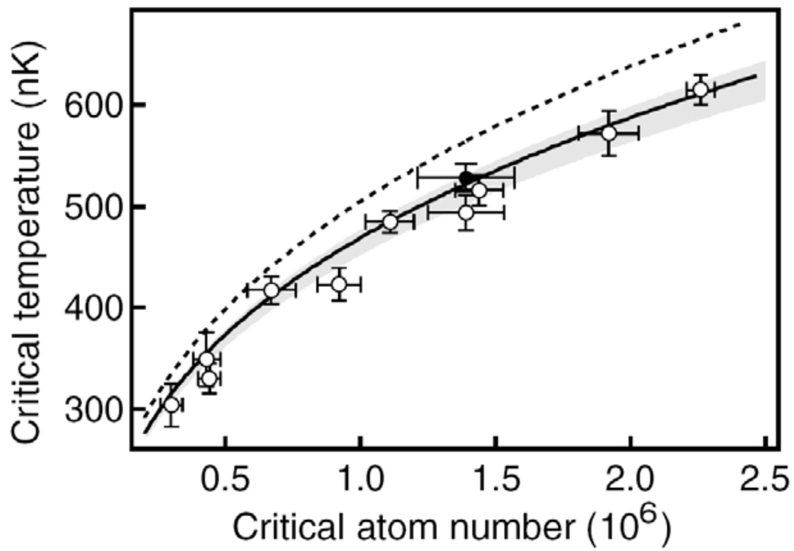
Condensate fraction and critical temperature of a trapped interacting Bose gas

S. Giorgini, L. P. Pitaevskii and S. Stringari

$$\frac{\Delta T_c}{T_c^{\,0}} \simeq -1.33\frac{a_0}{a_{ho}} N^{1/6}$$

where $a_{ho} = \sqrt{\frac{\hbar}{m\omega}}$ is the harmonic oscillator length.

Phys. Rev. Lett. 92, 030405,(2004)

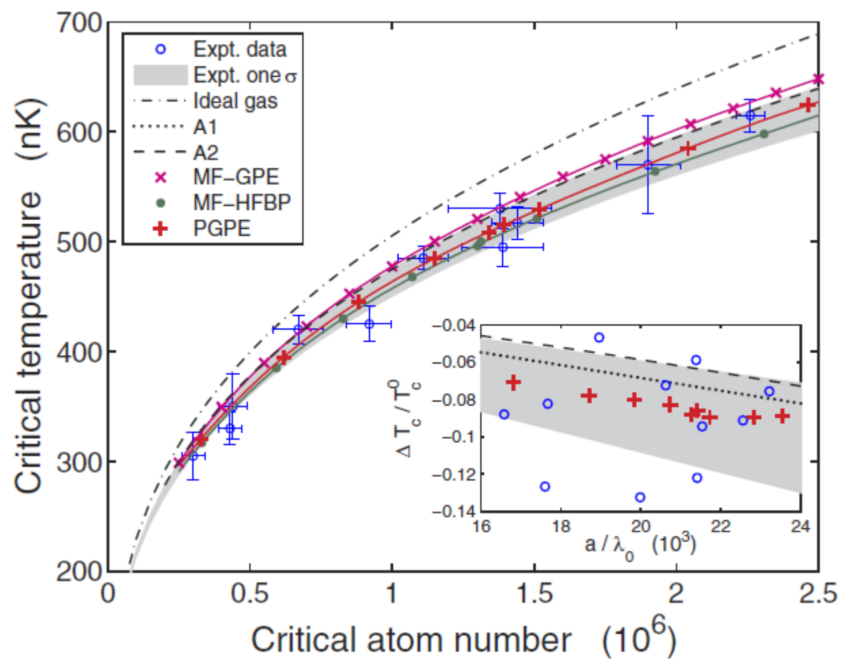


$\delta T_c/T_c^0 = \alpha N^{1/6}$,

Theo. α = - 0.07

Exp. α = - 0.09

Phys. Rev. Lett. 96, 060406,(2006)



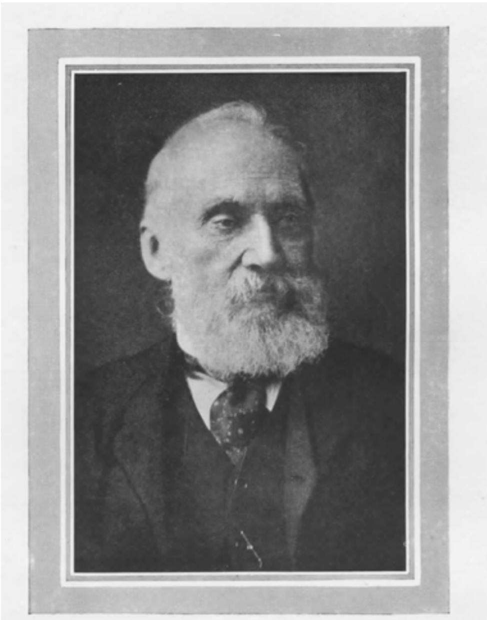
A1: This is the first-order analytic estimate of Giorgini et al.

A2: This is the full second-order result

Transition temperature of a weakly interacting Bose gas

$$\begin{cases} E_T = E_T^{\,ideal} + 2g\bar{n}_{TT} + 2g\bar{n}_{cT}, \\ E_c = E_c^{\,ideal} + 2g\bar{n}_{Tc} + g\bar{n}_{cc}, \end{cases}$$

$$\frac{\Delta T_c}{T_c^{\,0}} \simeq -(1.33+0.45)\frac{a_0}{a_{ho}} N^{1/6}$$



William Thomson, Lord Kelvin

1900年初，當時在英國皇家學會的新年致辭中，發表了題爲「籠罩在熱和光的動力理論上的十九世紀之雲」的著名演講。他說

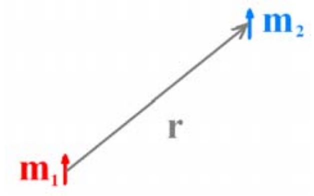
“物理的大廈已經落成，所剩的只是一些修飾的工作 **...**”

Dipolar BEC

PRL **98, 080407 (2007)**

Critical Temperature of Weakly Interacting Dipolar Condensates

Konstantin Glaum, Axel Pelster, Hagen Kleinert, and Tilman Pfau



$$U_{\mathrm{dd}}(\vec{r}) = \frac{\mu_0 \mu_{\mathrm{m}}^2}{4\pi r^3}\left(1 - \frac{3(\hat{e}_\mu \vec{r})^2}{r^2}\right)$$

$$\frac{\Delta T_c}{T_c^{(0)}} = -\frac{c_\delta a}{\lambda_c^{(0)}} + \left[3\cos^2\alpha - 1\right] f\left(\frac{\omega_\parallel}{\omega_\perp}\right)\frac{\mu_0 m^2 M c_\delta}{48\pi\hbar^2\lambda_c^{(0)}} .$$

PHYSICAL REVIEW A 75, 033607 2007

Transition temperature of the interacting dipolar Bose gas

Yee-Mou Kao and T. F. Jiang

$$\frac{\Delta T_c}{T_c^{0}} \simeq -(2.78+5.46)\frac{a_d}{a_{ho}} N^{1/6}\chi_0(\kappa)$$

International Journal of Modern Physics B
Vol. 17, No. 12 (2003) 2439-2446

World Scientific
www.worldscientific.com

CRITICAL TEMPERATURE OF A WEAKLY INTERACTING BOSE GAS IN A POWER-LAW POTENTIAL

HIDENORI SUZUKI and MASUO SUZUKI

Department of Physics, Graduate School of Science,Tokyo University of Science, 1-3, Kagurazaka, Shinjuku-ku, Tokyo, 162-8601, Japan j1202706@ed.kagu.tus.ac.jp

Received 20 December 2002

Mean field approximation $V_{ext}(\vec{r}) = Ar^n$

In three-dimensional space, we show that the shift of T_c changes its sign from a negative value for n < 3 to a positive one for n > 3, where n is the exponent of the power-low potential.

J. Phys. B: At. Mol. Opt. Phys. **37 (2004) 2593–2605**

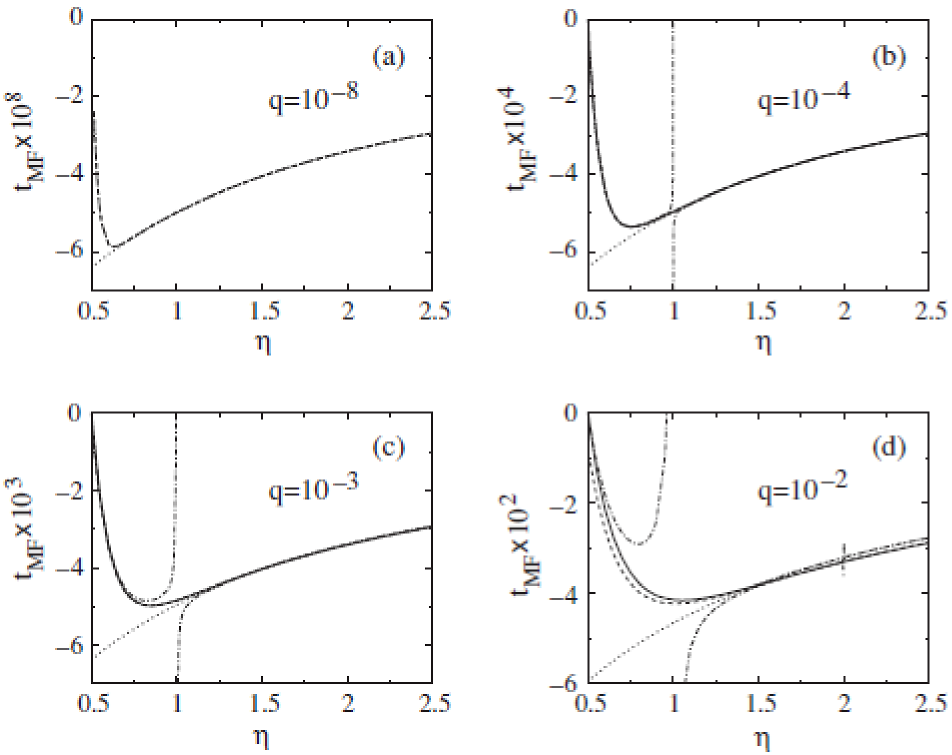
Mean-field analysis of Bose–Einstein condensation in general power-law potentials

O Zobay

Institut f¨ur Angewandte Physik, Technische Universit¨at Darmstadt, 64289 Darmstadt, Germany

Mean-field theory is applied to describe the condensation of dilute interacting Bose gases in general power-law potentials.

$$V_{ext}(\vec{r}) = E_1\left(\frac{x}{L_1}\right)^p + E_2\left(\frac{y}{L_2}\right)^l + E_3\left(\frac{z}{L_3}\right)^s, \quad \eta = \frac{1}{p}+\frac{1}{l}+\frac{1}{s}+\frac{1}{2}$$



Mean-field calculation of critical temperature

$$t = \frac{\Delta T_c}{T_c^{\,0}}, \quad q = \frac{a}{\lambda_T}$$

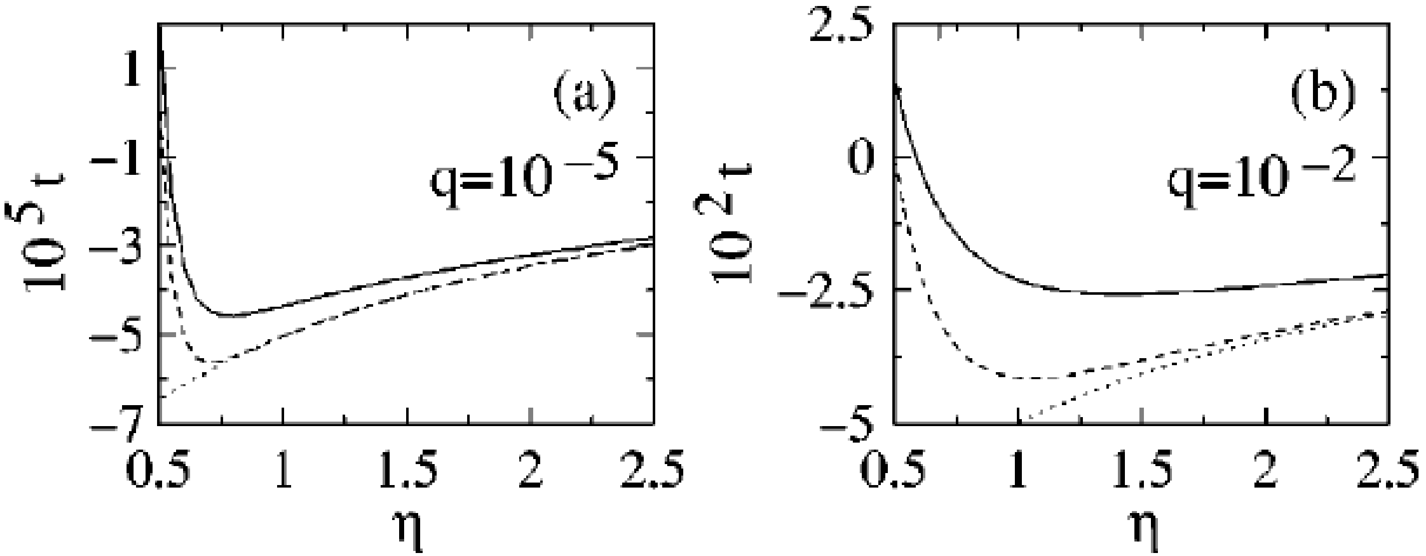
PHYSICAL REVIEW A **69, 063615 (2004)**

Phase transition of interacting Bose gases in general power-law potentials

O. Zobay, G. Metikas, and G. Alber

Institut für Angewandte Physik, Technische Universität Darmstadt, 64289 Darmstadt, Germany

Using energy-shell renormalization and the ε expansion



Bold curves: RG results with ε expansion. Dashed curves: mean-field LDA. Dotted curves: linear mean-field approximation

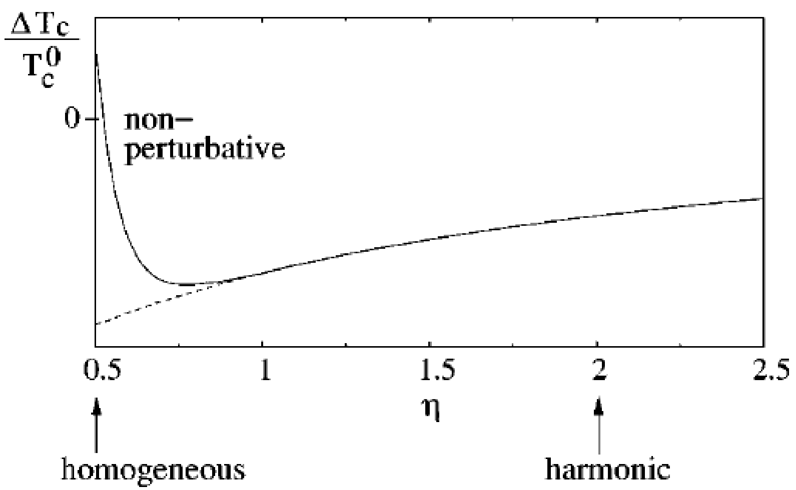
PHYSICAL REVIEW A **71, 043614 (2005)**

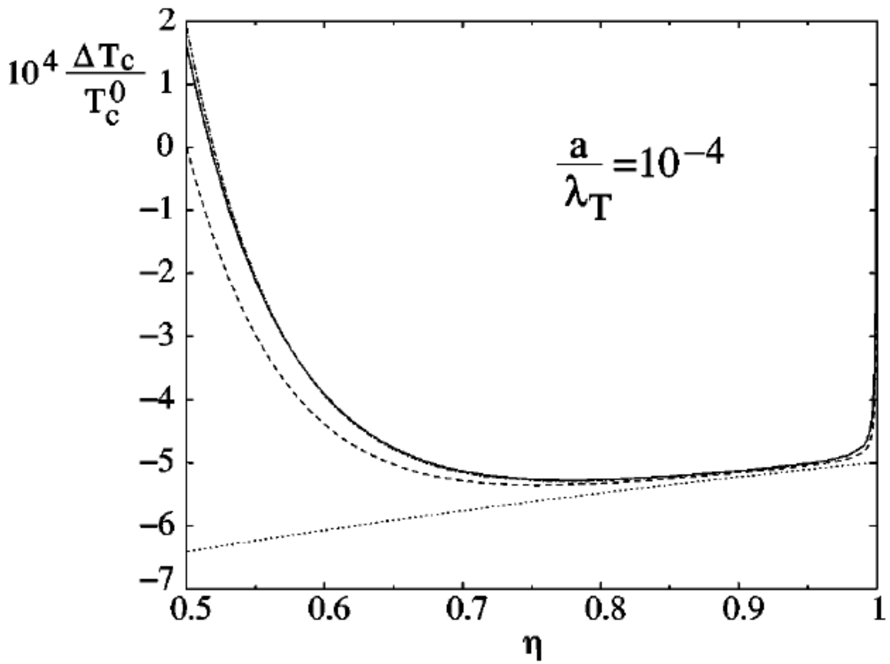
Nonperturbative effects on *T_c of interacting Bose gases in power-law traps*

O. Zobay, G. Metikas, and H. Kleinert

calculated with the help of variational perturbation theory.

Schematic diagram





Laser Physics, 2009, Vol. 19, No. 4, pp. 700–724.

Phase Transition of Trapped Interacting Bose Gases

O. Zobay

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A first calculation by Lee and Yang in 1957 [P.R.105,p.1119] obtained an increase in T_c compared to the ideal gas that is proportional to the square root $\sqrt{a_0}$ of the scattering length.

It is only since about 1999 that a consistent description of the critical Bose gas has emerged and has led to a generally accepted result for the critical temperature.

$$E_{\mathrm{int}} \begin{cases} k^{2-\eta}, & k \le k_c \\ k^2, & k \ge k_c \end{cases}$$

$$\frac{\Delta T_c}{T_c^{\,0}} = c a_0 n^{1/3}$$

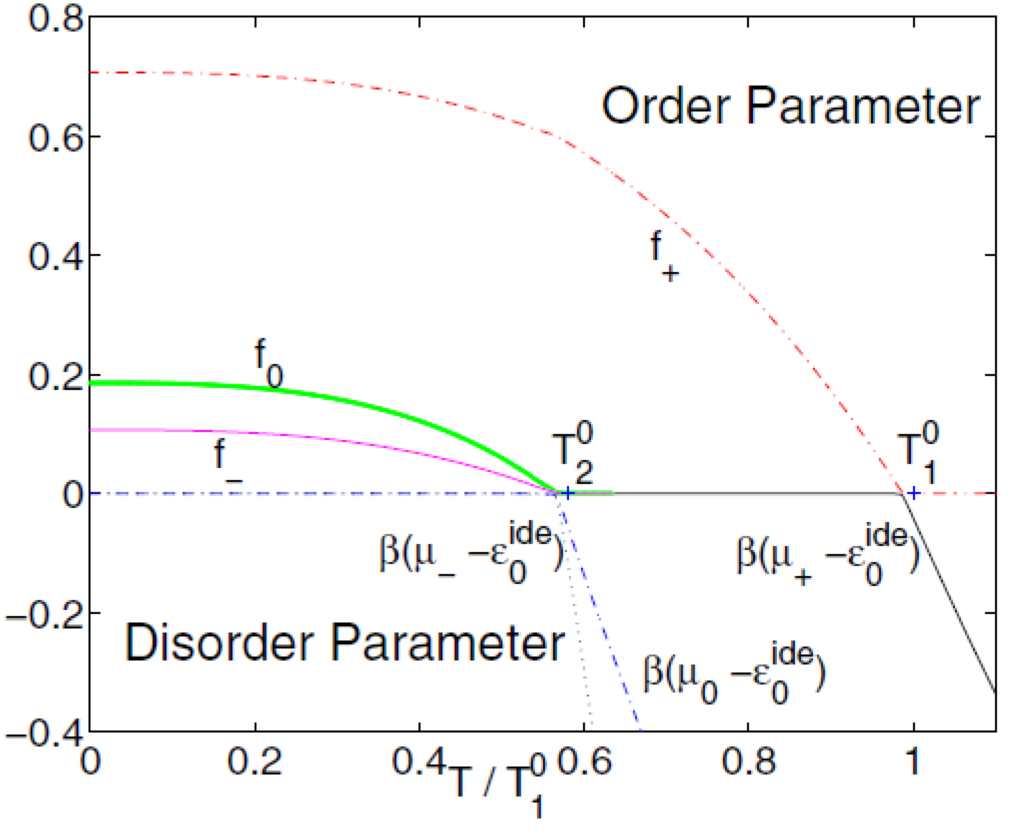
In weakly interacting,

Mean Field Theorem 是足以決定相變溫度 ！

Transition temperatures of the trapped ideal spinor Bose gas

particle number $N = N_+ + N_0 + N_-$

magnetization $M = N_+ - N_-$

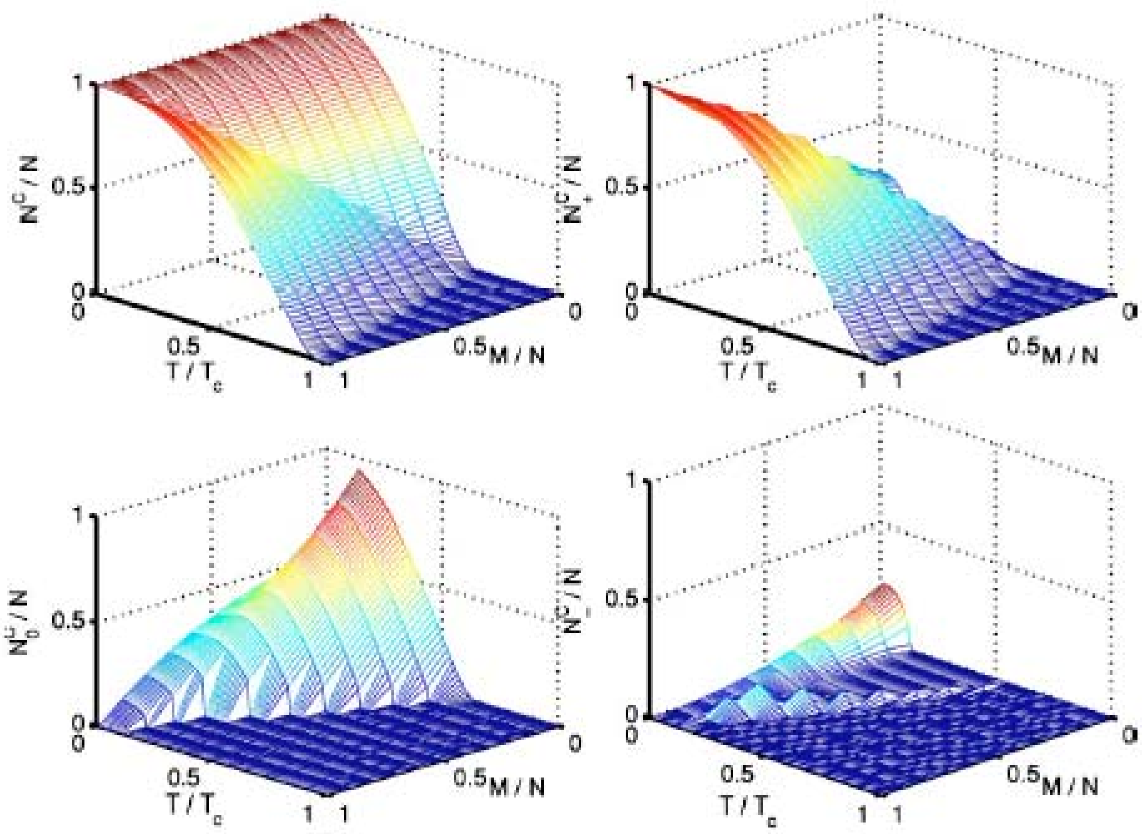


Interacting Spinor BEC

PHYSICAL REVIEW A 70, 043611 (2004)

Wenxian Zhang, Su Yi, and L. You

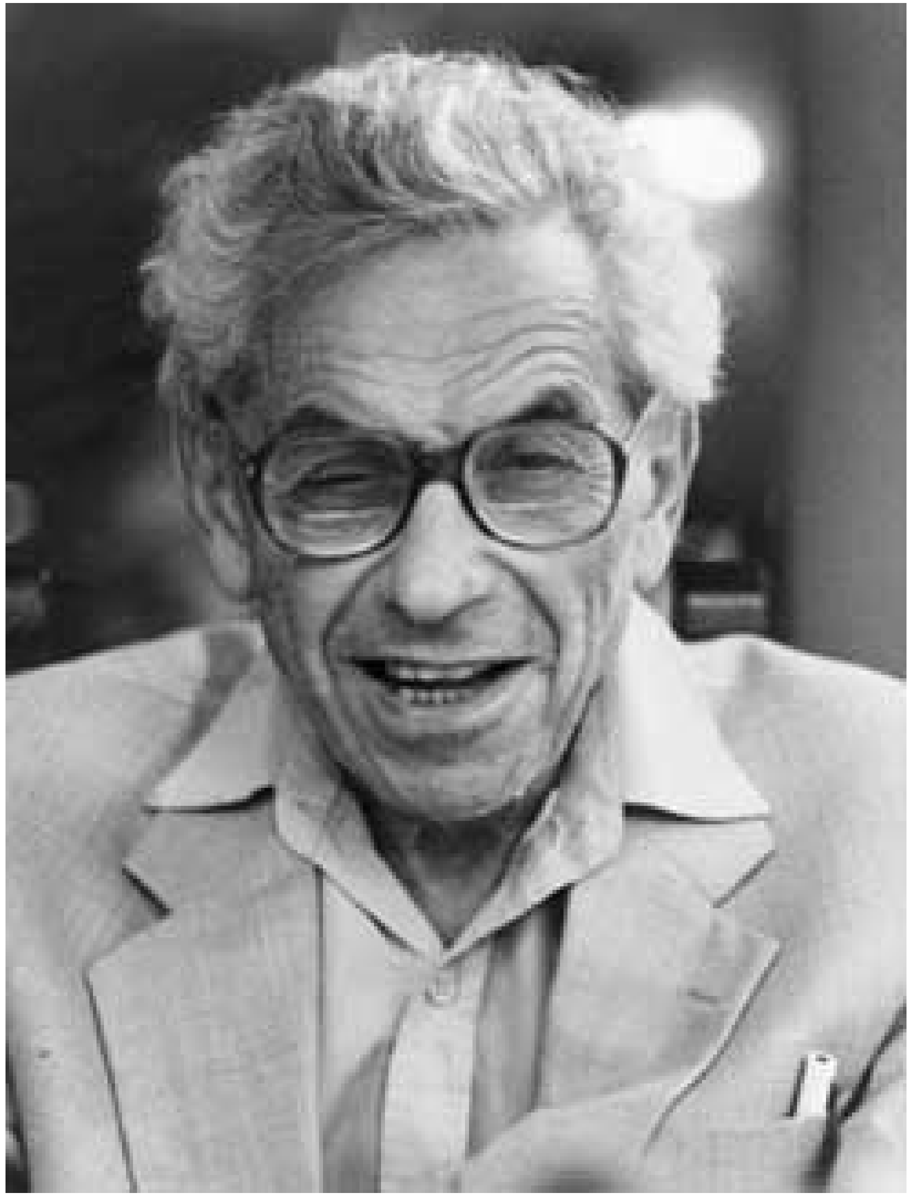
For a gas of ^{87}Rb atoms.



Several Phase Transition ?

Dr. Erdös defined the word
"mathematician"
as
"a machine for turning
coffee into theorems."

The End



Paul Erdös(1913~1996)