Quantum interference in the time-of-flight distribution for atomic Bose-Einstein condensates

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Time-of-flight measurement

- First evidence for Bose Einstein condensates (BEC's) emerged from TOF measurements.
- A technique of measuring the temperature of cold atomic samples is the TOF measurement.
- Cold atomic cloud is allowed for a thermal expansion after its release from the trap.
- TOF measurements are performed either by acquiring the absorption signal of the probe laser beam through the falling and expanding atomic cloud, or by measuring the fluorescence of the atoms excited by the resonant probe light.

Analyses of TOF measurements

- Initial probability distribution of finding an atom in the phase space

$$D(z_0, v_0)dz_0dv_0 = \frac{e^{-z_0^2/2\sigma_0^2}}{(2\pi\sigma_0^2)^{1/2}} \frac{e^{-v_0^2/2\sigma_v^2}}{(2\pi\sigma_v^2)^{1/2}} dz_0dv_0$$

where $\sigma_v^2 = k_B T / m$

- Newton's equation for ballistic motion of a particle in the gravitational field

$$v_0 = (z - z_0 + gt^2/2)/t$$

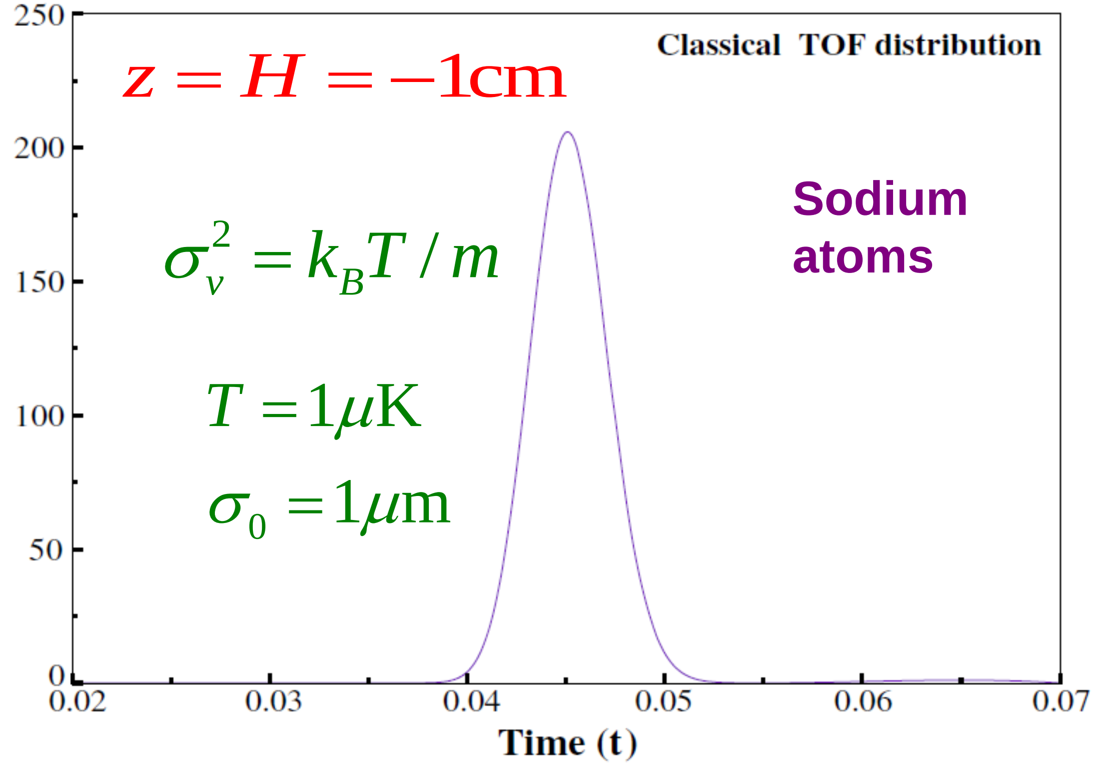
- Substituting the above expression for v_0 and then integrating over z_0, one can obtain the TOF distribution at an arbitrary distance $z = H$,

Classical TOF distribution in 1D

TOF distribution at an arbitrary distance $z = H$,

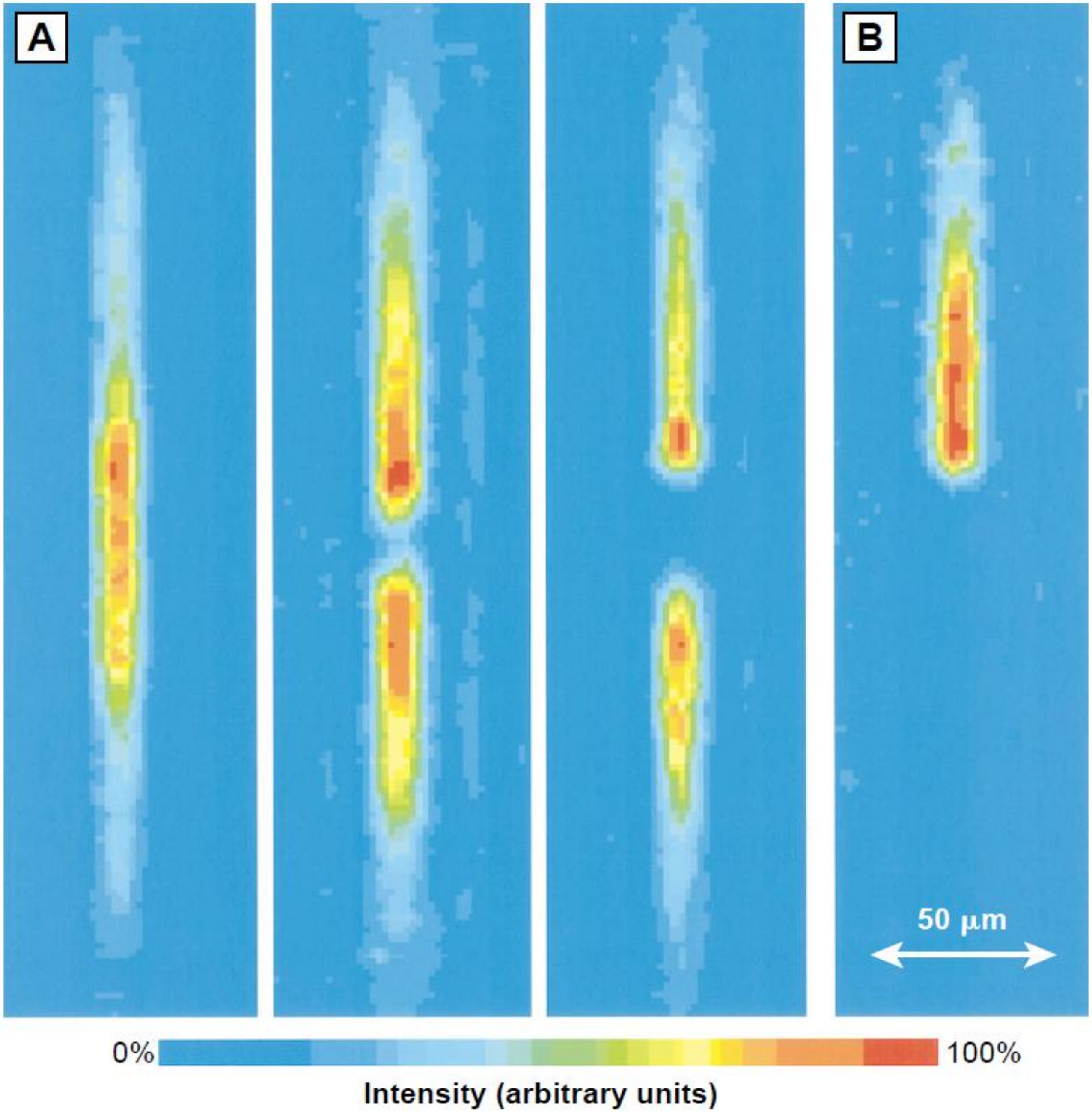
$$D(t)dt = \frac{[(gt^2/2)(2\sigma_0^2 + \sigma_v^2 t^2) - H\sigma_v^2 t^2]}{(\sigma_0^2 + \sigma_v^2 t^2)^{3/2}} \frac{e^{-(H+gt^2/2)^2/[2(\sigma_0^2+\sigma_v^2 t^2)]}}{(2\pi t^2)^{1/2}} dt$$

The temperature of the atomic cloud is determined by fitting the experimental result to the theoretically predicted TOF signal of the cloud.



Classical analysis of TOF distribution

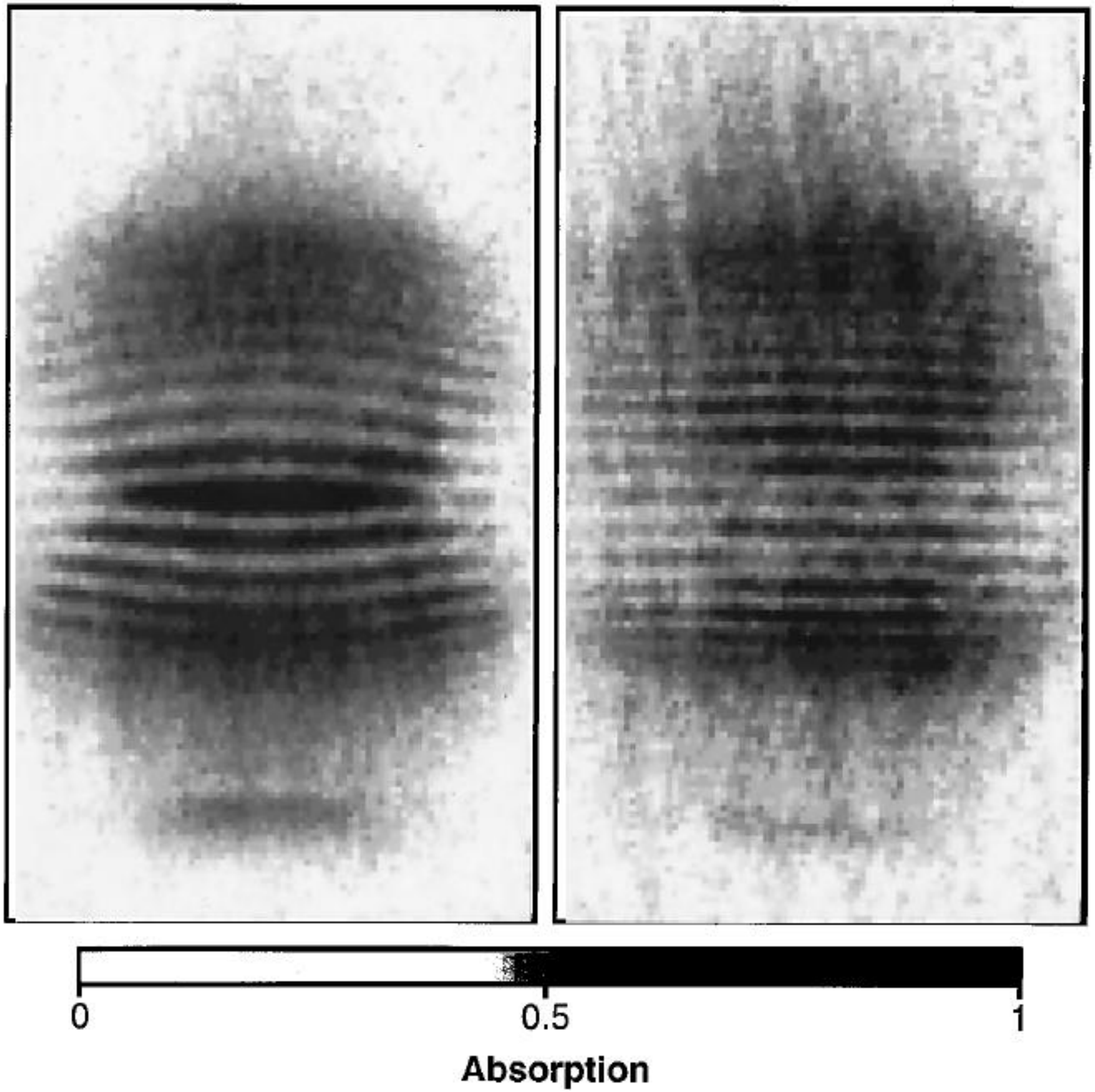
- The theoretical treatments of the TOF distribution that can be obtained using the Green's function method or any semiclassical method, however, are equivalent to the TOF distribution obtained by using Newton's equations for ballistic motion of particles.
- This kind of purely classical analyses are adopted in most of the discussions on TOF measurements where arrival time of atomic or sub-atomic particles is treated as an elementary well-defined, unique, and classical quantity.
- In the domain of small atomic masses and low temperatures where quantum mechanical effects should be significant, quantum TOF distribution can not be reproduced with classical or semiclassical analyses.
- Here provide an example in the context of BEC matter-wave interference, where **a quantum analysis for TOF is necessary.**



M. R. Andrews *et al., Science* ***275, 637 (1997).***

- **Phase contrast images of BEC's** of sodium atoms optically cooled , trapped, and then transferred into a double well potential.
- The distance between the two BEC's was varied by changing the power of the argon ion laser-light sheet from 7 to 43 mW.

Interference pattern of two expanding BEC'S



Observed after 40 ms time-of-flight for two different powers of the argon ion laser-light sheet (raw-data images).

The fringe periods were 20 and 15 mm, the powers were 3 and 5 mW.

M. R. Andrews *et al.*, *Science* ***275, 637 (1997).***

Interference of BEC's in space

- Interference between two freely expanding Bose-Einstein condensates (BEC) **in space** has been observed in a remarkable experiment [1].
- Coherent splitting of BEC atoms with optically induced Bragg diffraction have been done experimentally [2,3].
- The **spatial coherence** of a BEC is measured using interference technique by creating and recombining two spatially displaced, coherently diffracted copies of an original BEC [3].

1. M. R. Andrews *et al., Science* **275**, 637 (1997).
2. M. Kozuma *et al., Phys. Rev. Lett*. **82**, 871 (1999).
3. E. W. Hagley *et al., Phys. Rev. Lett.* **83**, 3112 (1999); *J. E. Simsarian et al. Phys. Rev. Lett.* **85**, 2040 (2000).

Time as an operator

- **Pauli's argument**: if there existed a self-adjoint time operator $\mathcal{T}$ canonically conjugate to the Hamiltonian

  $$[H, \mathcal{T}] = i\hbar,$$

  $e^{-iE_1\mathcal{T}/\hbar}|E\rangle$ would produce a new energy eigenstate $|E - E_1\rangle$

  so that he spectrum of *E* would necessarily extend continuously over the range [- ∞, ∞].
- In principle, this precludes the existence of a self-adjoint time operator for systems where the spectrum of the Hamiltonian is bounded, semibounded or discrete, i.e. for most of the systems of physical interest.

J. G. Muga and C. R. Leavens, *Phys. Rep.* ***338***, 353 (2000).

Postulates in Bohm's causal interpretation of quantum mechanics

- A quantum entity, such as an electron propagating, in a potential $V(x, t)$ is an actual point-like particle and an accompanying wave $\psi(x, t)$ which probes the potential and guides the particle's motion accordingly so that it has a well-defined position $x(t)$ and velocity $v(t)$ at each instant of time t.
- $\psi(x, t)$ satisfies the time-dependent Schrödinger equation.
- The particle's equation of motion is

$$v(t) \equiv dx(t)/dt = v(x,t)|_{x=x(t)}, \text{ where}$$

$$v(x,t) = \frac{J(x,t)}{|\psi(x,t)|^2}, \text{ with } J(x,t) = \frac{\hbar}{m}\mathrm{Im}[\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x}]$$

- The quantity $\psi(x,t)dx$ is the probability of the particle being between x and $x + dx$ *at time t even in the absence* of a position measurement.

Bohm's causal approach

- Uncertainty enters only through the probability distribution $|\psi(x^{(0)},0)|^2$ for the unknown initial position $x^{(0)}=x(t=0)$ of the particle.
- Nonintersection property of Bohm trajectories $x(x^{(0)},t)$ with different starting points $x^{(0)}$ [but the same initial wave function $\psi(x,0)$]: If $x^{(0)\prime} \neq x^{(0)}$ then $x(x^{(0)\prime},t) \neq x(x^{(0)},t)$ for any t.
- The probability distribution for a particle property f:

$$P(f) \equiv \int_{-\infty}^{\infty} dx^{(0)} \, |\psi(x^{(0)},0)|^2 \, \delta(f - f(x^{(0)},t)).$$

- So

$$|\psi(x,t)|^2 = \int_{-\infty}^{\infty} dx^{(0)} \, |\psi(x^{(0)},0)|^2 \, \delta(x - x(x^{(0)},t)).$$

Arrival time distribution

- Consider the complete set of starting points $x^{(0)}$ for each of which the associated trajectory $x(x^{(0)}, t)$ reaches $x=X$ at least once at some time(s) $T(x^{(0)})$ subsequent to $t=0$.
- Because, the trajectories do not cross or touch each other, this set must consist of a single continuous interval, say $[x_a^{(0)}, x_b^{(0)}]$.
- The arrival time distribution is

$$\Pi(T) = \int_{-\infty}^{\infty} dx^{(0)} \, |\psi(x^{(0)}, 0)|^2 \, \delta[T - T(x^{(0)})].$$

- Again because of the nonintersection property, there is one and only one value of $x^{(0)}$ *in the interval* $[x_a^{(0)}, x_b^{(0)}]$ for which the trajectory $x(x^{(0)},t)$ *reaches* X *at a particular value* of T.

Arrival time distribution in 1D

- In addition, of course, even if that trajectory reaches X more than once only one of its arrival times is equal to the specified value of T.

$$\delta[x(x^{(0)},t)-X]|_{t=T}=\left.\frac{\delta[t-T(x^{(0)})]}{|dx(x^{(0)},t)/dt|}\right|_{t=T}=\left.\frac{\delta[t-T(x^{(0)})]}{|v[x(x^{(0)},t),t]|}\right|_{t=T}$$

- So

$$\begin{aligned}\Pi(T)&=\int_{-\infty}^{\infty}dx^{(0)}\,|\psi(x^{(0)},0)|^2\,\delta[T-T(x^{(0)})]\\&=|v(X,T)|\int_{-\infty}^{\infty}dx^{(0)}\,|\psi(x^{(0)},0)|^2\,\delta[x(x^{(0)},T)-X]\\&=|v(X,T)||\psi(X,T)|^2\\&=|J(X,T)|.\end{aligned}$$

Arrival time distribution in 3D

- Continuity equation

$$\frac{\partial |\psi(x,y,z,t)|^2}{\partial t} + \nabla \cdot J(x,y,z,t) = 0, \text{ where}$$

$$J(x,y,z,t) = \frac{\hbar}{m} \mathrm{Im}[\psi^*(x,y,z,t) \nabla \psi(x,y,z,t)].$$

- Quantum TOF distribution for the atoms reaching a detector at a finite surface plane S is given by

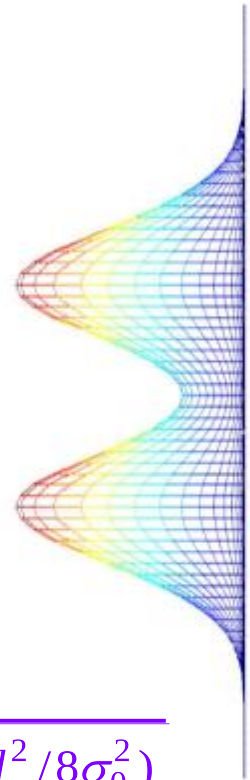
$$\Pi(t) = \left| \int_S J \cdot d\vec{S} \right| = \left| \int_S J \cdot \hat{n} dS \right|$$

TOF distribution or arrival time distribution

- Most of the experiments (particularly when matter-waves are associated with centre-of-mass motion) demonstrate matter-wave interference by showing the intensity variation **at an extended region of detection space at a fixed time**.
- We discuss here the BEC matter-wave interference in the time-of-flight (TOF) distribution or *arrival time distribution.*
- We predict and quantify the matter- wave interference in the center-of-mass motion by calculating ***the time distribution*** of matter-wave arrival probability **at some fixed spatial point.**

Vertical Setup 1

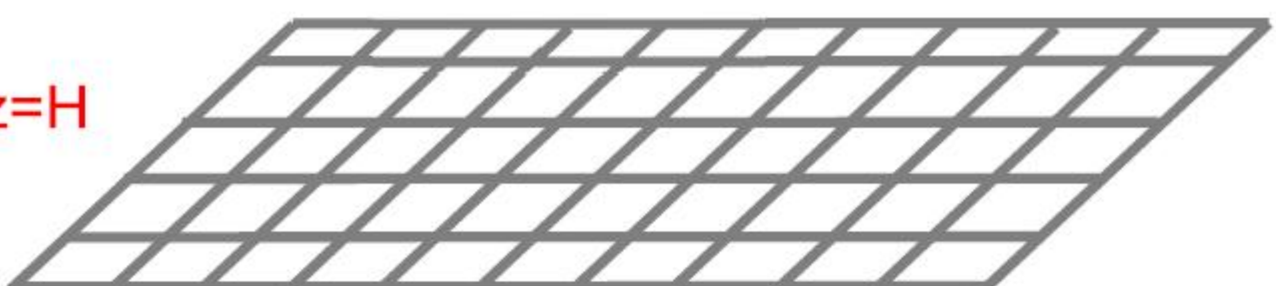
$$\Psi(x,y,z,t) = \frac{N}{\sqrt{2}}[\psi_1(x,y,z,0) + \psi_1(x,y,z,0)]$$



$$\psi_1(x,y,z,0) = \frac{e^{-(x^2+y^2+z^2)/4\sigma_0^2}}{(2\pi\sigma_0^2)^{3/4}}$$

$$\psi_2(x,y,z,0) = \frac{e^{-[x^2+y^2+(z+d)^2]/4\sigma_0^2}}{(2\pi\sigma_0^2)^{3/4}}$$

$$N = 1/\sqrt{1+e^{-(d^2/8\sigma_0^2)}}$$



Time evolution of condensate wavefunction

- use the Gross-Pitaevskii equation for the evolution of condensate wavefunction

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + mgz\psi + U|\psi|^2\psi$$

$$\psi_1(x,y,z,t) = \frac{e^{-[x^2+y^2+(z+gt^2/2)^2]/(4s_t\sigma_0)}}{(2\pi s_t^2)^{3/4}} e^{-i(m/\hbar)(gtz+g^2t^3/6)}$$

$$\psi_2(x,y,z,t) = \frac{e^{-[x^2+y^2+(z+d+gt^2/2)^2]/(4s_t\sigma_0)}}{(2\pi s_t^2)^{3/4}} e^{-i(m/\hbar)(gtz+g^2t^3/6)}$$

$$s_t = \sigma_0(1+i\hbar t/2m\sigma_0^2)$$

Quantum TOF distribution

$$\Pi_1(t) = \left|\int_S J \cdot \hat{n} dS\right| = \left|\int_{xy} J_z(x, y, z = H) dxdy\right|$$

$$= (N^2/2)\left|J_1(H,t) + J_2(H,t) + J_3(H,t) + J_3^*(H,t)\right|$$

$$J_1(H,t) = \left[\frac{\hbar^2 t}{4m^2\sigma_0^2\sigma^2}(H + \frac{gt^2}{2}) - gt\right]|\psi_1(H,t)|,$$

$$J_2(H,t) = \left[\frac{\hbar^2 t}{4m^2\sigma_0^2\sigma^2}(H + d + \frac{gt^2}{2}) - gt\right]|\psi_2(H,t)|,$$

$$J_3(H,t) + J_3^*(H,t) = 2P_{12}(z = H,t)(\eta_1 \cos\delta_1 - \lambda_1 \sin\delta_1),$$

$$P_{12}(z,t) = |\psi_1(z,t)||\psi_2(z,t)|,$$

Parameters

Position spread

$$\sigma^2 = |s_t|^2 = \sigma_0^2 (1 + \hbar^2 t^2 / 4m^2 \sigma_0^4),$$

$$\lambda_1 = \frac{\hbar d}{4m\sigma^2}, \quad \eta_1 = \frac{\hbar^2 t}{8m\sigma_0^2 \sigma^2}(2z + d + gt^2) - gt,$$

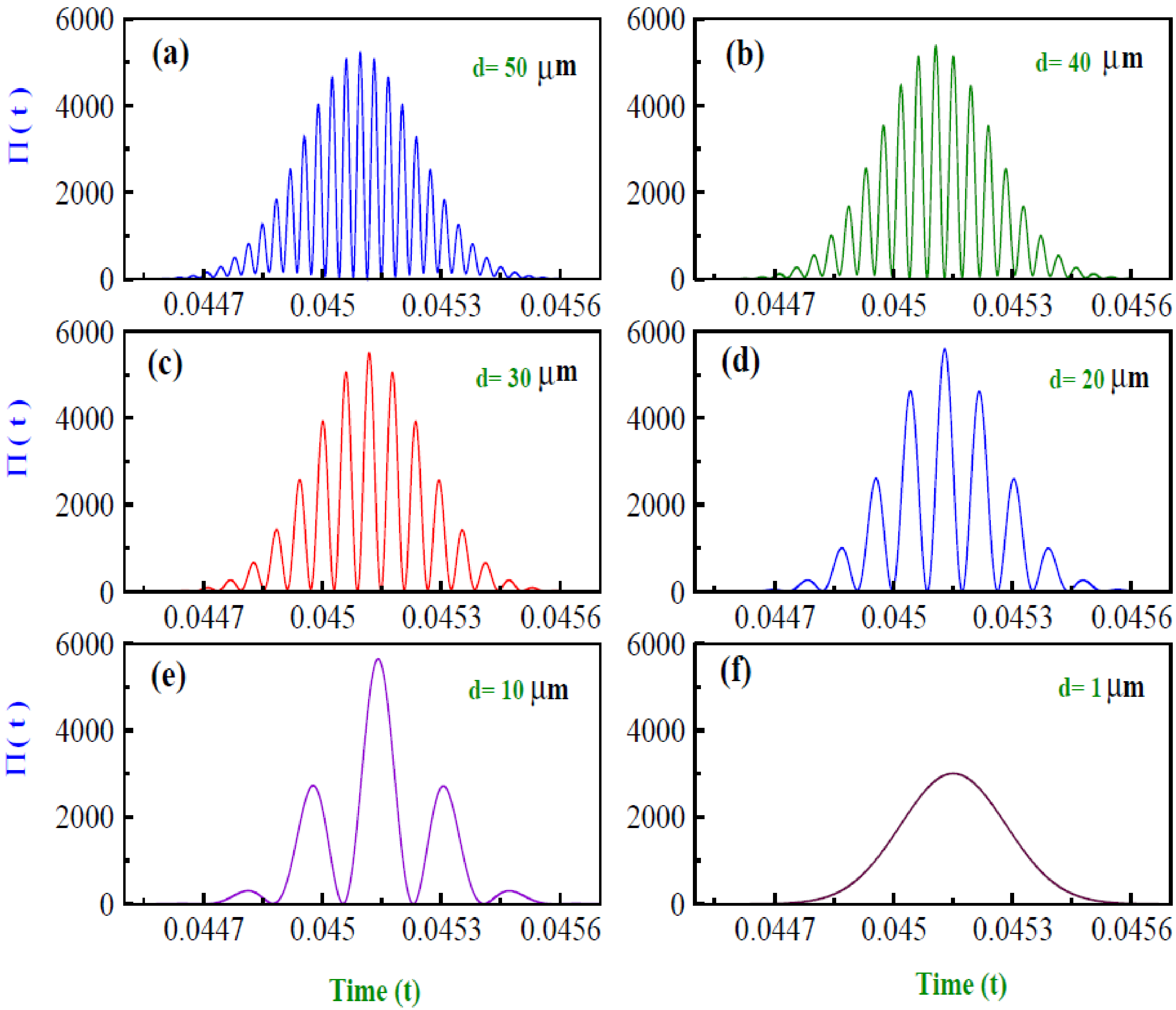
Oscillatory factor

$$\delta_1 = \frac{\hbar t(2zd + d^2 + dgt^2)}{8m\sigma_0^2\sigma^2} = \frac{\hbar t(2zd + d^2 + dgt^2)}{8m(\sigma_0^4 + \hbar^2 t^2 / 4m^2)},$$

$$|\psi_1(z,t)| = \frac{e^{-(z+gt^2/2)^2/(4\sigma^2)}}{(2\pi\sigma^2)^{1/4}},$$

$$|\psi_2(z,t)| = \frac{e^{-(z+d+gt^2/2)^2/(4\sigma^2)}}{(2\pi\sigma^2)^{1/4}}.$$

Quantum TOF distribution $\Pi_1(t)$



Sodium atoms

$$z = H = -1\text{cm}$$

$$\sigma_0 = 1\mu\text{m}$$

Square modulus of the wavefunction

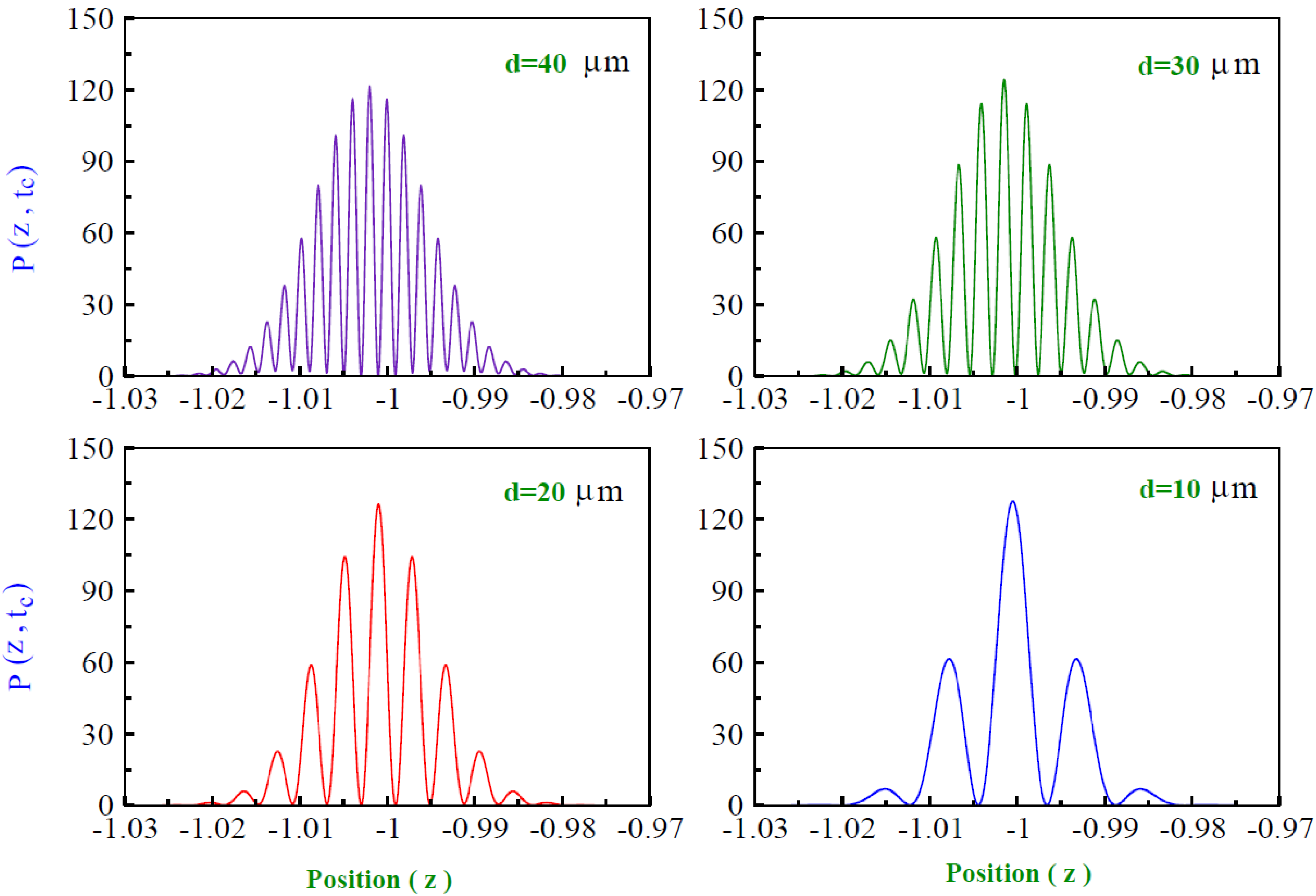
- One can also be tempted to calculate a time distribution from

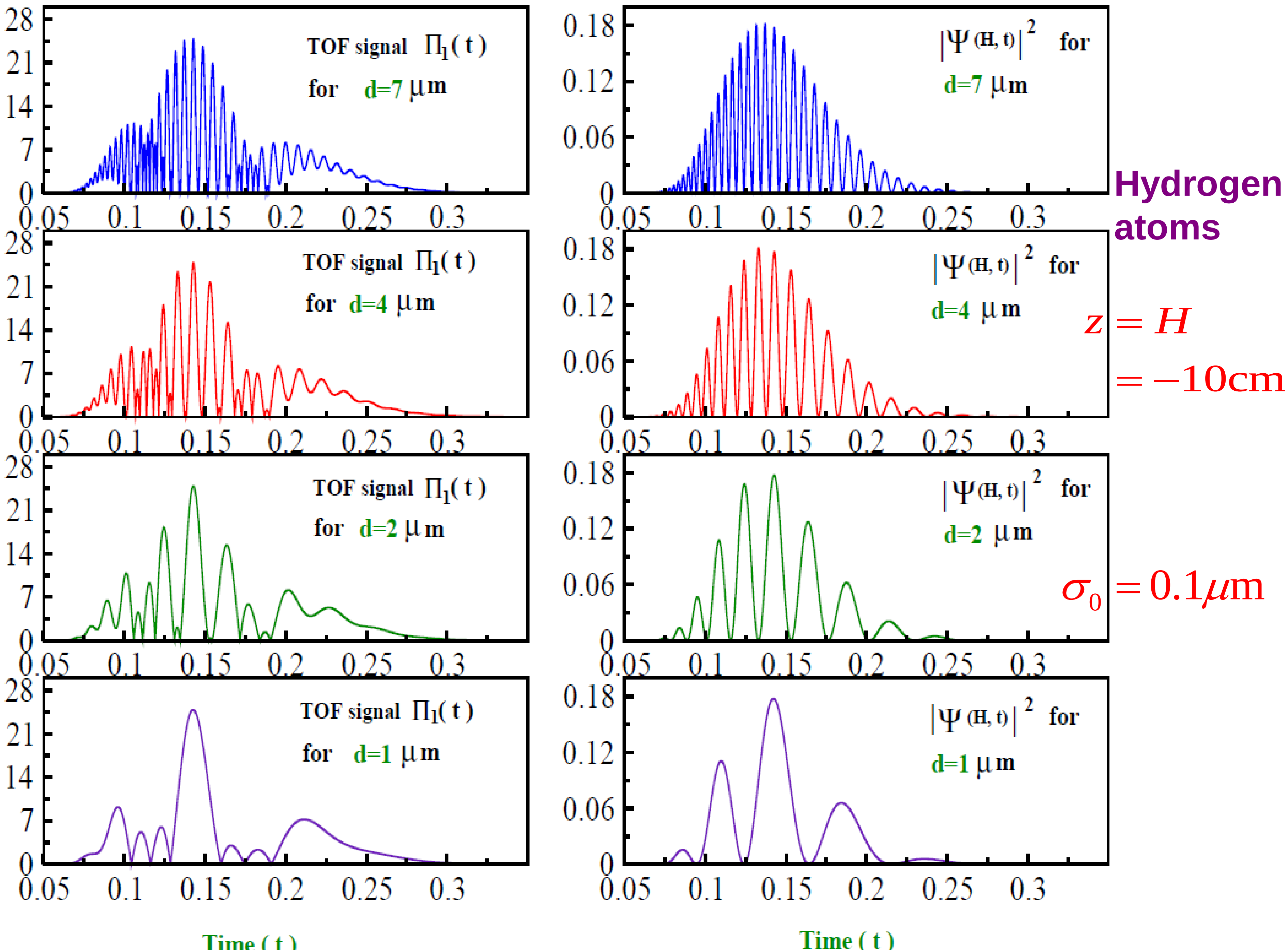
$$\left|\Psi(z,t)\right|^2 = \int_{xy} \left|\Psi(x,y,z,t)\right|^2 dxdy$$

$$= (N^2/2)[|\psi_1(z,t)| + |\psi_1(z,t)| + 2P_{12}(z,t)\cos\delta_1]$$

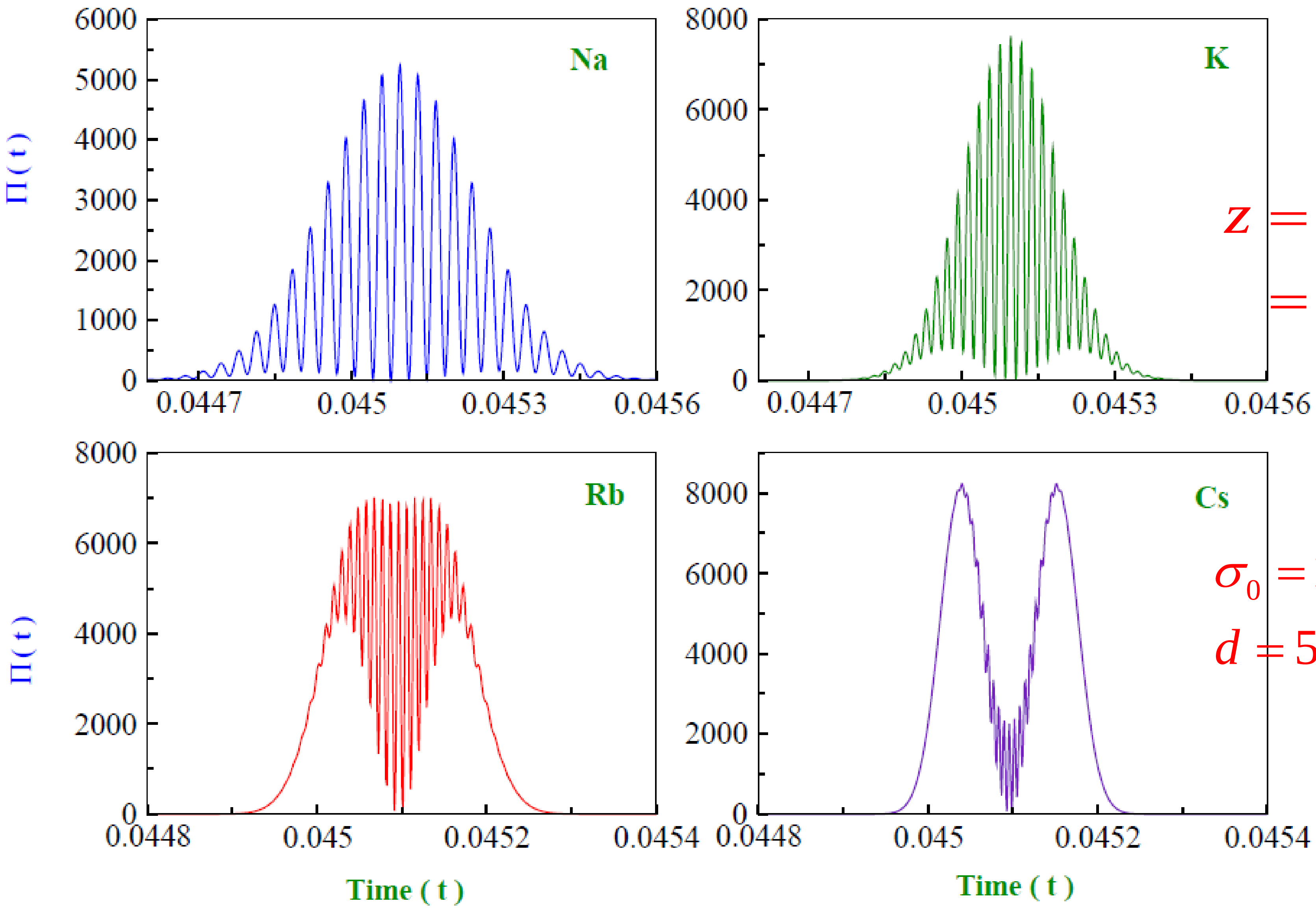
- But $|\Psi\ (H,t)|^2\ dt$ does not provides us a dimensionless probability.
- On the other hand, $\Pi_1(t)$ has the proper dimension (time^{-1}) for the time distribution since $\Pi_1(t)dt$ gives us the probability for the BEC atoms to have the TOF between t and $t + dt$.
- The characteristic behavior and magnitude of the two distribution functions $|\Psi\ (H,t)|^2$ and $\Pi_1(t)$ are not the same.

$$P(z,t_c) = |\Psi(z,t_c)|^2 \quad \text{at } t = t_c = \sqrt{2|H|/g}$$




TOF signal $\Pi_1(t)$ for d=7 μm
$|\Psi(H,t)|^2$ for d=7 μm
TOF signal $\Pi_1(t)$ for d=4 μm
$|\Psi(H,t)|^2$ for d=4 μm
TOF signal $\Pi_1(t)$ for d=2 μm
$|\Psi(H,t)|^2$ for d=2 μm
TOF signal $\Pi_1(t)$ for d=1 μm
$|\Psi(H,t)|^2$ for d=1 μm
0 7 14 21 28
0 0.06 0.12 0.18
0.05 0.1 0.15 0.2 0.25 0.3
Time (t)
Time (t)
Hydrogen atoms
$z = H = -10\text{cm}$
$\sigma_0 = 0.1\mu\text{m}$

Quantum TOF distribution $\Pi_1(t)$

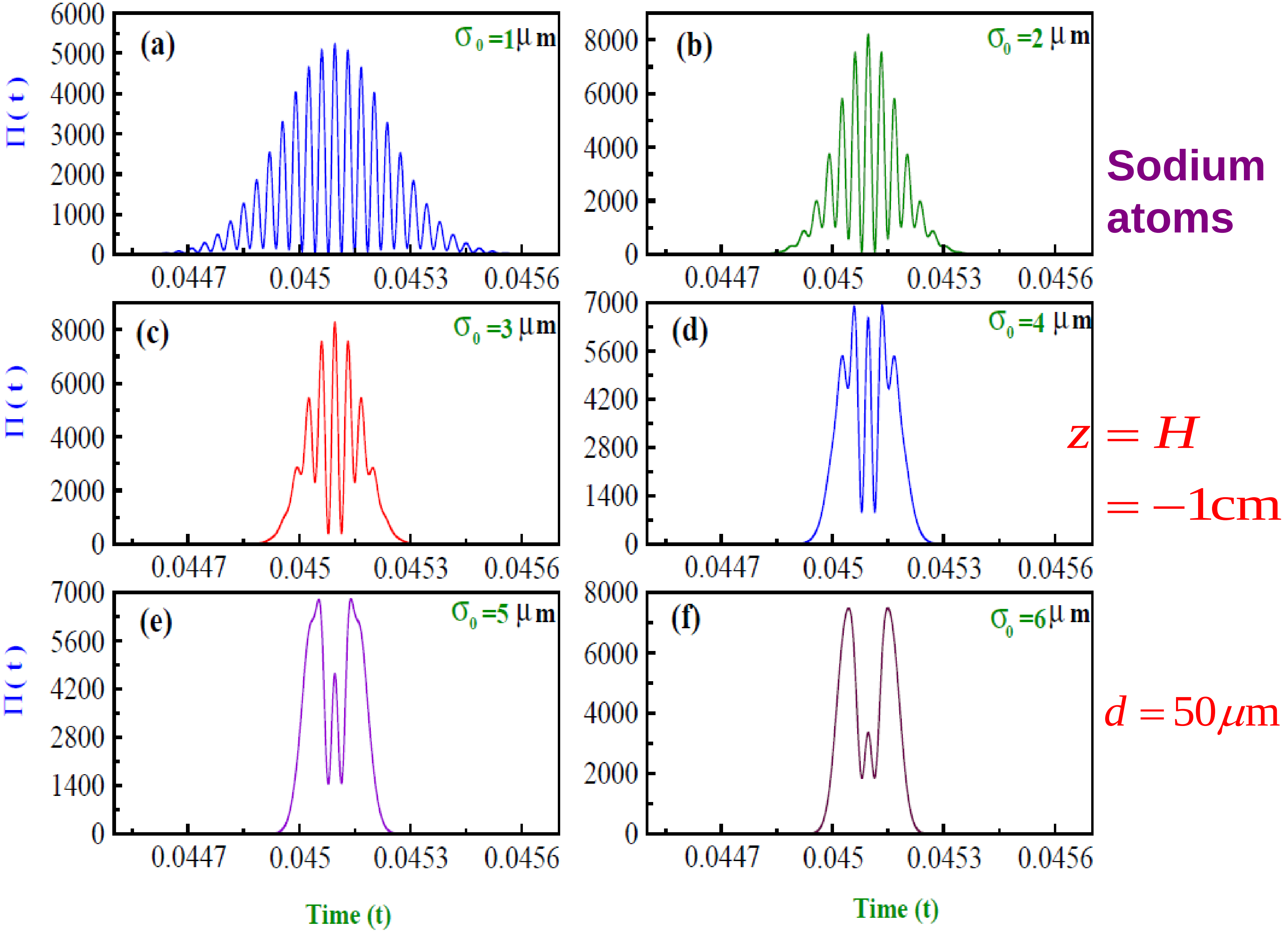


$z = H$

$= -1\text{cm}$

$\sigma_0 = 1\mu\text{m}$

$d = 50\mu\text{m}$



(a)
σ0 =1μm
(b)
σ0 =2μm
(c)
σ0 =3μm
(d)
σ0 =4μm
(e)
σ0 =5μm
(f)
σ0 =6μm
Π(t)
Time (t)
0.0447
0.045
0.0453
0.0456
Sodium atoms
z = H = −1cm
d = 50μm

Factors affecting interference in TOF distribution

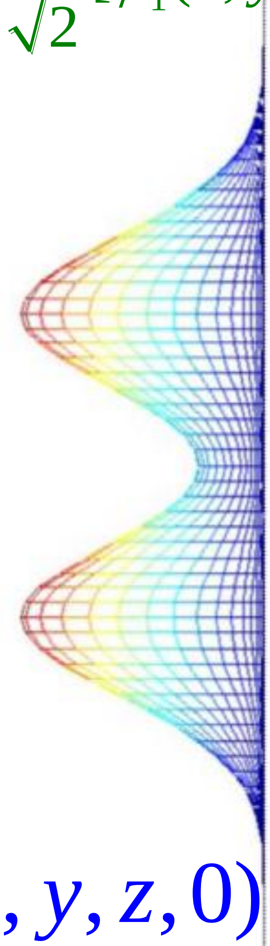
- Interference in $\Pi_1(t)$ arises mainly because of the temporal overlap $P_{12}(H, t)$ and the oscillatory factor δ_1.
- To increase the temporal overlap $P_{12}(H, t)$, one has to find the condition under which the spreading of the wave packet increases: small σ_0, lighter mass atoms (small m), distant detector location (large H) will be helpful in this regard to enhance this effect.
- The oscillatory factor δ_1 can be increased either by reducing the value of σ_0, or by increasing the parameters d and H.
- Higher value of σ_0 (or m) → small σ → small $P_{12}(H, t)$ (should increase H)
- Higher value of σ_0 → small δ_1 ($\sigma_0{}^4$ in denominator should increase H and d)

$$\Psi(x,y,z,t)=\frac{N}{\sqrt{2}}[\psi_1(x,y,z,0)+\psi_1(x,y,z,0)]$$

Vertical Setup 2

x=X

No interference in the TOF distribution



$$N=1/\sqrt{1+e^{-(d^2/8\sigma_0^2)}}$$

(0,0,0)

(0,0,-d)

$$\psi_1(x,y,z,0)=\frac{e^{-(x^2+y^2+z^2)/4\sigma_0^2}}{(2\pi\sigma_0^2)^{3/4}}$$

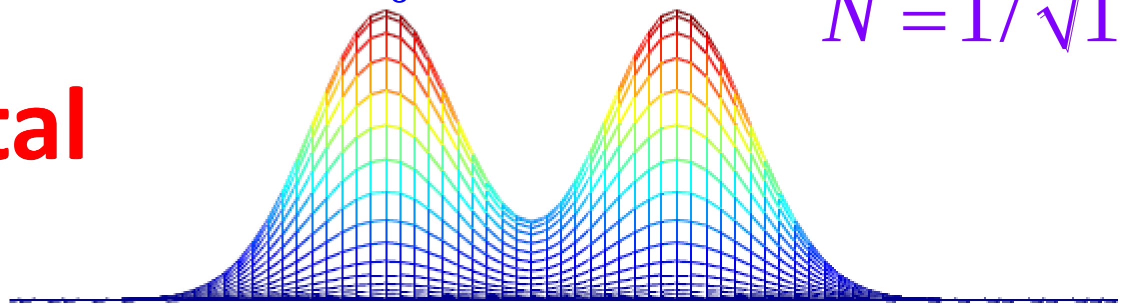
$$\psi_2(x,y,z,0)=\frac{e^{-[x^2+y^2+(z+d)^2]/4\sigma_0^2}}{(2\pi\sigma_0^2)^{3/4}}$$

$$\Psi(x,y,z,t) = \frac{N}{\sqrt{2}}[\psi_1(x,y,z,0) + \psi_1(x,y,z,0)]$$

$$\psi_2(x,y,z,0) = \frac{e^{-[(x+d)^2+y^2+z^2]/4\sigma_0^2}}{(2\pi\sigma_0^2)^{3/4}}$$

$$N = 1/\sqrt{1+e^{-(d^2/8\sigma_0^2)}}$$

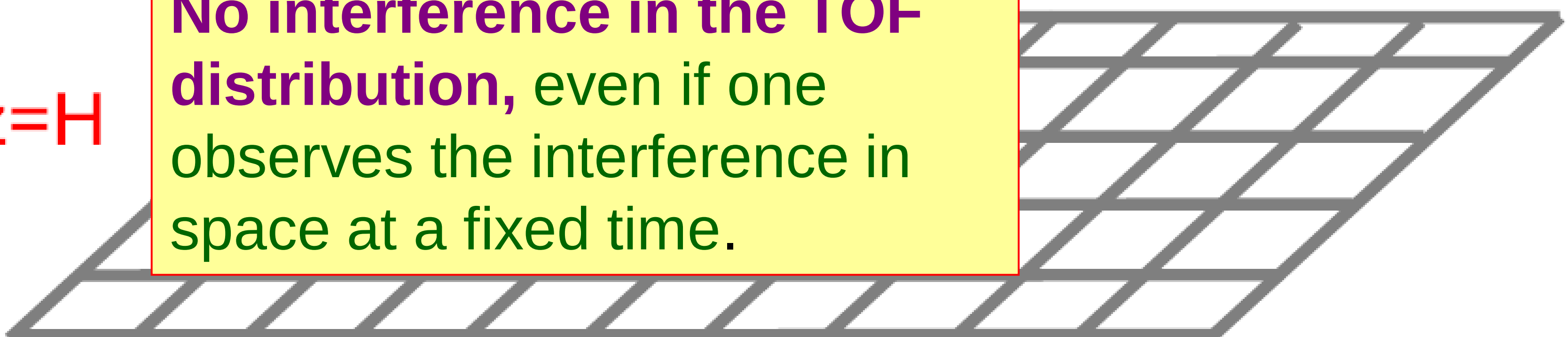
Horizontal Setup 1



(-d,0,0) (0,0,0)

$$\psi_1(x,y,z,0) = \frac{e^{-(x^2+y^2+z^2)/4\sigma_0^2}}{(2\pi\sigma_0^2)^{3/4}}$$

z=H

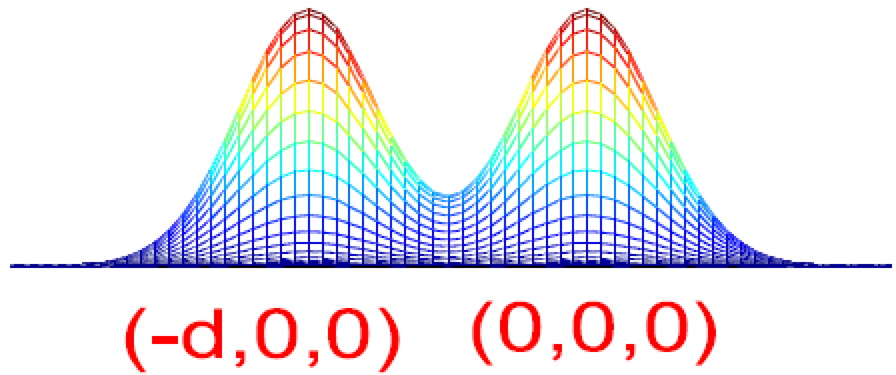


No interference in the TOF distribution, even if one observes the interference in space at a fixed time.

Horizontal Setup 2

$$\Psi(x,y,z,t) = \frac{N}{\sqrt{2}}[\psi_1(x,y,z,0) + \psi_1(x,y,z,0)]$$

$$N = 1/\sqrt{1+e^{-(d^2/8\sigma_0^2)}}$$



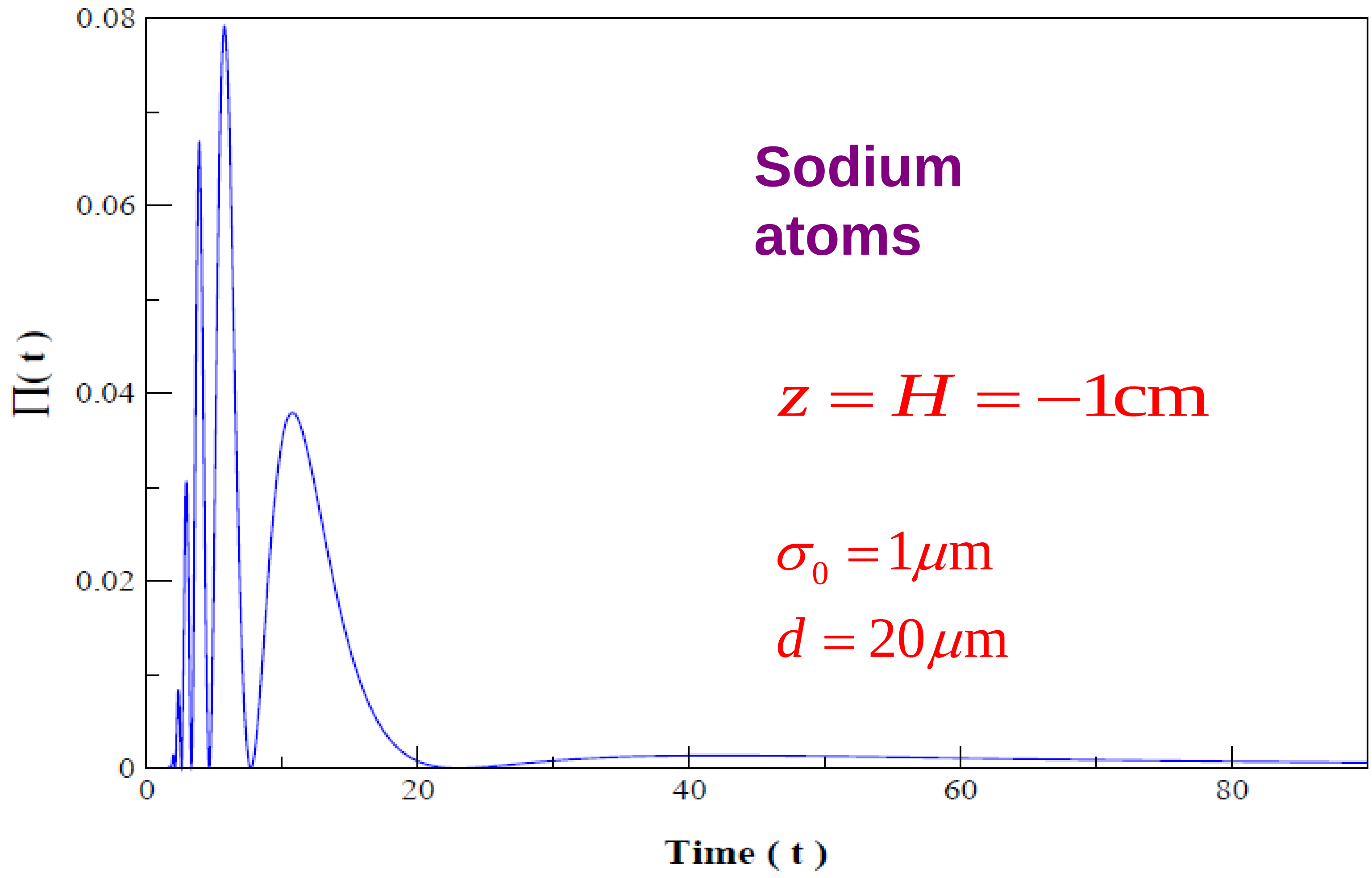
TOF distribution similar to Π_1(t) with no gravity ($g=0$).

x=X

$$\psi_1(x,y,z,0) = \frac{e^{-(x^2+y^2+z^2)/4\sigma_0^2}}{(2\pi\sigma_0^2)^{3/4}}$$

$$\psi_2(x,y,z,0) = \frac{e^{-[(x+d)^2+y^2+z^2]/4\sigma_0^2}}{(2\pi\sigma_0^2)^{3/4}}$$

Quantum TOF distribution $\Pi_1(t)$ in the absence of gravity



Magnitude is roughly 10^5 times smaller

Conclusions (I)

- We propose a scheme to experimentally observe matter-wave interference in the time domain, specifically in the TOF (arrival-time) distribution using atomic BEC.
- This experimentally testable scheme has the potential to empirically resolve ambiguities inherent in the theoretical formulations of the quantum arrival time distribution.
- We use the modulus of the probability current density approach to calculate the quantum TOF distributions for atomic BEC Schrödinger cat represented by superposition of macroscopically separated wave packets in space.

Conclusions (II)

- There is no classical analogue of this TOF distribution $\Pi_1(t)$ and this is purely a quantum distribution where we quantify the matter-wave interference in the quantum TOF signal.
- This approach also provides a proper classical limit as the interference and hence the coherence in the quantum TOF signal disappears in the large-mass limit.
- It will be interesting to see if our prediction of interference in time domain (TOF distribution) can be verified in actual experiments.
- Refs: Md. M. Ali and H.-S. Goan, *J. Phys. A: Math. Theor.* ***42***, 385303 (2009); Md. M. Ali and H.-S. Goan*, **invited book chapter** in “Bose-Einstein Condensates: Theory, Characteristics, and Current Research”, Ed. by P. E. Matthews (Nova Science, New York, 2010)