Lagrangians and Hamiltonians for the Electrodynamics of Wire-SRR & single-resonance-Chiral Metamaterials

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Motivation

- To find the formulas of the EM energy densities in a dispersive medium with finite loss

- To find the formulas of the Lagrangian and Hamiltonian density for studying the Quantum EM Phenomena of metamaterials

Part I

EM energy density in a dispersive medium

Methods for deriving the energy density formula (I)

- For a non-dispersive, linear medium, the EM energy density can be easily identified in the equation of Poynting's theorem (electrodynamics method, ED method).

- For a dispersive, linear, lossless medium, the time averaged energy density for time harmonic EM wave can be derived (**Landau & Lifshitz**)

Methods for deriving the energy density formula (II)

- For a dispersive, linear medium with loss, the time averaged energy density for time harmonic EM waves can be derived using the equivalent circuit (EC) method.

- For a dispersive, linear medium with loss, the instantaneous energy density can be derived using the electrodynamics (ED) method.

Energy density formulas (I)

- **Brillouin, Landau & Lifshitz (books): Energy density formula for dispersive-lossless medium. (harmonic wave, time averaged)**
- **R. Loudon (1970): Permittivity is of the Lorentz type, finite loss. ED approach.**
- **P. C. W. Fung and K. Young (1978): EC approach.**
- **Ruppin (2002), T. J. Cui & J. Au. Kong (2004): Dispersive-lossy metamaterials consisting of E&M dipoles having Lorentz resonance behaviors. (The magnetic dipoles in Ruppin's model are not realistic, because they behave exactly like the electric dipoles.) ED approach.**

Energy density formulas (II)

- **Tretyakov (2005,EC), Boardman & Marinov (2006,ED): Dispersive metamaterials. The magnetic dipoles are current loops (SRRs). The time average of the B&M's result is different from that of Tretyakov's. Tretyako's result does not approach the result of the Lifshitz &Landau's (LL) formula in the lossless limit.**

- **The results of T and B&M have been "unified" by Luan(2009). The resultant formula approaches LL formula in the lossless limit. Luan's Strategy: The power loss determines the energy density.**

- **The same strategy had been utilized to find the energy density formula for a single-resonance chiral metamaterial (Luan et al, 2011).**

Energy density in a nondispersive EM medium(I)

Maxwell's equations (Ampere's and Faraday's Laws)

give us: $-\nabla\cdot\mathbf{S}=\mathbf{E}\cdot\dfrac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\dfrac{\partial\mathbf{B}}{\partial t}+\mathbf{E}\cdot\mathbf{J}$

Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$

Constitutive relations: $\mathbf{D}=\varepsilon_0\varepsilon\mathbf{E}$, $\mathbf{B}=\mu_0\mu\mathbf{H}$

$\Rightarrow -\nabla\cdot\mathbf{S}=\dfrac{\partial W}{\partial t}+\mathbf{E}\cdot\mathbf{J}$ (Energy conservation law !)

(Naive) Energy density: $W=\dfrac{1}{2}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\right)=\dfrac{1}{2}\left(\varepsilon_0\varepsilon E^2+\mu_0\mu H^2\right)$

(Naive) Power loss: $P_{loss}=\mathbf{E}\cdot\mathbf{J}=J^2/\sigma$ (Joule Heat, if $\mathbf{J}=\sigma\mathbf{E}$)

If there is no way to dissipate the energy (eg. lossless plasma),

then $\mathbf{E}\cdot\mathbf{J}=\dfrac{\partial K}{\partial t}$, K is the kinetic energy density of the charged particles.

Effective energy density: $W_{tot}=W+K$

Energy density in a nondispersive EM medium (II)

Single frequency, complex field representation:

$$\bar{\mathbf{S}} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E}\times\mathbf{H}^*\right), \quad \bar{W} = \frac{1}{4}\left(\varepsilon_0\varepsilon\,|\,\mathbf{E}\,|^2 + \mu_0\mu\,|\,\mathbf{H}\,|^2\right)$$

$\bar{\mathbf{S}}$ and $\bar{W}$: time-averaged $\mathbf{S}$ and W, $-\nabla\cdot\bar{\mathbf{S}} = \bar{P}_{loss}$

Here $\left\langle \dfrac{\partial W}{\partial t} \right\rangle = \dfrac{1}{T}\displaystyle\int_0^T \dfrac{\partial W}{\partial t}\,dt = \dfrac{1}{T}\big(W(T) - W(0)\big) = 0$

The expression of $\bar{W}$ CANNOT be extracted from the time averaged equation $-\nabla\cdot\bar{\mathbf{S}} = \bar{P}_{loss}$.

EM energy density in dispersive media with negligible absorption

For monochromatic EM wave

$\mathbf{D} = \varepsilon_0 \varepsilon(\omega)\mathbf{E}, \quad \mathbf{B} = \mu_0 \mu(\omega)\mathbf{H}$

$$\bar{\mathbf{S}} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E}\times\mathbf{H}^*\right)$$

$$\bar{W} = \frac{1}{4}\left(\varepsilon_0 \frac{d}{d\omega}\left(\omega\varepsilon\left(\omega\right)\right)|\,\mathbf{E}\,|^2 + \mu_0 \frac{d}{d\omega}\left(\omega\mu\left(\omega\right)\right)|\,\mathbf{H}\,|^2\right)$$

–Brillouin, Landau, and Jackson

Only for the monochromatic and adiabatic cases.

Energy density formula-- lossless case (I)

Ref: Landau's ED book, Ch 9, pp. 274-276

$$\mathbf{E}_{real}(t)=\frac{1}{2}\left(\mathbf{E}(t)+\mathbf{E}^*(t)\right),\quad \mathbf{D}_{real}(t)=\frac{1}{2}\left(\mathbf{D}(t)+\mathbf{D}^*(t)\right)$$

$$\mathbf{E}(t)=\mathbf{E}_0(t)e^{-i\omega_0 t}=e^{-i\omega_0 t}\int \mathbf{E}_{0\alpha}e^{-i\alpha t}d\alpha$$

$$\mathbf{D}(t)=\varepsilon_0\int \varepsilon(\omega_0+\alpha)\mathbf{E}_{0\alpha}e^{-i(\omega_0+\alpha)t}d\alpha,\quad \omega=\omega_0+\alpha$$

$\boxed{\begin{array}{l}\mathbf{E}_0(t):\ \text{slowly varying quantity (w.r.t. } T=2\pi/\omega_0) \\ \mathbf{E}_{0\alpha}:\ \text{Foruier distribution (a narrow peak around } \alpha=0)\end{array}}$

Time avearging over $T=\dfrac{2\pi}{\omega_0}$:

$$-\nabla\cdot\overline{\mathbf{S}}=\frac{1}{4}\left\langle \mathbf{E}\cdot\frac{\partial\mathbf{D}^*}{\partial t}+\mathbf{E}^*\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}^*}{\partial t}+\mathbf{H}^*\cdot\frac{\partial\mathbf{B}}{\partial t}\right\rangle$$

Energy density formula-- lossless case (II)

$$\frac{\partial \mathbf{D}(t)}{\partial t} = \varepsilon_0 \int \left[-i(\omega_0 + \alpha)\varepsilon(\omega_0 + \alpha) \right] \mathbf{E}_{0\alpha} e^{-i(\omega_0+\alpha)t} d\alpha$$

$$\approx \varepsilon_0 \int \left[-i\omega_0 \varepsilon(\omega_0) \boxed{-i\alpha} \frac{\partial(\omega_0 \varepsilon(\omega_0))}{\partial \omega_0} \right] \mathbf{E}_{0\alpha} e^{-i(\omega_0+\alpha)t} d\alpha$$

$$= e^{-i\omega_0 t} \varepsilon_0 \left[-i\omega_0 \varepsilon(\omega_0) + \frac{\partial(\omega_0 \varepsilon(\omega_0))}{\partial \omega_0} \boxed{\frac{\partial}{\partial t}} \right] \int \mathbf{E}_{0\alpha} e^{-i\alpha t} d\alpha$$

$$= e^{-i\omega_0 t} \varepsilon_0 \left[-i\omega_0 \varepsilon(\omega_0) \mathbf{E}_0(t) + \frac{\partial(\omega_0 \varepsilon(\omega_0))}{\partial \omega_0} \frac{\partial \mathbf{E}_0(t)}{\partial t} \right]$$

Similarly we have

$$\frac{\partial \mathbf{B}(t)}{\partial t} = e^{-i\omega_0 t} \mu_0 \left[-i\omega_0 \mu(\omega_0) \mathbf{H}_0(t) + \frac{\partial(\omega_0 \mu(\omega_0))}{\partial \omega_0} \frac{\partial \mathbf{H}_0(t)}{\partial t} \right]$$

Energy density formula-- lossless case (III)

$$-\nabla\cdot\bar{\mathbf{S}}=\frac{1}{4}\left\langle \mathbf{E}\cdot\frac{\partial\mathbf{D}^*}{\partial t}+\mathbf{E}^*\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}^*}{\partial t}+\mathbf{H}^*\cdot\frac{\partial\mathbf{B}}{\partial t}\right\rangle_{T=2\pi/\omega_0}$$

$$=\frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{4}\frac{\partial(\omega\varepsilon)}{\partial\omega}|\mathbf{E}_0|^2+\frac{\mu_0}{4}\frac{\partial(\omega\mu)}{\partial\omega}|\mathbf{H}_0|^2\right)\equiv\frac{\partial\bar{W}}{\partial t}$$

(I have renamed ω_0 as ω)

Slowly varying

Using the fact $\mathbf{E}=\mathbf{E}_0e^{-i\omega t}$ and $\mathbf{H}=\mathbf{H}_0e^{-i\omega t}$, we obtain

$$\bar{W}=\frac{\varepsilon_0}{4}\frac{\partial(\omega\varepsilon)}{\partial\omega}|\mathbf{E}_0|^2+\frac{\mu_0}{4}\frac{\partial(\omega\mu)}{\partial\omega}|\mathbf{H}_0|^2$$

$$=\boxed{\frac{\varepsilon_0}{4}\frac{\partial(\omega\varepsilon)}{\partial\omega}|\mathbf{E}|^2+\frac{\mu_0}{4}\frac{\partial(\omega\mu)}{\partial\omega}|\mathbf{H}|^2}$$

When $\boxed{\Delta\alpha\to 0}$, $\bar{W}$ represents the energy density of harmonic wave

EM energy density in dispersive media with finite absorption (ED approach)

(R. Loudon, J. Phys. A 3, 233 1970)

$$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P},\quad \mathbf{B}=\mu_0\mathbf{H},\quad \mathbf{P}=N\mathbf{p},\quad \mathbf{p}=q\mathbf{r},$$

$$m\left(\ddot{\mathbf{r}}+\Gamma\dot{\mathbf{r}}+\omega_r^2\mathbf{r}\right)=q\mathbf{E}\Rightarrow \boxed{\ddot{\mathbf{P}}+\Gamma\dot{\mathbf{P}}+\omega_r^2\mathbf{P}=\varepsilon_0\omega_p^2\mathbf{E}}$$

$$-\nabla\cdot\mathbf{S}=\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}=\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\varepsilon_0E^2+\mu_0H^2\right)\right]+\boxed{\mathbf{E}}\cdot\frac{\partial\mathbf{P}}{\partial t}$$

$$=\frac{\partial W_0}{\partial t}+\frac{\partial}{\partial t}\left[\frac{1}{2\omega_p^2\varepsilon_0}\left(\dot{\mathbf{P}}^2+\omega_r^2\mathbf{P}^2\right)\right]+\frac{\Gamma}{\omega_p^2\varepsilon_0}\dot{\mathbf{P}}^2=\frac{\partial W}{\partial t}+\frac{\Gamma}{\omega_p^2\varepsilon_0}\dot{\mathbf{P}}^2$$

Energy density: $\boxed{W=W_0+\boxed{\frac{1}{2\omega_p^2\varepsilon_0}\left(\dot{\mathbf{P}}^2+\omega_r^2\mathbf{P}^2\right)}}$ Dipole energy

where $W_0=\frac{1}{2}\left(\varepsilon_0E^2+\mu_0H^2\right)$

EM energy density in metamaterials with finite absorption (Ruppin 2002)

The electric part is the same as that of Loudon's.

For the magnetic part, Ruppin considered the simplified permeability

$$\mu(\omega) = 1 + \chi_m(\omega) = 1 - \frac{F\omega_0^2}{\omega^2 - \omega_0^2 + i\Gamma_h\omega},$$

which corresponds to the equaion of motion

$$\ddot{\mathbf{M}} + \Gamma_h\dot{\mathbf{M}} + \omega_0^2\mathbf{M} = F\omega_0^2\mathbf{H}.$$

This leads to: $-\nabla\cdot\mathbf{S} = \dfrac{\partial W}{\partial t} + P_{loss},$ where

$$W = \frac{\varepsilon_0}{2}E^2 + \frac{\mu_0}{2}H^2 + \frac{1}{2\omega_p^2\varepsilon_0}\left(\dot{\mathbf{P}}^2 + \omega_r^2\mathbf{P}^2\right) + \frac{\mu_0}{2F\omega_0^2}\left(\dot{\mathbf{M}}^2 + \omega_0^2\mathbf{M}^2\right),$$

$$P_{loss} = \frac{\Gamma_e\dot{\mathbf{P}}^2}{\omega_p^2\varepsilon_0} + \frac{\mu_0\Gamma_h\dot{\mathbf{M}}^2}{F\omega_0^2}$$

(the Joule Heat caused by the currents in the wires and SRRs)

EM energy density in metamaterials with finite absorption (Ruppin 2002) (II)

Remarks: the simplified permeability implies that in this model the magnetic dipoles consist of $\pm$ monopoles, so this is not a realistic model.

The realistic magnetic dipoles are the currents loops, i.e., the SRRs ($m = IS,\ m$: magnetic dipole, I : current, S : area encircled)

The realistic permeability is

$$\mu(\omega) = 1 - \frac{F\omega^2}{\omega^2 - \omega_0^2 + i\Gamma_h \omega},$$

corresponding to the equaion of motion

$$\ddot{\mathbf{M}} + \Gamma_h \dot{\mathbf{M}} + \omega_0^2 \mathbf{M} = -F\ddot{\mathbf{H}},$$

EM energy density in metamaterials with finite absorption (I)

One can derive from Maxwell's equations

$$-\nabla\cdot\mathbf{S}=\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}=\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\varepsilon_0E^2+\mu_0H^2\right)\right]+\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t}+\mu_0\mathbf{H}\cdot\frac{\partial\mathbf{M}}{\partial t}$$

The equations of motion for **P** and **M**:

$$\boxed{\begin{aligned}&\ddot{\mathbf{P}}+\nu\dot{\mathbf{P}}=\varepsilon_0\omega_p^2\mathbf{E}\quad\Leftrightarrow\quad\varepsilon(\omega)=1-\frac{\omega_p^2}{\omega(\omega+i\nu)}\\&\dot{\mathbf{M}}+\gamma\mathbf{M}+\omega_0^2\int\mathbf{M}dt=-F\dot{\mathbf{H}}\quad\Leftrightarrow\quad\mu(\omega)=1-\frac{F\omega^2}{\omega^2-\omega_0^2+i\omega\gamma}\end{aligned}}$$

Using the method similar to that of Loudon's, one can derive the result

$$-\nabla\cdot\mathbf{S}=\frac{\partial}{\partial t}\left(W_e+W_b\right)+P_{loss}.\ \boxed{W_e=\frac{\varepsilon_0\mathbf{E}^2}{2}+\frac{\dot{\mathbf{P}}^2}{2\omega_p^2\varepsilon_0}},\ \boxed{P_{loss}^e=\frac{\nu\dot{\mathbf{P}}^2}{\omega_p^2\varepsilon_0}}$$

EM energy density in metamaterials with finite absorption (II)

$$\boxed{\dot{\mathbf{M}}+\gamma\mathbf{M}+\omega_0^2\int\mathbf{M}dt=-F\dot{\mathbf{H}}}$$

$$\mu_0\mathbf{H}\cdot\frac{\partial\mathbf{M}}{\partial t}=\frac{\partial}{\partial t}\left(\mu_0\mathbf{H}\cdot\mathbf{M}\right)-\mu_0\mathbf{M}\cdot\boxed{\frac{\partial\mathbf{H}}{\partial t}}$$

$$=\frac{\partial}{\partial t}\left(\mu_0\mathbf{H}\cdot\mathbf{M}\right)+\frac{\mu_0}{F}\mathbf{M}\cdot\left(\dot{\mathbf{M}}+\gamma\mathbf{M}+\omega_0^2\int\mathbf{M}dt\right)$$

$$=\frac{\partial}{\partial t}\left[\mu_0\mathbf{H}\cdot\mathbf{M}+\frac{\mu_0}{2F}\mathbf{M}^2+\frac{\mu_0\omega_0^2}{2F}\left(\int\mathbf{M}dt\right)^2\right]+\frac{\gamma\mu_0}{F}\mathbf{M}^2$$

Pi-Gang Luan, Phys. Rev. E 80, 046601 (2009)

EM energy density in metamaterials with finite absorption (III)

Magnetic energy density

$$\boxed{W_b = \frac{\mu_0 \mathbf{H}^2}{2} + \mu_0 \mathbf{H}\cdot\mathbf{M} + \frac{\mu_0}{2F}\mathbf{M}^2 + \frac{\mu_0\omega_0^2}{2F}\left(\int \mathbf{M}dt\right)^2}$$

magnetic part of power loss: $P_{loss}^b = \frac{\gamma\mu_0}{F}\mathbf{M}^2$ (pure Joule heat !)

Using $\omega_0^2 \int \mathbf{M}dt = -\left(\dot{\mathbf{M}} + F\dot{\mathbf{H}} + \gamma\mathbf{M}\right)$, W_b can also be expressed as

$$\boxed{W_b = \frac{\mu_0\left(1-F\right)\mathbf{H}^2}{2} + \frac{\mu_0}{2\omega_0^2 F}\left[\left(\dot{\mathbf{M}} + F\dot{\mathbf{H}} + \gamma\mathbf{M}\right)^2 + \omega_0^2\left(\mathbf{M} + F\mathbf{H}\right)^2\right]}$$

$\boxed{P_{loss} = \frac{\nu\dot{\mathbf{P}}^2}{\omega_p^2\varepsilon_0} + \frac{\gamma\mu_0\mathbf{M}^2}{F}}$ is the total power loss (of I^2R form)

P.G. Luan PRE (2009) -- Different from the result obtained

by Boardman *et*. *al*. [Phys. Rev. B 73, 165110 (2006)]

Energy density formula for lossless, isotropic-dispersive chiral media

Constitutive relations:

$$\mathbf{D} = \varepsilon_0 \varepsilon(\omega)\mathbf{E} + i\frac{\kappa(\omega)}{c}\mathbf{H}, \quad \mathbf{B} = -i\frac{\kappa(\omega)}{c}\mathbf{E} + \mu_0 \mu(\omega)\mathbf{H}$$

Adiabatic analysis leads to

$$\bar{W} = \frac{\varepsilon_0}{4}\frac{\partial(\omega\varepsilon)}{\partial\omega}|\mathbf{E}|^2 + \frac{\mu_0}{4}\frac{\partial(\omega\mu)}{\partial\omega}|\mathbf{H}|^2 + \frac{1}{2c}\frac{\partial(\omega\kappa)}{\partial\omega}\operatorname{Im}\left(\mathbf{E}\cdot\mathbf{H}^*\right)$$

$$= \frac{1}{4}\left(\sqrt{\varepsilon_0}\mathbf{E}^*, \sqrt{\mu_0}\mathbf{H}^*\right)\cdot\begin{pmatrix} \frac{\partial(\omega\varepsilon)}{\partial\omega} & i\frac{\partial(\omega\kappa)}{\partial\omega} \\ -i\frac{\partial(\omega\kappa)}{\partial\omega} & \frac{\partial(\omega\mu)}{\partial\omega} \end{pmatrix}\begin{pmatrix} \sqrt{\varepsilon_0}\mathbf{E} \\ \sqrt{\mu_0}\mathbf{H} \end{pmatrix} \equiv \frac{1}{4}V^\dagger M V$$

$$\bar{W} > 0 \Rightarrow \boxed{\frac{\partial(\omega\varepsilon)}{\partial\omega} + \frac{\partial(\omega\mu)}{\partial\omega} > 0 \text{ and } \frac{\partial(\omega\varepsilon)}{\partial\omega}\frac{\partial(\omega\mu)}{\partial\omega} > \left(\frac{\partial(\omega\kappa)}{\partial\omega}\right)^2}$$

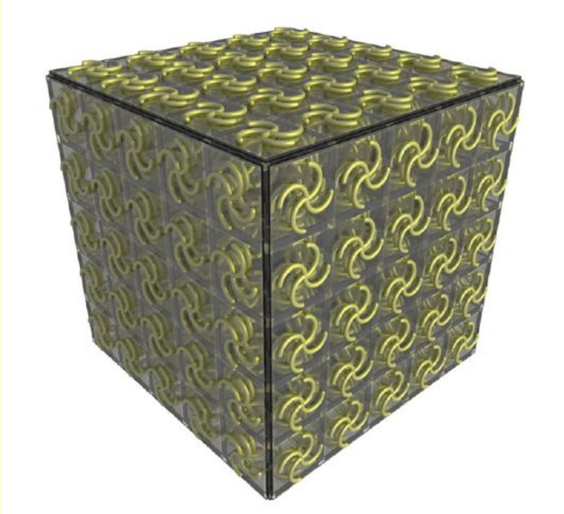
or $\operatorname{tr}(M) > 0$ and $\det(M) > 0$.

EM energy density in single-resonance chiral metamaterial with finite loss (I)

Isotropic chiral medium, single resonance case (eg. helix)

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i\Gamma\omega}, \quad \mu(\omega) = 1 - \frac{F\omega^2}{\omega^2 - \omega_0^2 + i\Gamma\omega}$$

$$\kappa(\omega) = \frac{A\omega}{\omega^2 - \omega_0^2 + i\Gamma\omega} \quad \Rightarrow \boxed{\chi_e(\omega)\chi_b(\omega) = \kappa^2(\omega)}$$



$$\boxed{\begin{aligned} &\ddot{\mathbf{P}} + \Gamma\dot{\mathbf{P}} + \omega_0^2\mathbf{P} = \varepsilon_0\omega_p^2\mathbf{E} + \frac{A}{c}\dot{\mathbf{H}} \\ &\dot{\mathbf{M}} + \Gamma\mathbf{M} + \omega_0^2\int \mathbf{M}dt = -F\dot{\mathbf{H}} - \frac{A}{\mu_0 c}\mathbf{E} \end{aligned}}$$

Both correspond to the following circuit equation

$$\boxed{L\frac{dI}{dt} + RI + \frac{q}{C} = V_e - \frac{d\Phi}{dt}}, \quad \dot{\mathbf{P}} = -\frac{\omega_p^2}{Ac}\mathbf{M}, \quad A = \pm\sqrt{F}\omega_p$$

EM energy density in isotropic chiral metamaterials with finite absorption (II)

The total energy density is given by

$$W = \frac{\varepsilon_0 \mathbf{E}^2}{2} + \frac{\mu_0 \mathbf{H}^2}{2} + \mu_0 \mathbf{H}\cdot\mathbf{M} + \frac{\dot{\mathbf{P}}^2}{2\varepsilon_0\omega_p^2} + \frac{\omega_0^2 \mathbf{P}^2}{2\varepsilon_0\omega_p^2}$$

$$= \frac{\varepsilon_0 \mathbf{E}^2}{2} + \frac{\mu_0 (1-F)\mathbf{H}^2}{2} + \frac{\omega_0^2 \mathbf{P}^2}{2\varepsilon_0\omega_p^2} + \frac{\mu_0 (\mathbf{M}+F\mathbf{H})^2}{2F},$$

Power loss $P_{loss} = \dfrac{\Gamma \dot{\mathbf{P}}^2}{\varepsilon_0 \omega_p^2} = \dfrac{\mu_0 \Gamma \mathbf{M}^2}{F}$

P.G. Luan et.al.Optics Letters 36, 675 (2011)

Term by term: $\dfrac{\varepsilon_0 \mathbf{E}^2}{2} + \dfrac{\mu_0 (1-F)\mathbf{H}^2}{2}$

(pure EM energy, the H part is stored outside the helix tube);

$\dfrac{\omega_0^2 \mathbf{P}^2}{2\varepsilon_0\omega_p^2}$: (electric in C); $\dfrac{\mu_0 (\mathbf{M}+F\mathbf{H})^2}{2F}$: (magnetic inside the helix tube)

Some remarks on effective medium energy

- 1. In an effective medium, the effective electric field energy also contains magnetic energy and vice versa.

- 2. In the thin-wire medium, the magnetic energy gathered around the wires is treated as the kinetic energy carried by the “massive electrons” in the effective medium (Pendry 96 PRL). (In the original Drude model such magnetic energy does not exist)

- 3. An SRR is an LC circuit, it stores both the magnetic (L) and electric (C) energies. (The “magnetic dipole energy” includes also the electric potential energy)

Summary: Part I

- Formulas for energy density and power loss of Wire-SRR metamaterial media are derived
- The derivation is based on the knowledge of the correct form of the power loss and the equations of motion of the resonators
- The effective electric energy also includes the magnetic energy. The effective magnetic energy includes the electric energy too.

Part II

The Lagrangian and Hamiltonian density for the dispersive-lossy EM system

Lagrangian density and dissipation function

Lagrangian Equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = -\frac{\partial F}{\partial \dot{q}_j}, \quad F \text{ is the dissipation function.}$$

Example: coupled circuits

$$L = \frac{1}{2}\sum_j L_j \dot{q}_j^2 + \frac{1}{2}\sum_{jk;\, j\neq k} M_{jk}\dot{q}_j \dot{q}_k - \sum_j \frac{q_j^2}{2C_j} + \sum_j E_j(t) q_j$$

$$F = \frac{1}{2}\sum_j R_j \dot{q}_j^2$$

$$L_j \frac{d^2 q_j}{dt^2} + \sum_{k;\, j\neq k} M_{jk}\frac{d^2 q_k}{dt^2} + R_j \frac{dq_j}{dt} + \frac{q_j}{C_j} = E_j(t)$$

Legendre transformation and Hamiltonian

Example: coupled circuits

$$L = \frac{1}{2}\sum_j L_j \dot{q}_j^2 + \frac{1}{2}\sum_{jk;\, j\neq k} M_{jk}\dot{q}_j \dot{q}_k - \sum_j \frac{q_j^2}{2C_j} + \sum_j E_j(t) q_j$$

Canonical momenta

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = L_j \dot{q}_j + \sum_{k;\, j\neq k} M_{jk}\dot{q}_k$$

Hamiltonian:

$$H = \sum_j p_j \dot{q}_j - L$$

Lagrangian equations for the fields

Lagrangian Equations:

$$\partial_t\left(\frac{\partial \mathrm{L}}{\partial\left(\partial_t\varphi_\alpha\right)}\right)+\sum_i \partial_i\left(\frac{\partial \mathrm{L}}{\partial\left(\partial_i\varphi_\alpha\right)}\right)-\frac{\partial \mathrm{L}}{\partial\varphi_\alpha}=-\frac{\partial F}{\partial\left(\partial_t\varphi_\alpha\right)},$$

L is the Lagrangian density

F is the dissipation function density

φ_α : the dynamical field variables

$$F=\frac{1}{2}\sum_{\alpha,\beta}R_{\alpha\beta}\dot{\varphi}_\alpha\dot{\varphi}_\beta$$

Hamiltonian and Quantization

Canonical momentum:

$$\Pi_\alpha = \frac{\partial \mathrm{L}}{\partial\left(\partial_t \varphi_\alpha\right)},$$

Hamiltonian density

$$H \equiv \sum_\alpha \Pi_\alpha \dot{\varphi}_\alpha - \mathrm{L} \equiv H\,(\varphi, \Pi);$$

Canonical quantization:

$$[\varphi_\alpha(\mathbf{r},t), \Pi_\beta(\mathbf{r}',t)] = i\hbar\delta^{(3)}\left(\mathbf{r}-\mathbf{r}'\right)$$

Lagrangian density for a plasma medium (I)

Dynamical variables: φ, $\mathbf{A}$, $\mathbf{P}$

$$L = \boxed{\frac{\varepsilon_0}{2}\mathbf{E}^2 - \frac{\mu_0}{2}\mathbf{H}^2} + \boxed{\frac{\dot{\mathbf{P}}^2}{2\omega_p^2\varepsilon_0}} + \boxed{\mathbf{P}\cdot\mathbf{E}\,\text{(interaction term)}}$$

$$= \frac{\varepsilon_0}{2}\left(\nabla\varphi + \dot{\mathbf{A}}\right)^2 - \frac{1}{2\mu_0}\left(\nabla\times\mathbf{A} - \dot{\mathbf{Q}}\right)^2 + \frac{\dot{\mathbf{P}}^2}{2\omega_p^2\varepsilon_0} - \mathbf{P}\cdot\left(\nabla\varphi + \dot{\mathbf{A}}\right)$$

Here $\mathbf{E} = -\nabla\varphi - \dot{\mathbf{A}}, \quad \mathbf{B} = \nabla\times\mathbf{A}$

Dissipation function density: $F = \dfrac{\nu\dot{\mathbf{P}}^2}{2\omega_p^2\varepsilon_0} = \dfrac{1}{2}P_{loss}$

Lagrangian density for a plasma medium (II)

Lagrangian Equations for the fields

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\varphi}}\right)+\nabla\cdot\left(\frac{\partial L}{\partial\left(\nabla\varphi\right)}\right)-\frac{\partial L}{\partial\varphi}=0, \qquad \Rightarrow\ \nabla\cdot\mathbf{D}=0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{A}_k}\right)+\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial\left(\partial A_k/\partial x_i\right)}\right)-\frac{\partial L}{\partial A_k}=0 \quad \Rightarrow\ \nabla\times\mathbf{H}=\dot{\mathbf{D}}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\mathbf{P}}}\right)+\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial\left(\partial \mathbf{P}/\partial x_i\right)}\right)-\frac{\partial L}{\partial \mathbf{P}}=-\frac{\partial F}{\partial\dot{\mathbf{P}}} \quad \Rightarrow\ \ddot{\mathbf{P}}+\nu\dot{\mathbf{P}}=\varepsilon_0\omega_p^2\mathbf{E}$$

Hamiltonian density for a plasma medium

Canonical momentum densities:

$$\Pi_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = 0, \quad \mathbf{\Pi_A} = \frac{\partial L}{\partial \dot{\mathbf{A}}} = -\mathbf{D}, \quad \mathbf{\Pi_P} = \frac{\partial L}{\partial \dot{\mathbf{P}}} = \frac{\dot{\mathbf{P}}}{\omega_p^2 \varepsilon_0},$$

Hamiltonian density:

$$H = \Pi_\varphi \dot{\varphi} + \mathbf{\Pi_A} \cdot \dot{\mathbf{A}} + \mathbf{\Pi_P} \cdot \dot{\mathbf{P}} - L$$

$$= \frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{\dot{\mathbf{P}}^2}{2\omega_p^2 \varepsilon_0} + \frac{\mu_0}{2}\mathbf{H}^2 + \nabla\varphi \cdot \mathbf{D} = W_e + W_b + \nabla \cdot (\varphi \mathbf{D})$$

The term $\nabla\varphi \cdot \mathbf{D} = \nabla \cdot (\varphi \mathbf{D})$ because $\nabla \cdot \mathbf{D} = 0$

$\nabla \cdot (\varphi \mathbf{D})$ is a surface term, which can be dropped.

$$\Rightarrow \quad H = W_e + W_b = \frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{\mu_0}{2}\mathbf{H}^2 + \frac{\dot{\mathbf{P}}^2}{2\omega_p^2 \varepsilon_0}$$

Lagrangian Density for WSM (I)

Dynamical variables: $\boxed{\varphi,\ \mathbf{A},\ \mathbf{P},\ \mathbf{Q}\ \equiv \mu_0 \int \mathbf{M}\, dt}$

$$\boxed{L = \frac{\varepsilon_0}{2}\mathbf{E}^2 - \frac{\mu_0}{2}\mathbf{H}^2} + \boxed{\frac{\dot{\mathbf{P}}^2}{2\omega_p^2\varepsilon_0} + \mathbf{P}\cdot\mathbf{E}} + \boxed{\frac{\mu_0}{2F}\mathbf{M}^2 - \frac{\mu_0\omega_0^2}{2F}\left(\int \mathbf{M}\, dt\right)^2}$$

$$= \frac{\varepsilon_0}{2}\left(\nabla\varphi + \dot{\mathbf{A}}\right)^2 - \frac{1}{2\mu_0}\left(\nabla\times\mathbf{A} - \dot{\mathbf{Q}}\right)^2$$

$$+ \frac{\dot{\mathbf{P}}^2}{2\omega_p^2\varepsilon_0} - \mathbf{P}\cdot\left(\nabla\varphi + \dot{\mathbf{A}}\right) + \frac{1}{2\mu_0 F}\left(\dot{\mathbf{Q}}^2 - \omega_0^2\mathbf{Q}^2\right)$$

Here $\mathbf{E} = -\nabla\varphi - \dot{\mathbf{A}}, \quad \mathbf{B} = \nabla\times\mathbf{A}$

Dissipation function density:

$$\boxed{F = \frac{\nu\dot{\mathbf{P}}^2}{2\omega_p^2\varepsilon_0} + \frac{\gamma\mu_0}{2F}\mathbf{M}^2 = \frac{1}{2}P_{loss}}$$

Lagrangian Density for WSM (II)

Lagrangian Equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\varphi}}\right)+\nabla\cdot\left(\frac{\partial L}{\partial(\nabla\varphi)}\right)-\frac{\partial L}{\partial\varphi}=0, \qquad \Rightarrow \boxed{\nabla\cdot\mathbf{D}=0}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{A}_k}\right)+\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial(\partial A_k/\partial x_i)}\right)-\frac{\partial L}{\partial A_k}=0 \quad \Rightarrow \boxed{\nabla\times\mathbf{H}=\dot{\mathbf{D}}}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\mathbf{P}}}\right)+\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial(\partial \mathbf{P}/\partial x_i)}\right)-\frac{\partial L}{\partial \mathbf{P}}=-\frac{\partial F}{\partial \dot{\mathbf{P}}} \Rightarrow \boxed{\ddot{\mathbf{P}}+\nu\dot{\mathbf{P}}=\varepsilon_0\omega_p^2\mathbf{E}}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\mathbf{Q}}}\right)+\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial(\partial \mathbf{Q}/\partial x_i)}\right)-\frac{\partial L}{\partial \mathbf{Q}}=-\frac{\partial F}{\partial \dot{\mathbf{Q}}} \Rightarrow \boxed{\dot{\mathbf{M}}+\gamma\mathbf{M}+\omega_0^2\int\mathbf{M}dt=-F\dot{\mathbf{H}}}$$

Hamiltonian Density for WSM

Canonical momentum densities:

$$\Pi_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = 0, \quad \mathbf{\Pi}_\mathbf{A} = \frac{\partial L}{\partial \dot{\mathbf{A}}} = -\mathbf{D}$$

$$\mathbf{\Pi}_\mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{P}}} = \frac{\dot{\mathbf{P}}}{\omega_p^2 \varepsilon_0}, \quad \mathbf{\Pi}_\mathbf{Q} = \frac{\partial L}{\partial \dot{\mathbf{Q}}} = \mathbf{H} + \frac{\mathbf{M}}{F} = \mathbf{H}_{in}$$

Hamiltonian density:

$$H = \Pi_\varphi \dot{\varphi} + \mathbf{\Pi}_\mathbf{A} \cdot \dot{\mathbf{A}} + \mathbf{\Pi}_\mathbf{P} \cdot \dot{\mathbf{P}} + \mathbf{\Pi}_\mathbf{Q} \cdot \dot{\mathbf{Q}} - L$$

$$= \boxed{\frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{\dot{\mathbf{P}}^2}{2\omega_p^2 \varepsilon_0}} + \boxed{\frac{\mu_0}{2}\mathbf{H}^2 + \mu_0 \mathbf{H}\cdot\mathbf{M} + \frac{\mu_0}{2F}\mathbf{M}^2 + \frac{\mu_0 \omega_0^2}{2F}\left(\int \mathbf{M}\, dt\right)^2}$$

$$+\nabla\varphi \cdot \mathbf{D} = W_e + W_b + \nabla\cdot(\varphi\mathbf{D})$$

The term $\nabla\varphi\cdot\mathbf{D} = \nabla\cdot(\varphi\mathbf{D})$ because $\nabla\cdot\mathbf{D} = 0$

$\nabla\cdot(\varphi\mathbf{D})$ is a surface term, which can be dropped.

$\Rightarrow$ $\boxed{H = W_e + W_b = \text{total energy density}}$

Lagrangian Density for the SRCM (I)

Dynamical variables: φ, $\mathbf{A}$, $\mathbf{Q} \equiv \mu_0 \int \mathbf{M}\, dt$

Here $\mathbf{P} = \xi \mathbf{Q}$, $\quad \xi = -\dfrac{A}{F Z_0}$, $\quad Z_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}$

$$L = \frac{\varepsilon_0}{2}\mathbf{E}^2 - \frac{\mu_0}{2}\mathbf{H}^2 + \mathbf{P}\cdot\mathbf{E} + \frac{\mu_0}{2F}\mathbf{M}^2 - \frac{\mu_0 \omega_0^2}{2F}\left(\int \mathbf{M}\, dt\right)^2$$

$$= \frac{\varepsilon_0}{2}\left(\nabla\varphi + \dot{\mathbf{A}}\right)^2 - \frac{1}{2\mu_0}\left(\nabla\times\mathbf{A} - \dot{\mathbf{Q}}\right)^2 - \xi\mathbf{Q}\cdot\left(\nabla\varphi + \dot{\mathbf{A}}\right) + \frac{1}{2\mu_0 F}\left(\dot{\mathbf{Q}}^2 - \omega_0^2\mathbf{Q}^2\right)$$

Here $\mathbf{E} = -\nabla\varphi - \dot{\mathbf{A}}$, $\quad \mathbf{B} = \nabla\times\mathbf{A}$

Dissipation function density:

$$F = \frac{\Gamma\dot{\mathbf{P}}^2}{2\omega_p^2\varepsilon_0} = \frac{\Gamma\mu_0}{2F}\mathbf{M}^2 = \frac{\Gamma}{2\mu_0 F}\dot{\mathbf{Q}}^2 = \frac{1}{2}P_{loss}$$

Lagrangian Density for the SRCM(II)

The equation of motion (with disipassion effect) is written as

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\phi}_a}\right)+\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial\left(\partial \phi_a / \partial x_i\right)}\right)-\frac{\partial L}{\partial \phi_a}=-\frac{\partial F}{\partial \dot{\phi}_a}$$

Explicitly, we have

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\varphi}}\right)+\nabla\cdot\left(\frac{\partial L}{\partial\left(\nabla \varphi\right)}\right)-\frac{\partial L}{\partial \varphi}=0 \quad \Rightarrow \quad \nabla\cdot\mathbf{D}=0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{A}_k}\right)+\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial\left(\partial A_k / \partial x_i\right)}\right)-\frac{\partial L}{\partial A_k}=0 \quad \Rightarrow \quad \nabla\times\mathbf{H}=\dot{\mathbf{D}}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\mathbf{Q}}}\right)+\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial\left(\partial \mathbf{Q} / \partial x_i\right)}\right)-\frac{\partial L}{\partial \mathbf{Q}}=-\frac{\partial F}{\partial \dot{\mathbf{Q}}}$$

$$\Rightarrow \dot{\mathbf{M}}+\gamma\mathbf{M}+\omega_0^2\int\mathbf{M}dt=-F\dot{\mathbf{H}}-\frac{A}{Z_0}\mathbf{E}$$

Hamiltonian Density for SRCM

Canonical momentum densities:

$$\Pi_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = 0, \quad \mathbf{\Pi}_\mathbf{A} = \frac{\partial L}{\partial \dot{\mathbf{A}}} = -\mathbf{D}, \quad \mathbf{\Pi}_\mathbf{Q} = \frac{\partial L}{\partial \dot{\mathbf{Q}}} = \mathbf{H} + \frac{\mathbf{M}}{F} = \mathbf{H}_{in}$$

Hamiltonian density:

$$H = \Pi_\varphi \dot{\varphi} + \mathbf{\Pi}_\mathbf{A} \cdot \dot{\mathbf{A}} + \mathbf{\Pi}_\mathbf{Q} \cdot \dot{\mathbf{Q}} - L$$

$$= \frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{\mu_0}{2}\mathbf{H}^2 + \mu_0 \mathbf{H}\cdot\mathbf{M} + \frac{\mu_0}{2F}\mathbf{M}^2 + \frac{\mu_0\omega_0^2}{2F}\left(\int \mathbf{M}\, dt\right)^2 + \nabla\varphi\cdot\mathbf{D}$$

$$= W + \nabla\cdot\left(\varphi\mathbf{D}\right)$$

The term $\nabla\varphi\cdot\mathbf{D} = \nabla\cdot\left(\varphi\mathbf{D}\right)$ because $\nabla\cdot\mathbf{D} = 0$

$\nabla\cdot\left(\varphi\mathbf{D}\right)$ is an irrelevant surface term, which can be dropped.

$\Rightarrow \quad H = W \equiv$ total energy density

Note that $\dfrac{\mu_0}{2F}\mathbf{M}^2 + \dfrac{\mu_0\omega_0^2}{2F}\left(\int \mathbf{M}\, dt\right)^2 = \dfrac{\dot{\mathbf{P}}^2}{2\varepsilon_0\omega_p^2} + \dfrac{\omega_0^2\mathbf{P}^2}{2\varepsilon_0\omega_p^2}$

Summary: Part II

- The Lagrangian and Hamiltonian density of the EM fields in a wire-SRR medium are derived
- The Hamiltonian obtained is the same as the energy density derived in Part I
- It is possible to study the quantum phenomena in a metamaterial if we quantize the electrodynamics system

Thank you for your attention!

Metamaterial