Valence-bond ground states of isotropic quantum antiferromagnets for universal quantum computation

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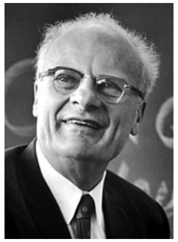
Refs. (1) Wei, Affleck & Raussendorf,
PRL 106, 070501 (2011) and arXiv:1009.2840

(2) Wei & Raussendorf, in preparation

(3) Li, Browne, Kwek Raussendorf
& Wei, arXiv:1102.5153

NTHU, April 11th, 2011

Quantum spin systems



- Bethe solution (1931) on Heisenberg chain

$$H = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$

- Lieb, Schultz & Mattis (1961): XY & Ising-Heisenberg chains & spectral gap

- Haldane (1983): Spectral gap in AF Heisenberg chain is finite for integer spin-S

Quantum spin systems

- Active research in condensed matter, statistical physics & high-energy physics

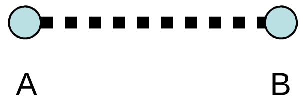
- Rich features:

  - Fluctuations and frustration may prevent Néel order
  - AFM closely related to high-Tc superconductivity
  - Spin liquid
  - Can be simulated by untracold atoms
  - …….

Our focus on antiferromagnets

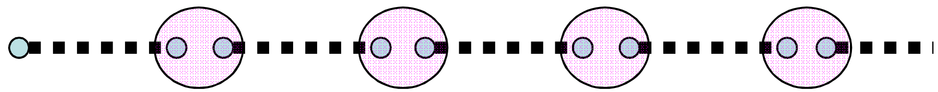
- Valence-bond ground states
  - Simplest valence-bond of two spin-1/2 ➔ singlet state

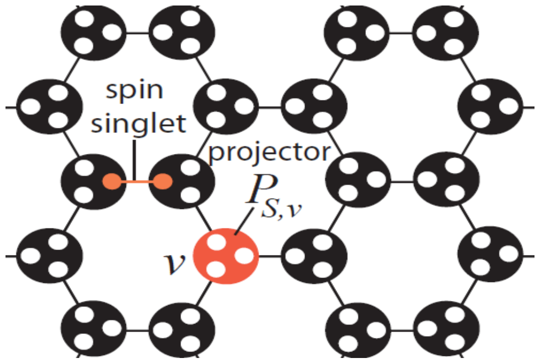
$$|\psi\rangle = |\uparrow_A \downarrow_B\rangle - |\downarrow_A \uparrow_B\rangle = |01\rangle - |10\rangle$$



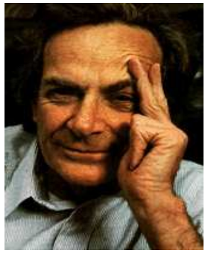
- E.g.:1D and 2D structure

[*AKLT* '87,88]





Quantum computation



Feynman ('81): "Simulating Physics with (Quantum) Computers"

➔ Idea of quantum computer further developed by Deutsch ('85), Lloyd ('96), …

1st conference on Physics and Computation, 1981

Quantum computation

Shor ('94): quantum mechanics enables fast factoring

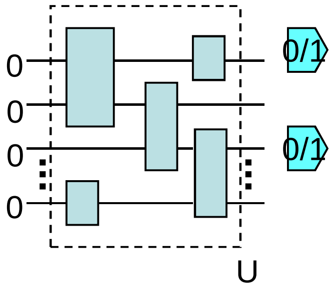
18070820886874048059516561644059055662781025167694013491701270214
5005666254024404838734112759081230337178188796656318201321488055

=(3968599945959745429016112616288378606757644911281006483255157243)
x (4553449864673597218840368689727440886435630126320506960099904459)

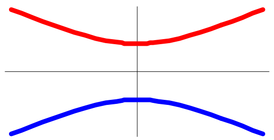
➔ Ever since: rapid growing field of quantum information & computation

- Quantum computational models

1. Circuit model:

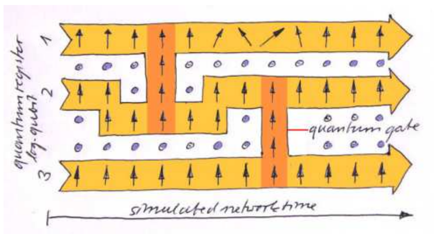


2. Adiabatic QC:



[*Farhi, Goldstone, Gutmann & Sipster* '00]

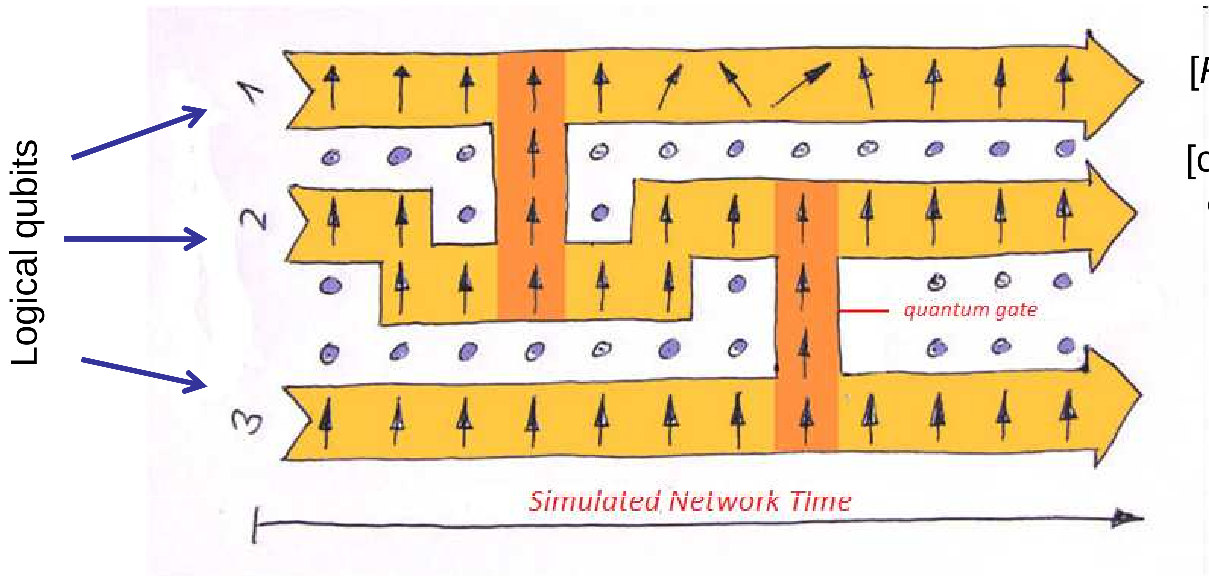
3. Measurement-based:



[*Raussendorf &Briegel* '01]

[c.f. *Gottesman & Chuang*, '99 *Childs, Leung & Nielsen* '04]

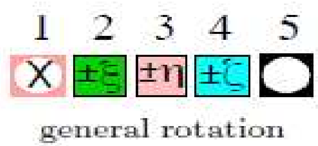
Quantum computation by measurement

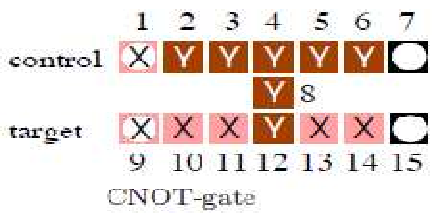


[*Raussendorf & Briegel* '01]

[c.f. *Gottesman & Chuang*, '99
Childs, Leung & Nielsen '04]

- Use cluster state $|\mathcal{C}\rangle$ as computational resource
- Information is written on to $|\mathcal{C}\rangle$, processed and read out <u>all</u> by single spin measurements
- Key points: measurement patterns for 1- and 2-qubit gates (universal gates)





Two-qubit universal gates

- Four by four unitary matrices (acting on the two qubits)

  - ✓ Control-NOT gate:

    0 0 → 0 0
    0 1 → 0 1
    1 0 → 1 1
    1 1 → 1 0

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & X \end{array}\right)$$

  - ✓ Control-Phase gate:

    0 0 → 0 0
    0 1 → 0 1
    1 0 → 1 0
    1 1 → -1 1

$$\mathrm{CP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & Z \end{array}\right)$$

- Generate entanglement

$$|+\rangle|+\rangle = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \xrightarrow{\text{CP}} \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle) \neq |\phi_1\rangle|\phi_2\rangle$$

Entanglement: state preparation

- Entangled state has strong correlation

$$|+\rangle_1|+\rangle_2 \xrightarrow{CP} |\psi\rangle_{12} = |+\rangle_1|0\rangle_2 + |-\rangle_1|1\rangle_2$$

- Measurement on 1st qubit in basis

$$|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$$

✓ If outcome = +: the second qubit becomes

$${}_1\langle+| \cdot |\psi\rangle_{12} \sim |0\rangle_2$$

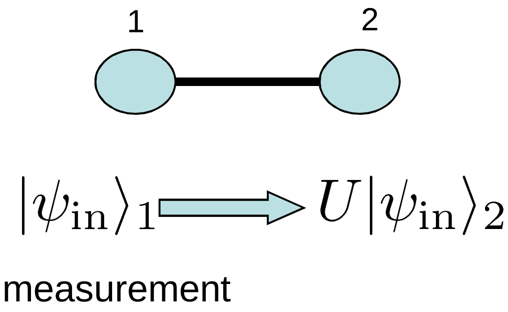
✓ If outcome = -: the second qubit becomes

$${}_1\langle-| \cdot |\psi\rangle_{12} \sim |1\rangle_2 = X|0\rangle_2$$

Unitary operation by measurement?

- Intuition: entanglement

$$(a|0\rangle + b|1\rangle)\,|+\rangle \xrightarrow{CP} |\psi\rangle = a|0\rangle|+\rangle + b|1\rangle|-\rangle$$



  - Measurement on 1st qubit in basis

$$|\pm\xi\rangle \equiv (|0\rangle \pm e^{i\xi}|1\rangle)/\sqrt{2}$$

✓ If outcome=+ξ: an effective rotation applied:

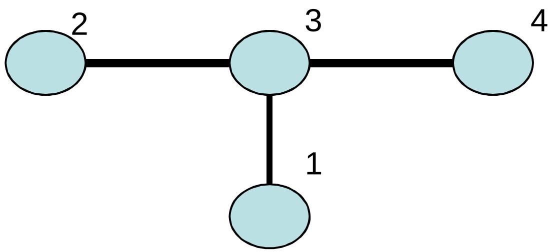
$$_1\langle +\xi|\psi\rangle_{12} \sim a|+\rangle_2 + b\,e^{-i\xi}|-\rangle_2 = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\xi} \\ 1 & -e^{-i\xi} \end{pmatrix}(a|0\rangle_2 + b|1\rangle_2)$$

CNOT by measurement

- Consider initial state

$$(a|0\rangle + b|1\rangle)_1 \, (c|0\rangle + d|1\rangle)_2 \, |+\rangle_3 \, |+\rangle_4$$

$$\xrightarrow{CP_{23}\, CP_{13}\, CP_{34}} \quad |\psi\rangle_{1234}$$



$$|\psi_{\rm in}\rangle_{12} \Longrightarrow U|\psi_{\rm in}\rangle_{14}$$

  - Measurement on 2nd and 3rd qubits in basis

$$|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$$

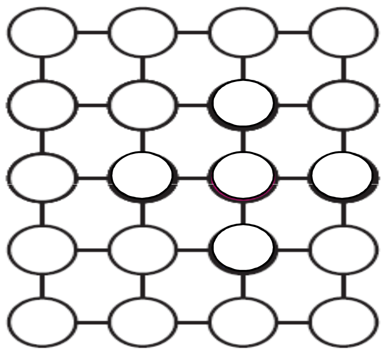
If outcome=++: an effective CNOT applied:

$${}_{23}\langle + + |\psi\rangle_{1234} \sim {\rm CNOT}_{14}(a|0\rangle_1 + b|1\rangle_1)(c|0\rangle_4 + d|1\rangle_4)$$

- Note the action of CP gates can be pushed up front

Cluster state is a resource for quantum computation

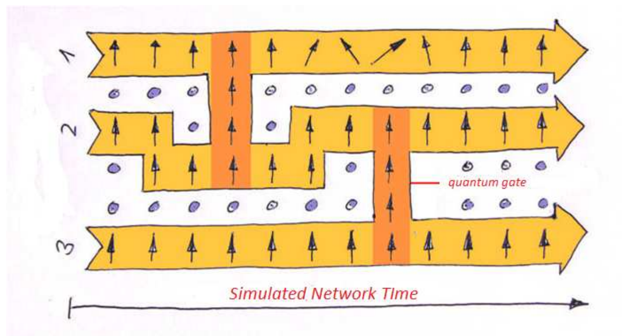
- All the "linking" (by CP) can be done in the beginning; this gives rise to a 2D cluster state:



$$|\mathcal{C}\rangle = \prod_{\text{edge } \langle i,j \rangle} CP_{ij} \left(|+\rangle |+\rangle \cdots |+\rangle \right)$$

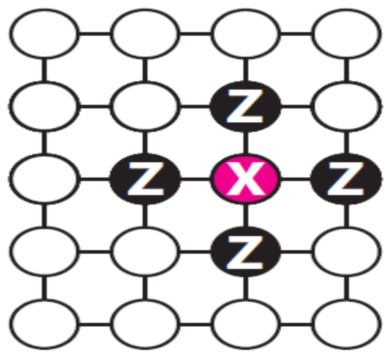
- The whole entangled state is created first and subsequent operations are single-qubit measurements

➔ pattern of measurement gives computation (i.e. simulates a circuit)



Cluster state: special case of graph states

$$|\mathcal{C}\rangle = \prod_{\text{edge }\langle i,j\rangle} CP_{ij}\left(|+\rangle|+\rangle\cdots|+\rangle\right)$$



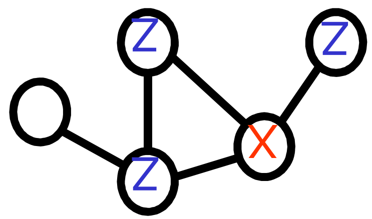
- Use the equality

$$CP_{uv}\, X_u = X_u \otimes Z_v\, CP_{uv}$$

➔ $$\Big(\prod_{\text{edge }\langle i,j\rangle} CP_{ij}\Big) X_u = \Big(X_u \prod_{v\in \mathrm{Nb}(u)} Z_v\Big) \prod_{\text{edge }\langle i,j\rangle} CP_{ij}$$

- Apply to cluster state*

$$\Big(X_u \prod_{v\in \mathrm{Nb}(u)} Z_v\Big)|\mathcal{C}\rangle = |\mathcal{C}\rangle$$

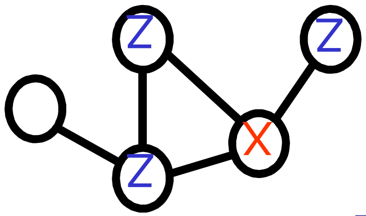


- This definition on graph ➔ graph state

*Note: $X|+\rangle = |+\rangle$

Cluster and graph states as ground states

❑ Graph state: defined on a graph [*Hein, Eisert & Briegel* 04']



$$H_G = -\sum_{\mathrm{site}\, v} K_v \quad \text{with} \quad K_v \equiv X_v \bigotimes_{u \in \mathrm{Nb}(v)} Z_u$$

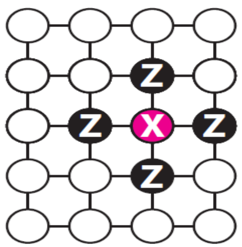
neighbors

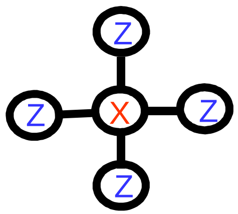
➔ Graph state is the unique ground state of H_G

$$K_v|G\rangle = |G\rangle, \ \forall\, \mathrm{site}\, v$$

❑ Cluster state |C › = graph state on square lattice

[*Raussendorf &Briegel*, 01']





$$|\mathcal{C}\rangle = |\mathcal{C}\rangle$$

- Cluster state is the unique ground state of five-body interacting Hamiltonian (cannot be that of two-body)☹ [*Nielsen* '04]
- Where to look for other resource states?

Note: X, Y & Z are Pauli matrices

Search for universal resource states?

- The first known resource state is the 2D cluster state

[*Raussendorf & Briegel* 01']

- Other known examples:
  - Any other 2D *graph* states* on regular lattice: triangular, honeycomb, kagome, etc. [*Van den Nest et al.* '06]
  - MPS & PEPS framework

    [*Verstraete & Cirac* '04] [*Gross & Eisert* '07, '10]

- Can universal resource states be ground states?

→ Create resources by cooling!

→ Desire simple and short-ranged (nearest nbr) 2-body Hamiltonians

Outline

I. Introduction: quantum spins & quantum computation

II. Valence-bond states (AKLT construction)

- Ground states of isotropic antiferromagnet

III. Valence-bond ground states for quantum computation

- Preprocessing: generalized measurement
- Give rise to graph states and cluster states
  (graph properties and percolation argument)

IV. Summary and outlook

A new direction: valence-bond ground states of isotropic antiferromagnet

- = AKLT (Affleck-Kennedy-Lieb-Tasaki) states [*AKLT* '87,88]

- States of spin 1,3/2, or higher (defined on any lattice)

  ➔ Unique* ground states of two-body isotropic Hamiltonians

$$H = \sum_{\langle i,j \rangle} f(\vec{S}_i \cdot \vec{S}_j)$$ *f(x)* is a polynomial

- Important progress on 1D AKLT states:

  [*Gross & Eisert, PRL* '07] [*Brennen & Miyake, PRL* '09]

  [*Miyake*, *PRL* '10] [*Bartlett et al. PRL* '10]

  ➔ Can be used to implement rotations on single-qubits

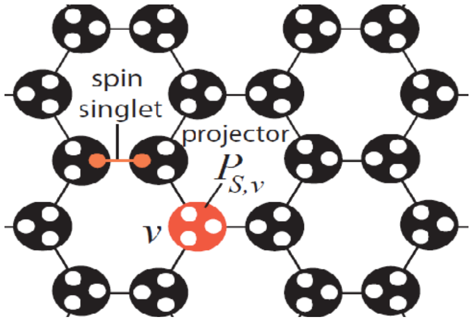
*with appropriate boundary conditions

1D: Single-qubit rotations not sufficient

- Key question: can any of AKLT states provide a resource for universal quantum computation?
  - Need 2D structure
  - We show that the spin-3/2 2D AKLT state on honeycomb lattice *is* such a resource state

    [<u>*Wei*</u>*, Affleck & Raussendorf,*
    arXiv1009.2840 and PRL106, 070501 (2011)]

    [*Alternative proof: Miayke, arXiv 1009.3491*]



  - Results beyond honeycomb & 3D
    (including thermal noise & fault tolerance)

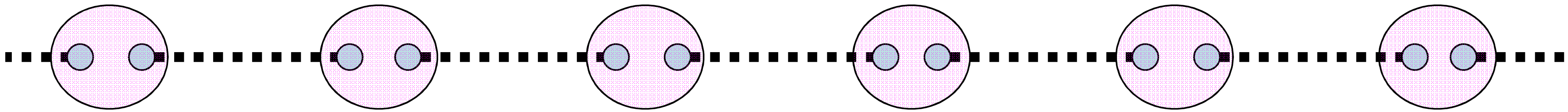
    [<u>*Wei*</u> *& Raussendorf,* in preparation]

    [*Li, Browne, Kwek, Raussendorf,*
    <u>*Wei*</u>, arXiv:1102.5153]

1D AKLT state

[*AKLT* '87,'88]

- Spin-1 chain: two virtual qubits per site



$$\frac{1}{2}\otimes\frac{1}{2}=0\oplus 1$$

Project into symmetric subspace of two spin-1/2 (qubits)

$$|00\rangle \Rightarrow |1,1\rangle$$

$$|11\rangle \Rightarrow |1,-1\rangle$$

$$(|01\rangle+|10\rangle)/\sqrt{2} \Rightarrow |1,0\rangle$$

singlet

$$|01\rangle-|10\rangle=|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle$$

- Ground state of two-body interacting Hamiltonian (with a gap)

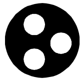
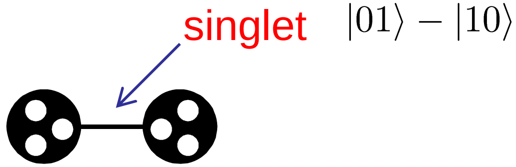
$$H=\sum_i\left[\vec{S}_i\cdot\vec{S}_{i+1}+\frac{1}{3}(\vec{S}_i\cdot\vec{S}_{i+1})^2+\frac{2}{3}\right]=2\sum_i \hat{P}^{(S=2)}_{i,i+1}$$

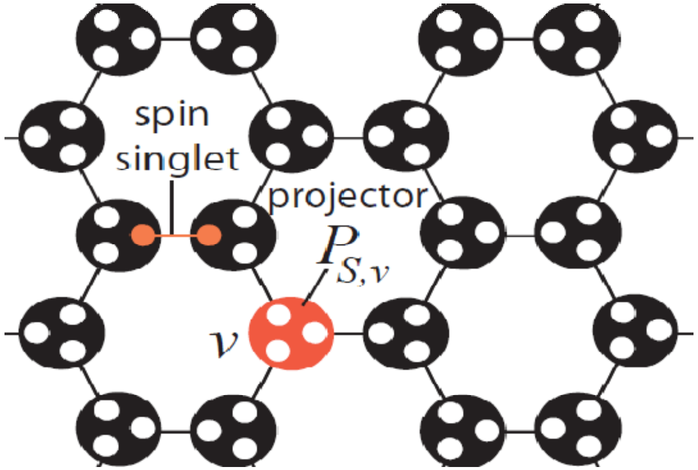
projector onto S=2

➔ Can realize rotation on one logical qubit by measurement
(not sufficient for universal QC)

[*Gross & Eisert, PRL* '07] [*Brennen & Miyake, PRL* '09]

2D universal resource: Spin-3/2 AKLT state on honeycomb

- Each site contains three virtual qubits 
- Two virtual qubits on an edge form a singlet 



Spin 3/2 and three virtual qubits

- Addition of angular momenta of 3 spin-1/2's

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \qquad \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$$

Symmetric subspace

- The four basis states in the symmetric subspace

$$|000\rangle \leftrightarrow \left|\frac{3}{2}, \frac{3}{2}\right\rangle$$

$$|W\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \leftrightarrow \left|\frac{3}{2}, \frac{1}{2}\right\rangle$$

$$|\overline{W}\rangle \equiv \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle) \leftrightarrow \left|\frac{3}{2}, -\frac{1}{2}\right\rangle$$

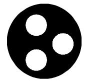
$$|111\rangle \leftrightarrow \left|\frac{3}{2}, -\frac{3}{2}\right\rangle$$

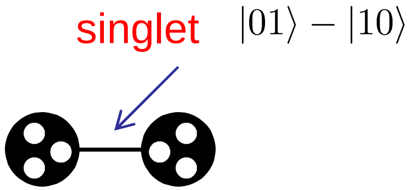
Effective 2 levels of a qubit

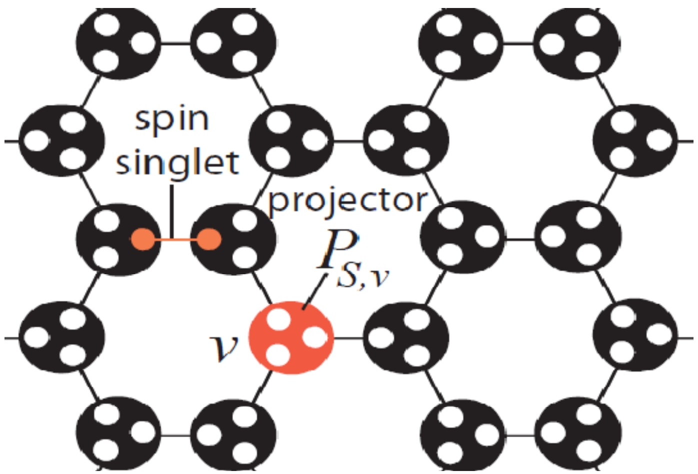
- Projector onto symmetric subspace

$$P_{S,v} = |000\rangle\langle 000| + |111\rangle\langle 111| + |W\rangle\langle W| + |\overline{W}\rangle\langle\overline{W}| \leftrightarrow I_{3/2}$$

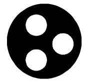
Spin-3/2 AKLT state on honeycomb

- Each site contains three virtual qubits 

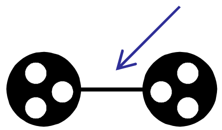
- Two virtual qubits on an edge form a singlet 



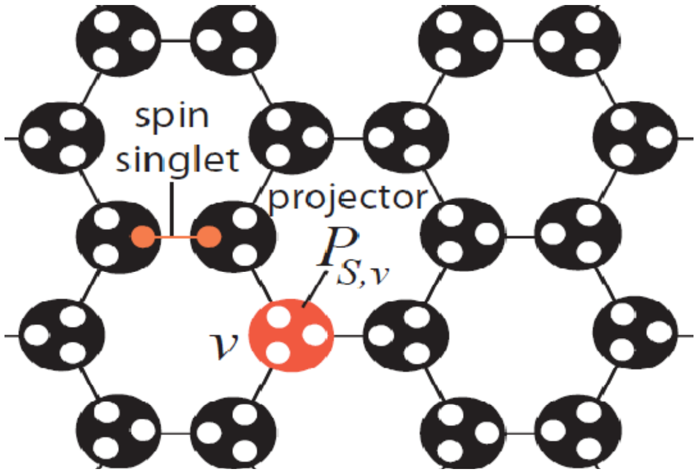
Spin-3/2 AKLT state on honeycomb

- Each site contains three virtual qubits 

singlet $|01\rangle - |10\rangle$

- Two virtual qubits on an edge form a singlet 

- Projection ($P_{S,v}$) onto symmetric subspace of 3 qubits at each site & relabeling with spin-3/2 (four-level) states



$$P_{S,v} = |000\rangle\langle 000| + |111\rangle\langle 111| + |W\rangle\langle W| + |\overline{W}\rangle\langle \overline{W}|$$

$$|000\rangle \leftrightarrow \left|\frac{3}{2}, \frac{3}{2}\right\rangle \qquad |111\rangle \leftrightarrow \left|\frac{3}{2}, -\frac{3}{2}\right\rangle$$

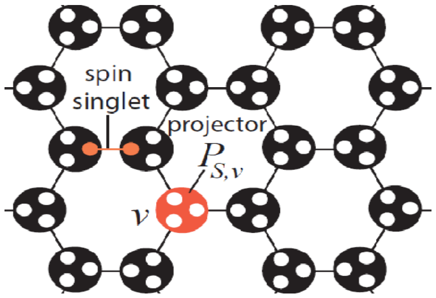
$$|W\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \leftrightarrow \left|\frac{3}{2}, \frac{1}{2}\right\rangle$$

$$|\overline{W}\rangle \equiv \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle) \leftrightarrow \left|\frac{3}{2}, -\frac{1}{2}\right\rangle$$

Spin-3/2 AKLT state on honeycomb

- (With appropriate BC) unique ground state of [*AKLT* '87,'88]

$$H = \sum_{\text{edge } \langle i,j \rangle} \hat{P}^{(S=3)}_{i,j} = \sum_{\text{edge } \langle i,j \rangle} \left[\vec{S}_i \cdot \vec{S}_j + \frac{116}{243} (\vec{S}_i \cdot \vec{S}_j)^2 + \frac{16}{243} (\vec{S}_i \cdot \vec{S}_j)^3 + \frac{55}{108} \right]$$



- Quantum disordered state (w/o Néel order): via mapping to a 2D classical model at finite T

[*Arovas, Auerbach & Haldane* '88,
Parameswaran, Sondhi & Arovas '09]

$$H_{\text{cl}} = - \sum_{\text{edge } \langle i,j \rangle} \ln \left(\frac{1 - \hat{n}_i \cdot \hat{n}_j}{2} \right)$$

$\hat{n}_i$: classical unit vector

- Exponential decay of correlation (gap not proved yet) [*AKLT* '87,'88]

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  (graph properties and percolation argument)

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Our strategy for universality

Show the 2D AKLT state can be locally converted to a 2D cluster state (known resource state)

- Spin 3/2 (4 levels) → Spin ½ (2 levels)?

  ➔ Need "projection" into smaller subspace

- We use generalized measurement (or POVM)

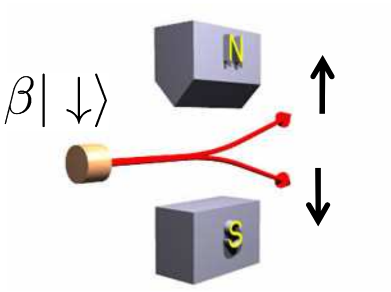
  ➔ Random outcome gives 2D graph state (graph modified from honeycomb)

- Use percolation argument :
  ➔ typical random graph state converted to cluster state

Measurement

[See e.g. *Nielsen & Chuang*]

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$



$$F_\uparrow = |\uparrow\rangle\langle\uparrow|,\ F_\downarrow = |\downarrow\rangle\langle\downarrow|$$

$$|\psi\rangle|\mathrm{blank}\rangle_m \xrightarrow{\text{unitary}} F_\uparrow|\psi\rangle \otimes |0\rangle_m + F_\downarrow|\psi\rangle \otimes |1\rangle_m$$

$$= \alpha|\uparrow\rangle \otimes |0\rangle + \beta|\downarrow\rangle \otimes |1\rangle_m$$

- Reading “measuring device” value ➔ infer outcome

- Conservation of probability $\sum F_a^\dagger F_a = I \qquad p_a = \langle\psi|F_a^\dagger F_a|\psi\rangle$

- *F* ’s need not be orthogonal

  ➔ generalized measurement (POVM)

- Outcome *a* ➔ state becomes $|\psi\rangle \longrightarrow F_a|\psi\rangle$

The POVM (spin-3/2 version)

$$F_{v,z} = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_z + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_z\right) = \frac{1}{\sqrt{6}}\left(S_z^2 - \frac{1}{4}\right)$$

$$F_{v,x} = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_x + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_x\right) = \frac{1}{\sqrt{6}}\left(S_x^2 - \frac{1}{4}\right)$$

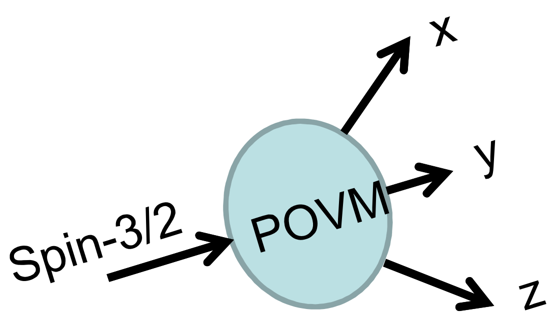
v: site index

$$F_{v,y} = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_y + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_y\right) = \frac{1}{\sqrt{6}}\left(S_y^2 - \frac{1}{4}\right)$$

[*Wei*,*Affleck & Raussendorf* '10; *Miyake* '10]

➔ Three elements satisfy: $F^\dagger_{v,x}F_{v,x} + F^\dagger_{v,y}F_{v,y} + F^\dagger_{v,z}F_{v,z} = I_v$

❑ POVM outcome (*x*,*y*, or *z*) is random (a_v ={x,y,z} ϵ *A* for all sites v)



➔ effective 2-level system

$$\left|\frac{3}{2}\right\rangle_{a_v} \leftrightarrow \left|000\right\rangle, \quad \left|-\frac{3}{2}\right\rangle_{a_v} \leftrightarrow \left|111\right\rangle$$

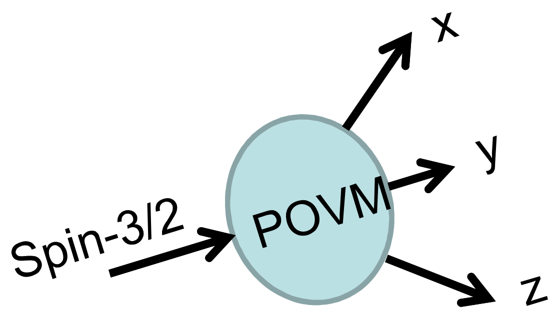
➔ a_v : new quantization axis

$$\overline{Z} \equiv \left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_{a_v} - \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_{a_v} \qquad \overline{X} \equiv \left|\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_{a_v} + \left|-\frac{3}{2}\right\rangle\left\langle+\frac{3}{2}\right|_{a_v}$$

➔ state becomes $|\Phi\rangle \longrightarrow F_{v,a_v}|\Phi\rangle$

Post-POVM state

- Outcome a_v ={x,y,z} ϵ *A* for all sites v



$$|\Psi(\mathcal{A})\rangle = \bigotimes_v F_{v,a_v} |\Phi_{\rm AKLT}\rangle$$

$$\sim \bigotimes_v \left(S^2_{v,a_v} - \frac{1}{4}\right) |\Phi_{\rm AKLT}\rangle$$

[*Wei, Affleck & Raussendorf* , arxiv'10 & PRL'11]

➔ What is this state?

The random state is an encoded graph state

[*Wei, Affleck & Raussendorf* , arxiv'10 & PRL'11]

- Outcome a_v ={x,y,z} ϵ *A* for all sites v

$$|\Psi(\mathcal{A})\rangle = \bigotimes_v F_{v,a_v} |\Phi_{\mathrm{AKLT}}\rangle \sim \bigotimes_v \left(S^2_{v,a_v} - \frac{1}{4} \right) |\Phi_{\mathrm{AKLT}}\rangle$$

- Encoding: effective two-level (qubit) is delocalized to a few sites

  ➔ Property of AKLT ("antiferromagnetic" tendency) gives us insight on encoding

- What is the graph? Isn't it honeycomb?

  ➔ Due to delocalization of a "logical" qubit, the graph is modified

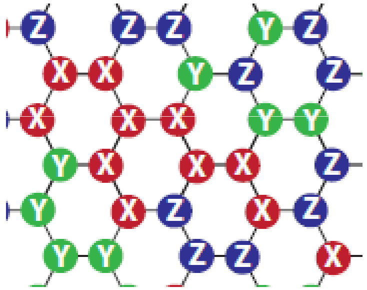
Encoding of a qubit: AFM ordering

- AKLT: Neighboring sites cannot have the same $S_a = \pm 3/2$

[*AKLT* '87,'88]

➔ Neighboring sites with same POVM outcome *a* = x, y or z: only two AFM orderings (call these site form a ***domain***):

$$|\bar{0}\rangle \equiv \left|\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}, \cdots\right\rangle_a \quad \text{or} \quad |\bar{1}\rangle \equiv \left|-\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, \cdots\right\rangle_a$$



➔ Form the basis of a qubit

- Effective Pauli Z and X operators become (extended)

$$\bar{Z} = |\bar{0}\rangle\langle\bar{0}| - |\bar{1}\rangle\langle\bar{1}| \qquad \bar{X} = |\bar{0}\rangle\langle\bar{1}| + |\bar{1}\rangle\langle\bar{0}|$$

- A domain can be reduced to a single site by measurement

➔ Regard a domain as a single qubit

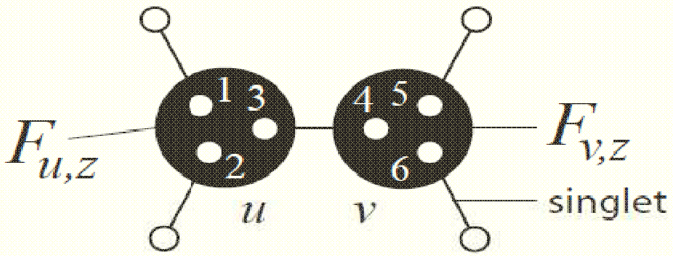
Qubit Encoding: Stabilizer formalism*

[*Gottesman '97]

$$|\Psi(\mathcal{A})\rangle = \bigotimes_{v\in V(\mathcal{L})} F_{v,a_v} |\Phi_{\text{AKLT}}\rangle = \bigotimes_{v\in V(\mathcal{L})} F_{v,a_v} \bigotimes_{e\in E(\mathcal{L})} |\phi\rangle_e$$

singlet $|01\rangle - |10\rangle$

- Example:



$a_u = z$ $a_v = z$

$$F_{u/v,z} = \sqrt{\frac{2}{3}}(|000\rangle\langle 000| + |111\rangle\langle 111|)$$

1. Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6
   - Stabilizers of $|\Psi(\mathcal{A})\rangle$

   as, e.g.,

   $$Z_1Z_2F_{u,z} = F_{u,z}$$

2. $-Z_3Z_4$ stabilizer of singlet (3,4) $|0_3 1_4\rangle - |1_3 0_4\rangle$ & commutes w. *F*'s

➔ Stabilizer of $|\Psi(\mathcal{A})\rangle$

3. Stabilizers of 1&2 give rise to one-qubit encoding:

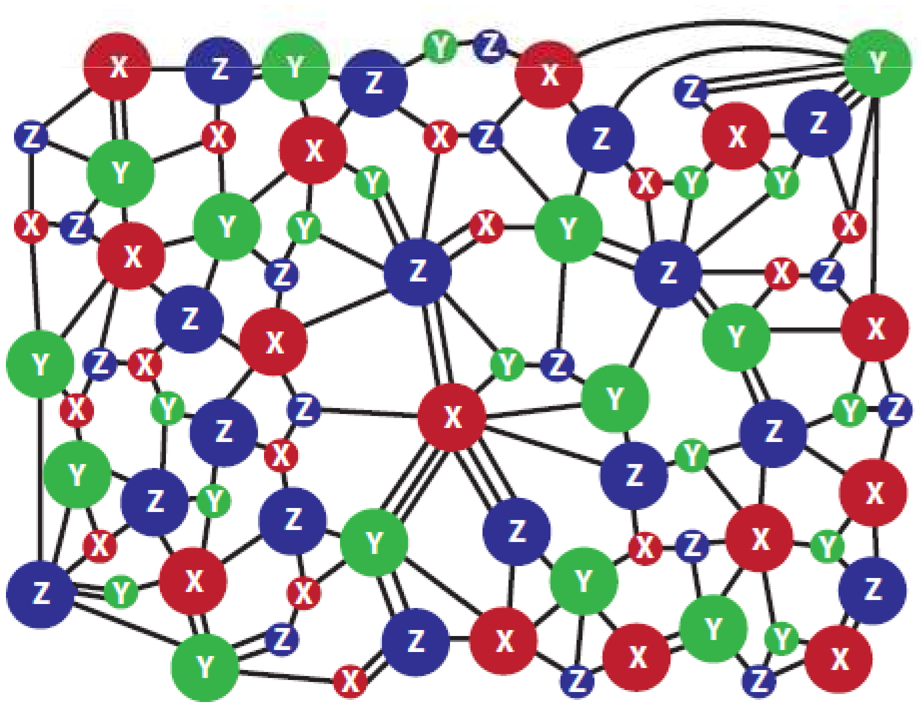
$$\alpha|(000)_u(111)_v\rangle + \beta|(111)_u(000)_v\rangle$$ ➔ AFM order among groups

Rule 1: merging sites of same outcome

- Post-POVM state $|\Psi(\mathcal{A})\rangle = \bigotimes_{v\in V(\mathcal{L})} F_{v,a_v} |\Phi_{\mathrm{AKLT}}\rangle$

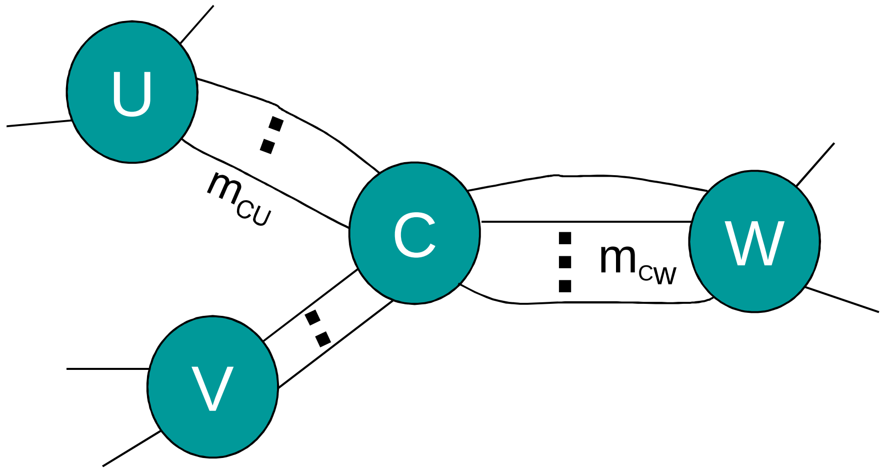
  with POVM outcome a_v = x, y or z

- Neighboring sites w. same POVM ➔ merged to a domain ➔ qubit



Each domain represents a qubit

- Graph structure of domains is modified from honeycomb
- Two domains can have more than one shared edges

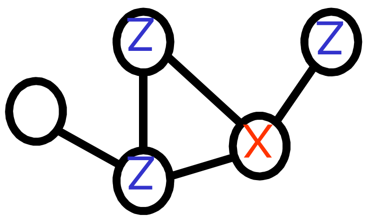


- We show that the post-POVM state satisfies

$$\overline{K}_C\,|\psi(\mathcal{A})\rangle = |\psi(\mathcal{A})\rangle \quad \forall C \in \text{ domains} \qquad \overline{K}_C = \overline{X}_C \bigotimes_{U\in \mathrm{Nb}(V)} (\overline{Z}_U)^{m_{CU}}$$

Recall graph states

- Graph state: defined on a graph [*Hein, Eisert & Briegel* 04']



$$H_G = -\sum_{\text{site}\, v} K_v \quad \text{with} \quad K_v \equiv X_v \bigotimes_{u \in \text{Nb}(v)} Z_u$$

neighbors

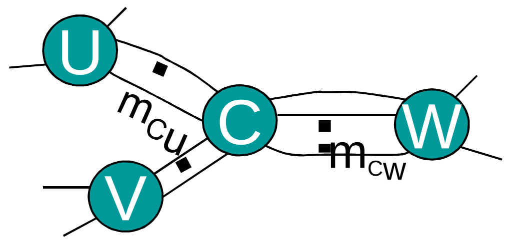
➔ Graph state is the unique ground state of H_G

$$K_v |G\rangle = |G\rangle, \ \forall\, \text{site}\, v$$

- Post-POVM state is a graph state

$$\overline{K}_C\, |\psi(\mathcal{A})\rangle = |\psi(\mathcal{A})\rangle \quad \forall C \in \text{ domains}$$

$$\overline{K}_C = \overline{X}_C \bigotimes_{U \in \text{Nb}(V)} (\overline{Z}_U)^{m_{CU}}$$



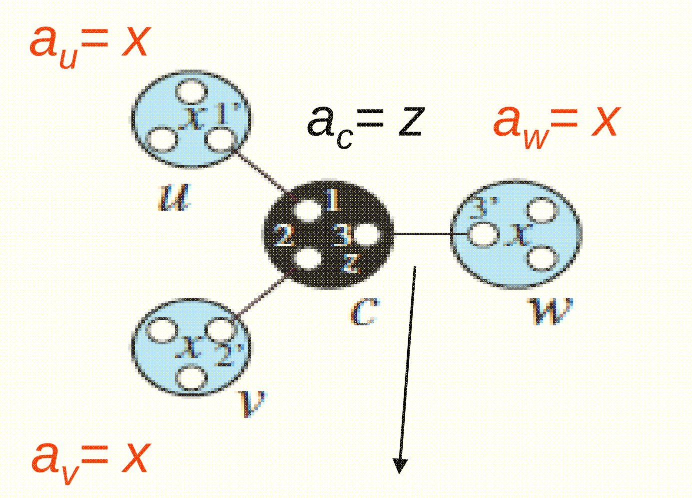
Notice the even & odd: $(\overline{Z}_U)^2 = I$ ➔ even: effectively no edge

➔ odd : effectively one edge

Proving stabilizer of graph states

$$|\Psi(\mathcal{A})\rangle = \bigotimes_{v\in V(\mathcal{L})} F_{v,a_v}\,|\Phi_{\mathrm{AKLT}}\rangle = \bigotimes_{v\in V(\mathcal{L})} F_{v,a_v} \bigotimes_{e\in E(\mathcal{L})} |\phi\rangle_e$$

[example]



$$-X_1 X_{1'}|\phi\rangle_{11'} = |\phi\rangle_{11'}$$

1. Stabilizer of underlying singlets:

$$-X_1X_{1'}\,,\quad -X_2X_{2'}\,,\quad -X_3X_{3'}$$

2. Commute with $F_{u,x}$, $F_{v,x}$, $F_{w,x}$

3. Product $\mathcal{O} \equiv -X_1X_{1'}X_2X_{2'}X_3X_{3'}$

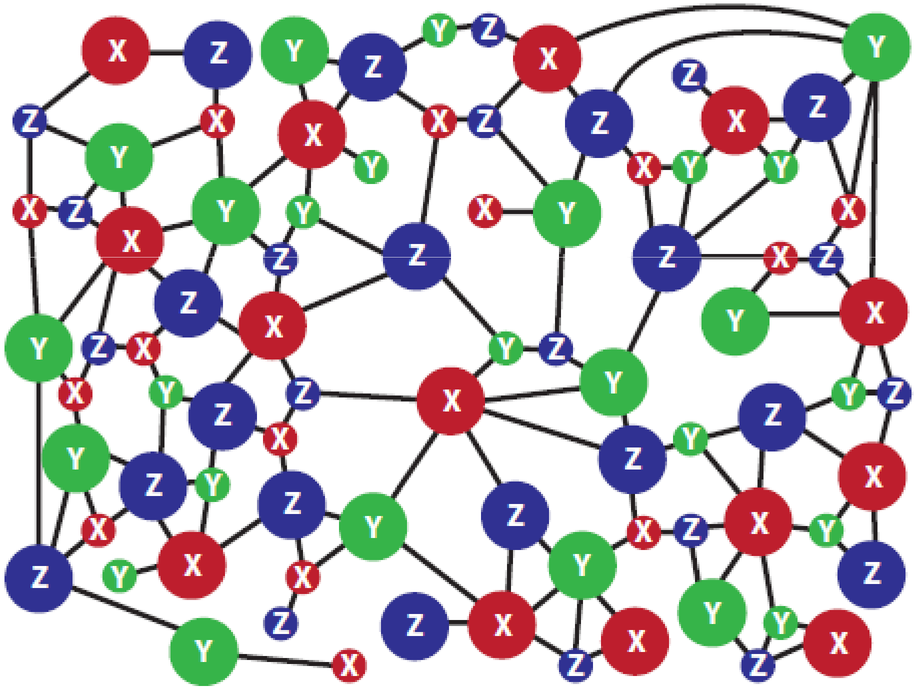
commutes* with $F_{c,z}$ ➔ stabilizer of $|\Psi(\mathcal{A})\rangle$

➔ $$\mathcal{O} \equiv -\underbrace{X_1X_2X_3}_{F_{c,z}}\,X_{1'}X_{2'}X_{3'} = \pm\overline{X}_c\overline{Z}_u\overline{Z}_v\overline{Z}_w$$

* $X\otimes X\otimes X(|000\rangle\langle 000| + |111\rangle\langle 111|) = (|000\rangle\langle 000| + |111\rangle\langle 111|)X\otimes X\otimes X$

Rule 2: modulo-2 on inter-domain edges

$$\overline{K}_C \,|\psi(\mathcal{A})\rangle = |\psi(\mathcal{A})\rangle \quad \forall C \in \text{ domains} \qquad \overline{K}_C = \overline{X}_C \bigotimes_{U \in \mathrm{Nb}(V)} (\overline{Z}_U)^{m_{CU}}$$



➔ The graph of the graph state

➔ So we have shown the AKLT state
is converted to some graph state by POVM

Outline

I. Introduction: quantum spins & quantum computation

II. Valence-bond states (AKLT construction)

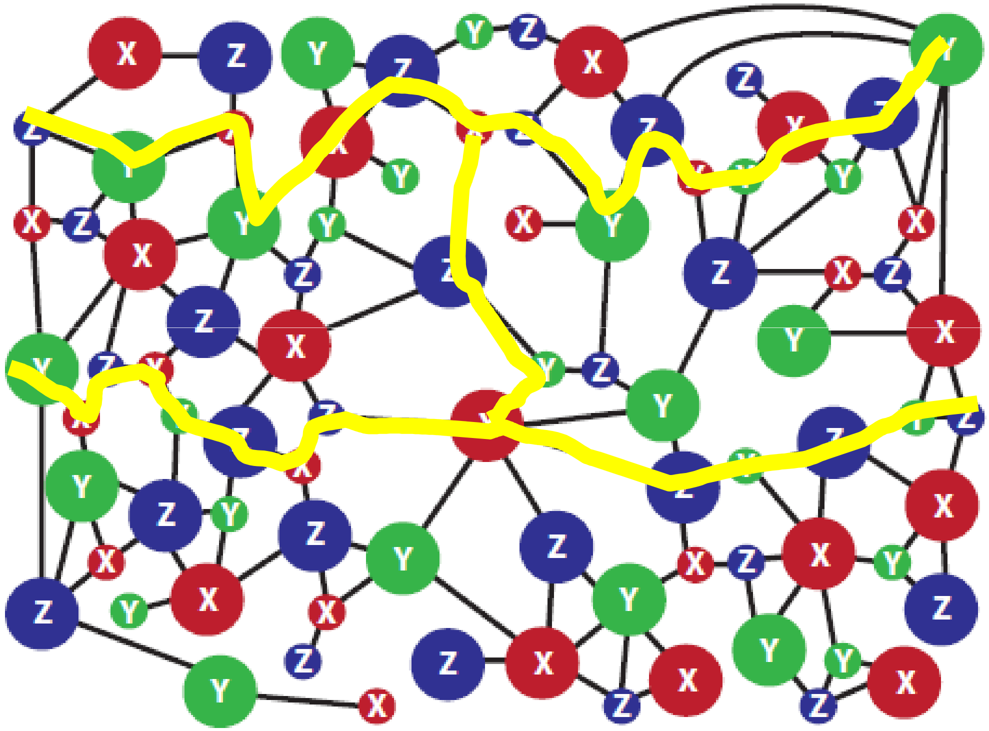
- Ground states of isotropic antiferromagnet

III. Valence-bond ground states for quantum computation

- Preprocessing: generalized measurement
- Give rise to graph states and cluster states (graph properties and percolation argument)

IV. Summary and outlook

➔ Quantum computation can be implemented on such a (random) graph state



- Wires define logical qubits, links give CNOT gates
- Sufficient number of wires if graph is supercritical (percolation)

Average graph properties: use of Monte Carlo method

- Each site has 3 possible POVM outcomes (x,y,z)
  ➔ N sites have 3^N possible combinations

- Probability of each combination $A=\{a_v\}$ is an N-point correlation fcn

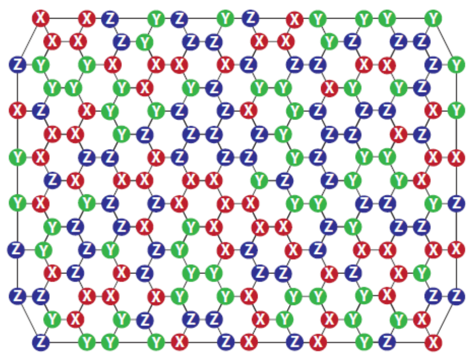
$$p_{\mathcal{A}} \sim \langle \Phi_{\mathrm{AKLT}} | \bigotimes_{\mathrm{sites}\, v} \left(S^2_{v,a_v} - \frac{1}{4} \right) | \Phi_{\mathrm{AKLT}} \rangle$$

➔ This correlation is related to the structure of the graph

- Monte Carlo simulations to investigate graph properties
  - Metropolis flip of local POVM outcome

$$p_{\mathrm{accept}} = \min \left\{ 1, 2^{|V'|-|\mathcal{E}'|-|V|+|\mathcal{E}|} \right\}$$

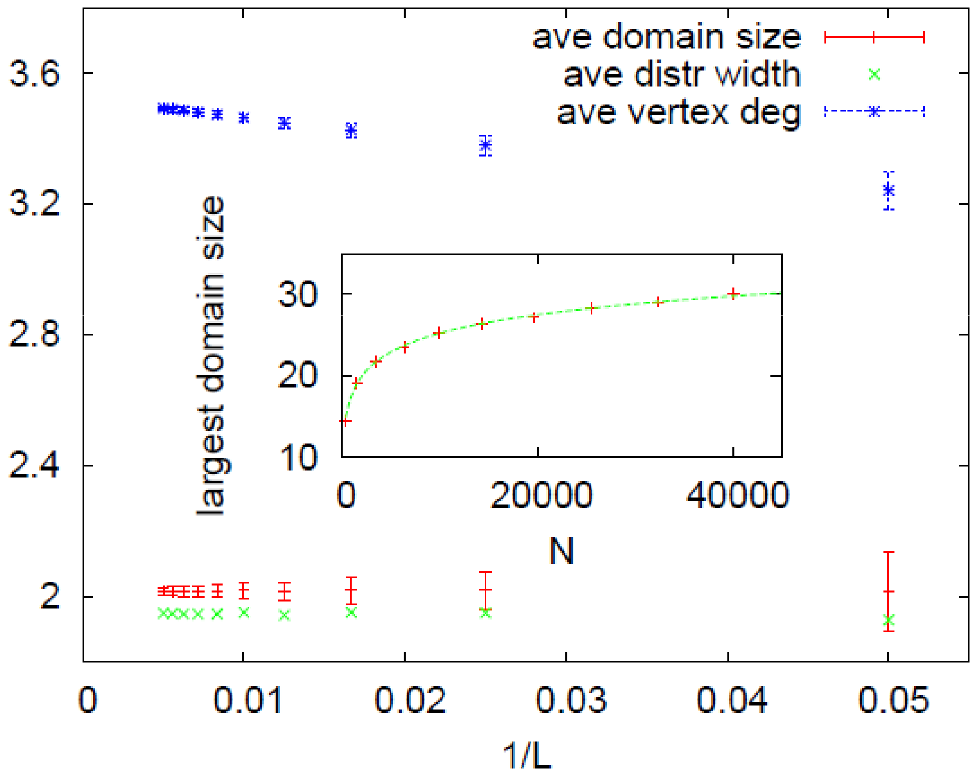
V: set of domains,
ε: set of inter-domain edges (before mod-2)



Graph properties of typical graphs

- Degree, vertices, edges and independent loops

  - Honeycomb: deg=3
    |E|=1.5|V|, B*=0.5|V|
  - Typical graphs: deg=3.52
    |E|=1.76|V|, B=0.76|V|
  - Square lattice: deg=4
    |E|=2|V|, B=|V|

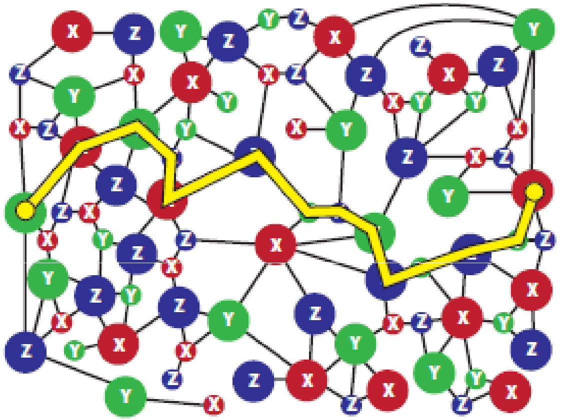


- Domain size is not macroscopic

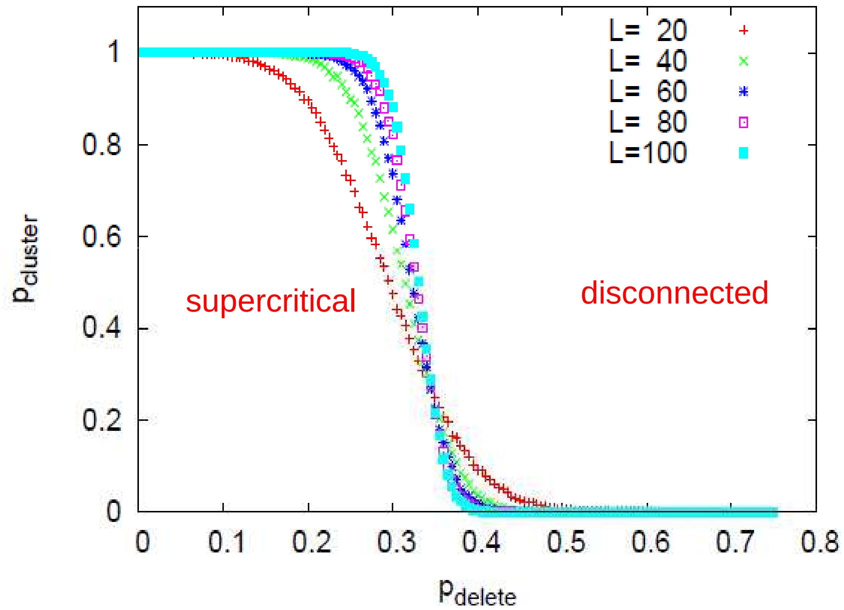
* B is # of independent loops, e.g. hexagons or squares

Robustness: finite percolation threshold

- Typical graphs are in percolated (or supercritical) phase



Site percolation by deletion

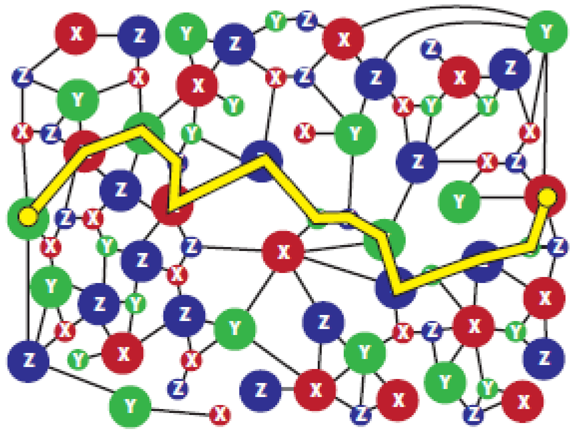


- C.f. Site perc threshold:
  Square: 0.593, honeycomb:0.697

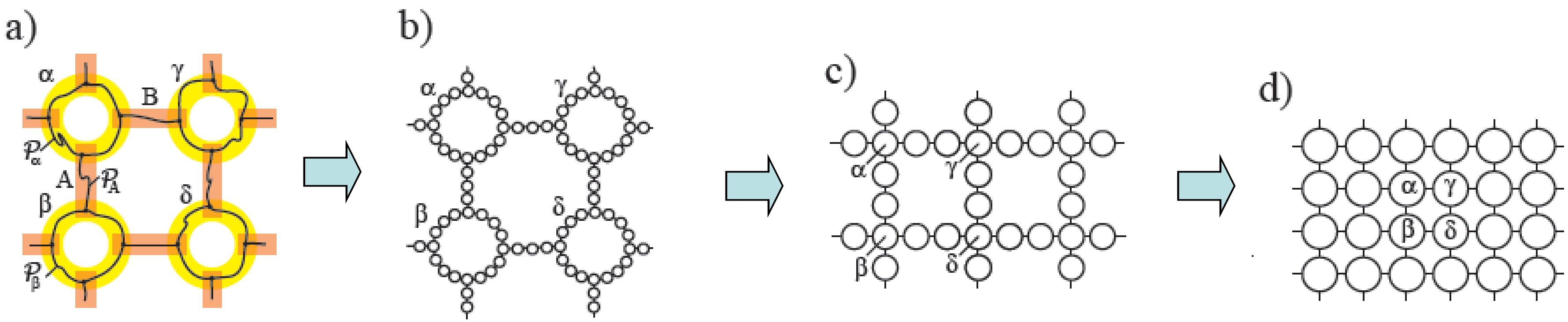
➔ threshold ≈1-0.33=0.67

➔ Sufficient (macroscopic) number of traversing paths exist

Convert graph states to cluster states



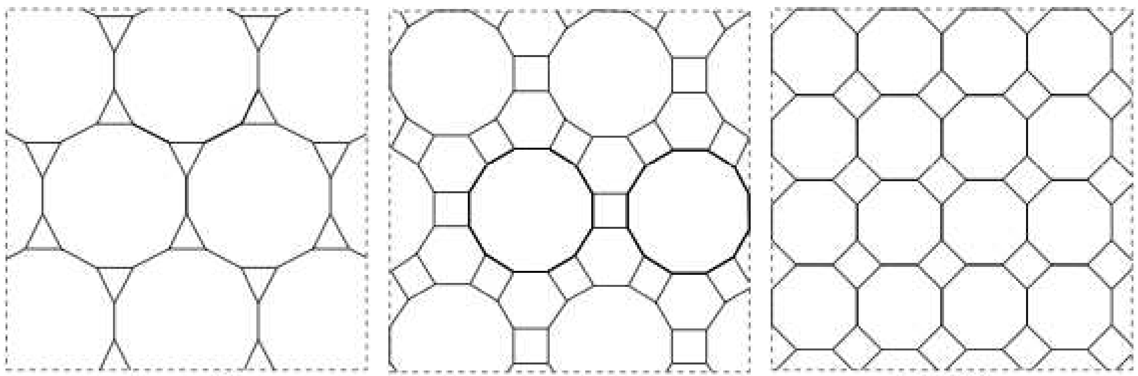
- Can identity graph structure and trim it down to square



➔ Thus we have shown the 2D AKLT state is a universal computational resource

Other 2D AKLT states expected to be universal resources

- Trivalent Achimedean lattices (in addition to honeycomb):

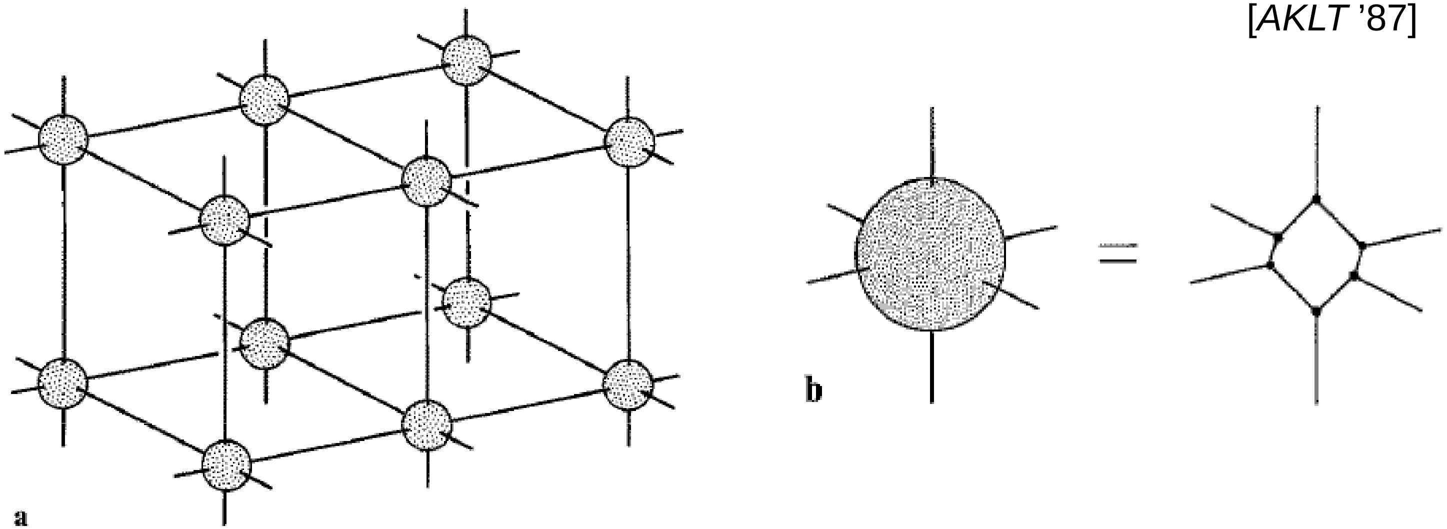


	(3, 12²)	(4, 6, 12)	(4, 8²)
Bond percolation threshold > 2/3:	≈0.7404	≈0.694	≈0.677

- Expect to be quantum disordered w/o Néel order

3D trivalent AKLT state

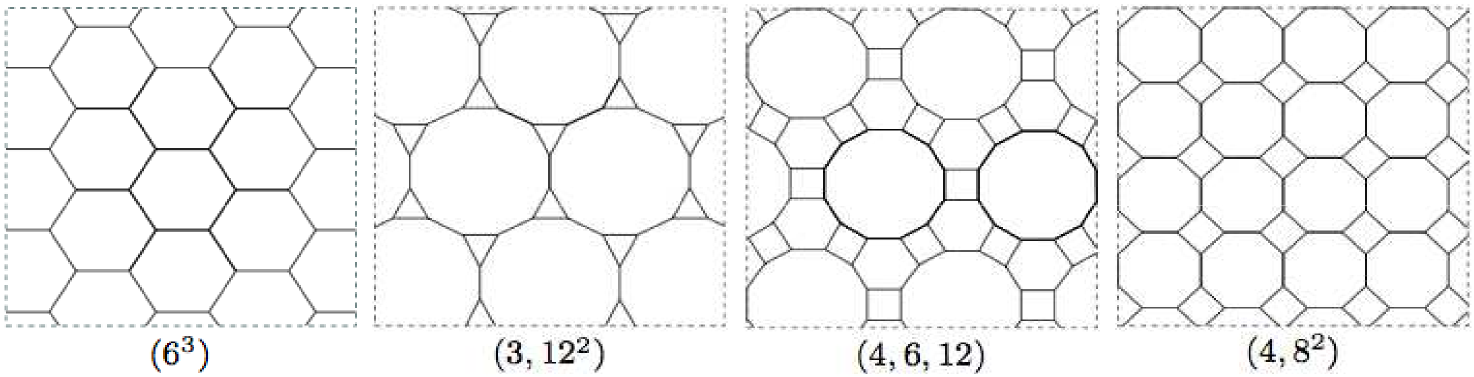
[*AKLT* ’87]



- Expect to be quantum disordered w/o Néel order
- Why? small number (3) of neighbors [c.f. *Parameswaran, Sondhi* & Arovas ’09]
  - AKLT state on cubic lattice (6 neighbors) has Néel order
  - AKLT state on diamond lattice (4 neighbors) is disordered

Conclusion

- Spin-3/2 valence-bond ground states on some 2D lattices are universal resource for quantum computation



➔ Design a generalized measurement

➔ Convert to graph states and then cluster states (←universal)

- Can extend to 3D as well

Collaborators

Ian Affleck

Robert Raussendorf (UBC)

AKLT & Quantum Computation
➔Wei, Affleck & Raussendorf
PRL106, 070501 (2011)
& arXiv:1009.2840
Wei & Raussendorf, in preparation

Kwek (CQT)

Ying Li (CQT)

Dan Browne (UCL)

Further extension (thermal state and always-on Hamiltonian)
➔ arXiv:1102.5153

Outlook

1. Implementation of spin-3/2 2D AKLT models & phase diagram?

   ✓ Spin-1 AKLT: use “spin-1” bosons on optical lattice

   [*Imambekov et al. PRA ‘03; Garcia-Ripoll et al. PRL ‘04; Rizzi et al., PRL ‘05*]

   ➔ 2D: spin-3/2: use of “spin-3/2” bosons?

2. Spectral gap of 2D AKLT models?

   ✓ Only exponential decay correlations

   ➔ Need techniques beyond AKLT & Knabe

   ➔ PEPS or Tensor Product States? Analytic or numeric

3. Spin-2 AKLT on square lattice universal? Other lattices?