The Physics of Metamaterials

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- Definition and Examples of Metamaterials
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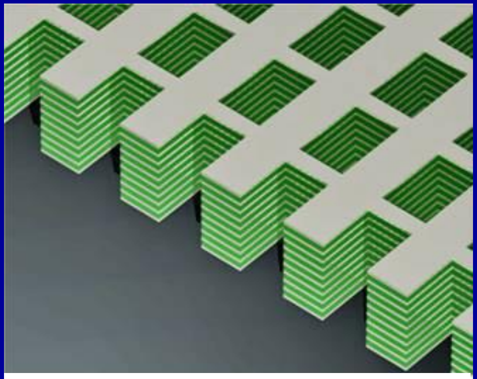
What are Metamaterials ?

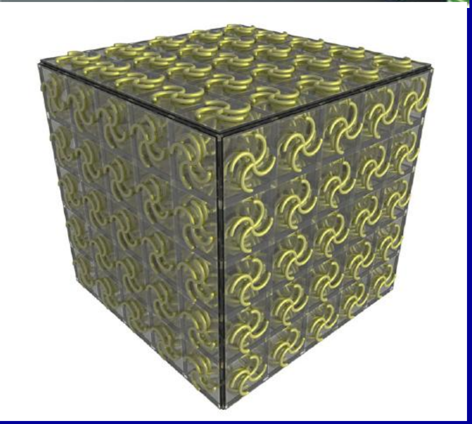
- Metamaterials are artificial materials engineered to have properties that may not be found in nature.

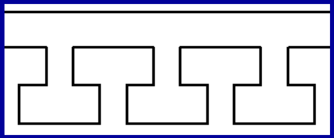
- They usually gain their properties from structure rather than composition, using small inhomogeneities to create effective macroscopic behavior.

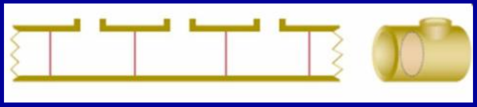
Examples

- Thin wire array (negative ε medium)
- Split-ring resonator (SRR) array (negative μ medium)
- Wire-SRR array (negative *n* medium)
- Metal-dielectric multilayer structure (indefinite medium)
- Conducting Helix/Gamada (卍字形) array (chiral medium)
- Fishnet structure (negative refraction medium)
- Helmholtz resonator/side hole array (negative modulus medium)









Constitutive relations and effective parameters of metamaterials

Effective Permittivity and Permeability

Constitutive relations: $\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}$, $\mathbf{B} = \mu \mu_0 \mathbf{H}$

$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$, $\mathbf{P} =$ electric dipole density $= N\mathbf{p}$

$\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M} \right)$, $\mathbf{M} =$ magnetic dipole density $= N\mathbf{m}$

$N =$ number density

1. In a metal, the "free electrons" are driven by the applied $\mathbf{E}$ field, moving long distances.

When $\mathbf{P} > \mathbf{D}$, we have $\mathbf{D} \cdot \mathbf{E} < 0$, which means $\varepsilon < 0$

2. There is no magnetic charge, thus resonance effect is utilized to make $\mathbf{M}$ large.

When $\mu_0 \mathbf{M} > \mathbf{B}$, we have $\mathbf{B} \cdot \mathbf{H} < 0$ and $\mu < 0$

Left-Handed Media

D. R. Smith *et. al*., Physics Today, 17, **May** (2000).

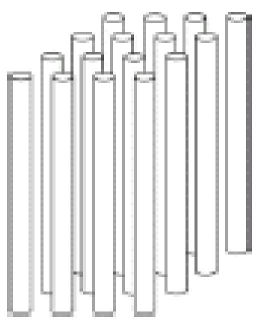
Phys. Rev. Lett. **84**, 4184 (2000) ; Science, **292**, 77 (2001)

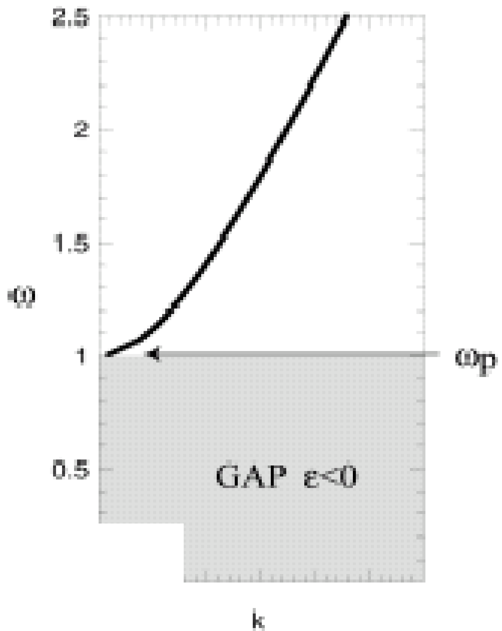
The **Building Blocks** of LHM

Electric Dipoles
(high inductance + dilute carrier density
→low plasma frequency)

+

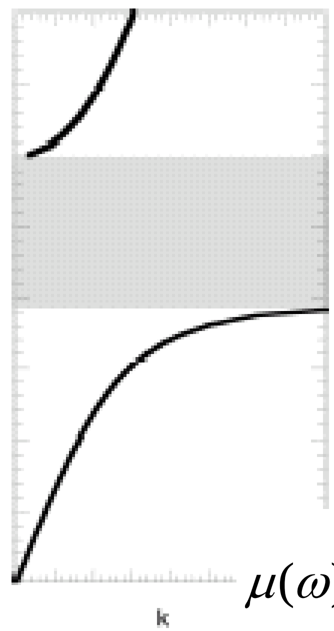
Magnetic Dipoles
(LC resonance of ring current
→ resonance of magnetization)

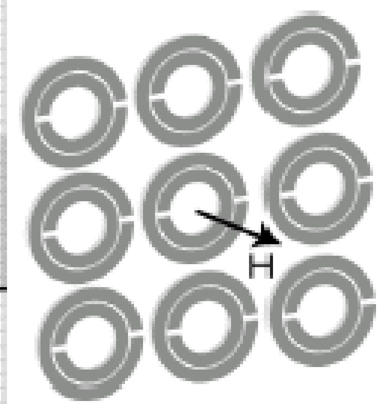




$$\varepsilon(\omega)=1-\frac{\omega_p^2}{\omega^2}$$

(Plasmon-like dispersion)

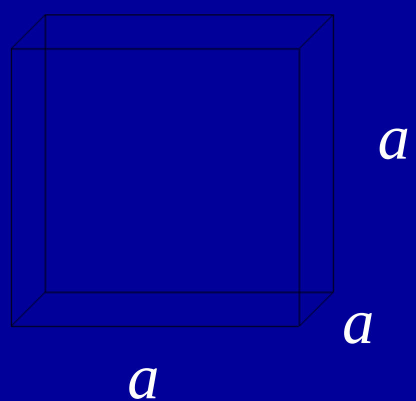




$$\mu(\omega)=1-\frac{F\omega^2}{\omega^2-\omega_0^2}$$

(Polariton-like dispersion)

Effective ‘Material Parameters’



Ampere & Faraday’s laws

$$\int_c \boldsymbol{H}\cdot d\boldsymbol{l} = +\frac{\partial}{\partial t}\int_s \boldsymbol{D}\cdot d\boldsymbol{S}$$
$$\int_c \boldsymbol{E}\cdot d\boldsymbol{l} = -\frac{\partial}{\partial t}\int_S \boldsymbol{B}\cdot d\boldsymbol{S}$$

Averaged H field

$$(H_{\text{ave}})_x = a^{-1}\int_{\boldsymbol{r}=(0,0,0)}^{\boldsymbol{r}=(a,0,0)} \boldsymbol{H}\cdot d\boldsymbol{r}$$
$$(H_{\text{ave}})_y = a^{-1}\int_{\boldsymbol{r}=(0,0,0)}^{\boldsymbol{r}=(0,a,0)} \boldsymbol{H}\cdot d\boldsymbol{r}$$
$$(H_{\text{ave}})_z = a^{-1}\int_{\boldsymbol{r}=(0,0,0)}^{\boldsymbol{r}=(0,0,a)} \boldsymbol{H}\cdot d\boldsymbol{r}.$$

Averaged B field

$$(B_{\text{ave}})_x = a^{-2}\int_{S_x} \boldsymbol{B}\cdot d\boldsymbol{S}$$
$$(B_{\text{ave}})_y = a^{-2}\int_{S_y} \boldsymbol{B}\cdot d\boldsymbol{S}$$
$$(B_{\text{ave}})_z = a^{-2}\int_{S_z} \boldsymbol{B}\cdot d\boldsymbol{S}.$$

Effective permeability

$$(\mu_{\text{eff}})_x = (B_{\text{ave}})_x/(\mu_0 H_{\text{ave}})_x$$
$$(\mu_{\text{eff}})_y = (B_{\text{ave}})_y/(\mu_0 H_{\text{ave}})_y$$
$$(\mu_{\text{eff}})_z = (B_{\text{ave}})_z/(\mu_0 H_{\text{ave}})_z.$$

IEEE Trans. Microwave Theory Tech. 47, 2075 (1999)

Dynamical Equation of Polarization P (I)

The current equation $L\frac{dI}{dt}+RI=V=El \Rightarrow \frac{d}{dt}\left(\frac{LI^2}{2}\right)+RI^2=IV$

can be identified as $m_{eff}\frac{dv}{dt}+bv=qE \Rightarrow \frac{d}{dt}\left(\frac{m_{eff}v^2}{2}\right)+bv^2=qEv$

if $m_{eff}v\frac{l}{q}=LI$ or $N\pi r^2 l\frac{m_{eff}v^2}{2}=\frac{LI^2}{2}$, N is the charge concentration.

Using the relation $I=\pi r^2\left(Nqv\right)$, we find $m_{eff}=\frac{N\pi r^2 L}{l}q^2$

$$L=\frac{\Phi}{I}=\frac{\mu_0}{I}\int Hda=\frac{\mu_0}{I}\int_r^a HldR=\frac{\mu_0 l}{I}\int_r^a\frac{I}{2\pi R}dR=\frac{\mu_0 l}{2\pi}\ln\left(\frac{a}{r}\right)$$

$$\Rightarrow \quad m_{eff}=\frac{Nr^2\mu_0 q^2}{2}\ln\left(\frac{a}{r}\right)=\frac{\mu_0 e^2}{2}Nr^2\ln\left(\frac{a}{r}\right)$$

J. B. Pendry, Phys. Rev. Lett.76, 4773 (1996)

Dynamical Equation of Polarization P (II)

Effective concerntration of charge : $N_{eff} = \dfrac{\pi r^2}{a^2} N$

Polarization: $\mathbf{P} = N_{eff} q\mathbf{r},$ where $\mathbf{r} = \int_0^t \mathbf{v}\, dt$

From $m_{eff} \dfrac{d\mathbf{v}}{dt} + b\mathbf{v} = q\mathbf{E},$ where $m_{eff} = \dfrac{Nr^2 \mu_0 q^2}{2} \ln\left(\dfrac{a}{r}\right)$

$\Rightarrow$ Equation of motion $\dfrac{d^2}{dt^2}\mathbf{P} + \nu \dfrac{d}{dt}\mathbf{P} = \omega_p^2 \varepsilon_0 \mathbf{E}$

Plasma frequency: $\omega_p = \sqrt{\dfrac{N_{eff} q^2}{m_{eff} \varepsilon_0}}$, Dissipation coefficient: $\nu = \dfrac{b}{m_{eff}}$

Permittivity: $\varepsilon(\omega) = \dfrac{D}{\varepsilon_0 E} = 1 - \dfrac{P}{\varepsilon_0 E} = 1 - \dfrac{\omega_p^2}{\omega(\omega + i\nu)}$

J. B. Pendry, Phys. Rev. Lett.76, 4773 (1996)

Effective Plasma vs. Real Plasma

金屬線的二維周期陣列所形成的等效電漿介質，其電漿頻率 ω_p 是由公式

$$\omega_p^2 = \frac{n_{eff}\, e^2}{\varepsilon_0 m_{eff}}$$

所決定　。其中 $n_{eff} = n\dfrac{\pi r^2}{a^2}$ 是金屬線的電子濃度 n 乘以面積比率 $f = \dfrac{\pi r^2}{a^2}$；這部份的確是按照比例的平均。但公式中的電子等效質量

$$m_{eff} = \frac{\mu_0\, n\, e^2 r^2}{2}\ln(a/r)$$

並不是電子原來的質量，而是與電子濃度 n 以及金屬線半徑 r 與晶格常數 a 的比值有關。仔細比較細金屬線陣列與原來的 Drude 模型可發現這兩者的差異：細金屬線具有很大的自感 (self inductance)，亦即金屬線的感應電流在其周圍產生很強的磁場，而這是在原來的 Drude 模型中是不存在的 (因其電荷密度與電流密度皆均勻分布)。若將對應於此磁場的磁場能 (magnetic field energy) 等效為一個假想的動能，就又能以 Drude 模型處理了，但此時電荷原來的質量必須被上述的等效質量所取代。

Dynamical Equation of Magnetization M (I)

Faraday's law+effect of depolarization field

$$\Rightarrow (1-F)L\frac{dI}{dt}+RI+\frac{q}{C}=-\frac{d\Phi}{dt},$$

F : filling fraction of the ring (SRR) in one unit cell.

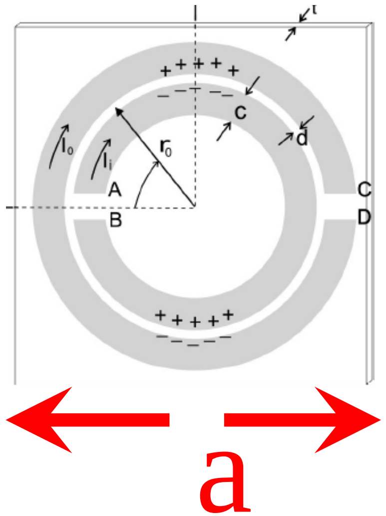
Magenetic fields inside/outside the ring: H_{in}/H, Applied field: $H_0=\dfrac{B}{\mu_0}$

$H_{in}=H+\dfrac{I}{l}$ (from Ampere's law),

$FH_{in}+(1-F)H=H_0$ (flux averaging)

$$\Phi=\pi r^2 B=\mu_0 F a^2 H_0,\quad L=\frac{\pi r^2\mu_0 I/l}{I}=\mu_0\frac{\pi r^2}{l}=\mu_0 F\frac{a^2}{l}$$

$$M=\frac{\pi r^2 I}{a^2 l}=F\frac{I}{l}=\frac{LI}{\mu_0 a^2}\quad\Rightarrow\quad H_0=H+M,\quad LI=\mu_0 a^2 M$$



IEEE Trans. Microwave Theory Tech. 47, 2075 (1999)
J. Appl. Phys. 100, 024915 (2006)

Dynamical Equation of Magnetization M (II)

Effective inductance of a SRR is $(1-F)L$

The term $-FL$ is caused by the dipolarization field

$$(1-F)L\frac{dI}{dt}+RI+\frac{q}{C}=-F\mu_0 a^2\frac{dH_0}{dt}$$

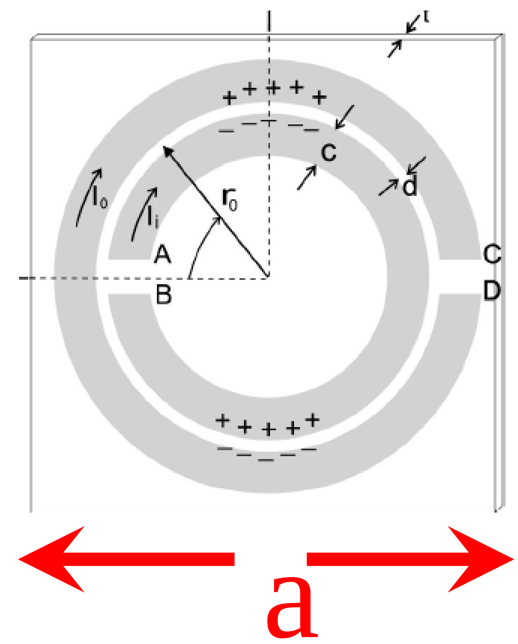
$$(1-F)\mu_0 a^2\frac{dM}{dt}+\frac{R}{L}\mu_0 a^2 M+\frac{\mu_0 a^2}{LC}\int Mdt$$

$$=-F\mu_0 a^2\frac{dH_0}{dt}$$

Dynamical equation of $\mathbf{M}$:

$$\frac{d\mathbf{M}}{dt}+\gamma\mathbf{M}+\omega_0^2\int\mathbf{M}dt=-F\frac{d\mathbf{H}}{dt}$$

where $\gamma=\frac{R}{L},\quad \omega_0^2=\frac{1}{LC}$

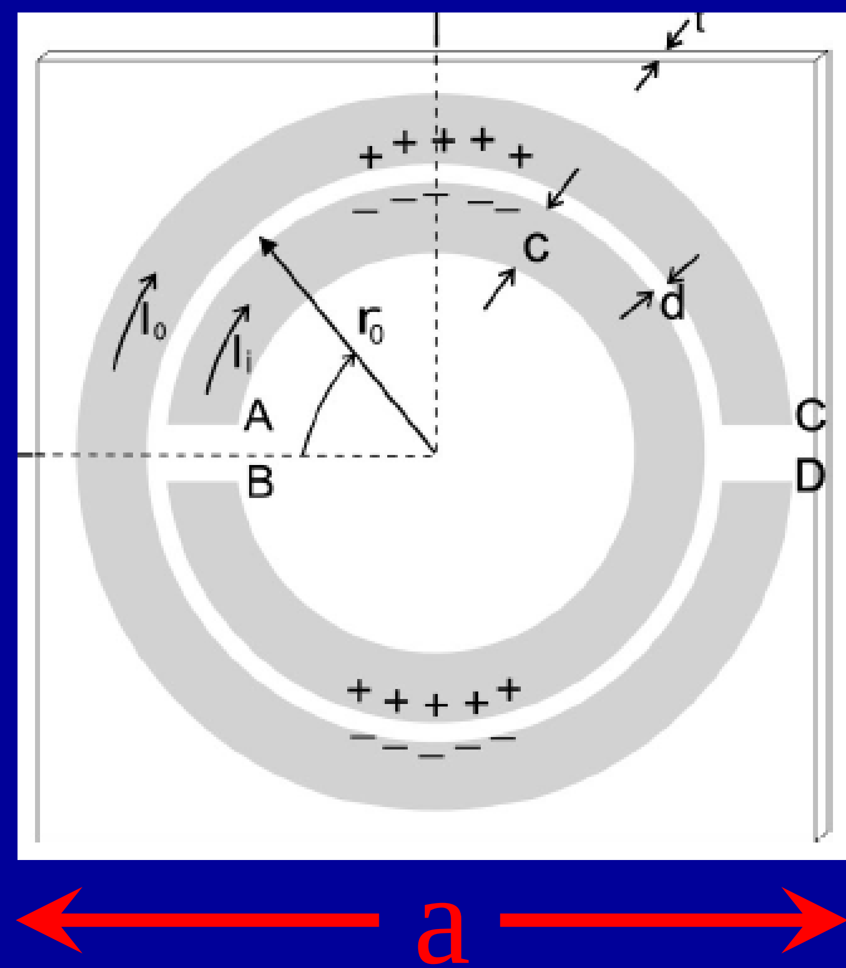


IEEE Trans. Microwave Theory Tech. 47, 2075 (1999)

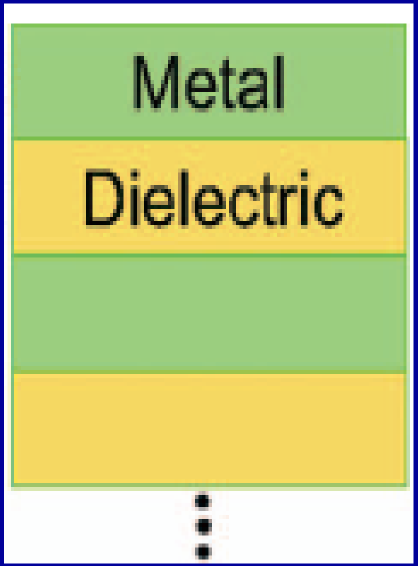
J. Appl. Phys. 100, 024915 (2006)

Size determines the Response

- 以 SRR (雙層環) 陣列為例，假定每個裂環 的半徑為 r。要讓這個環 (LC 電路) 在外加時變磁場的影響下有強烈反應 (非斷路)，外加磁場的波長就應接近此環周長的兩倍，即 $4\pi r$。此時圍繞此環的電流分布 (對角度) 近似於一個正弦函數，在兩個裂口處為 0；電荷分布則是餘弦函數，亦即半圈為正，半圈為負，且內外環電荷相反。若此環直 徑 2r 近似於晶格常數 a (環要夠大才能圍住夠多磁通)， 則估計的波長是 $2\pi a$，大約是晶格常數 a 的 6 倍，與目前已知的實驗結果相吻合。



Indefinite Medium (I)



1D dispersion relation

$$\cos Ka = \cos k_m a_m \cos k_d a_d - \frac{1}{2}\left(\frac{k_m \varepsilon_d}{k_d \varepsilon_m} + \frac{k_d \varepsilon_m}{k_m \varepsilon_d}\right)\sin k_m a_m \sin k_d a_d$$

Long wavelength limit

$$\frac{K^2}{\langle \varepsilon \rangle} + \left\langle \frac{1}{\varepsilon} \right\rangle k_y^2 = \frac{\omega^2}{c^2}, \text{ 其中 } \langle \varepsilon \rangle = \frac{a_m \varepsilon_m + a_d \varepsilon_d}{a}, \ \left\langle \frac{1}{\varepsilon} \right\rangle = \frac{a_m / \varepsilon_m + a_d / \varepsilon_d}{a}$$

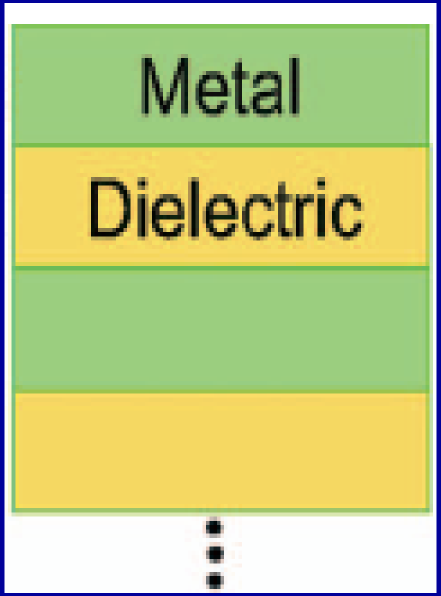
Define $k_{\parallel} = k_y$, $k_{\perp} = K$, $\varepsilon_{\parallel} = \langle \varepsilon \rangle$, $1/\varepsilon_{\perp} = \langle 1/\varepsilon \rangle$

$$\rightarrow \frac{k_{\perp}^2}{\varepsilon_{\parallel}} + \frac{k_{\parallel}^2}{\varepsilon_{\perp}} = \frac{\omega^2}{c^2}$$

If $\varepsilon_{\parallel} > 0, \ \varepsilon_{\perp} < 0$

$$\rightarrow \frac{k_{\perp}^2}{\varepsilon_{\parallel}} - \frac{k_{\parallel}^2}{|\varepsilon_{\perp}|} = \frac{\omega^2}{c^2}$$

Indefinite Medium (II)



A. Tangential field

$$P_{//} = \frac{a_d}{a} P_d + \frac{a_m}{a} P_m, \quad \boxed{E_{//} = E_d = E_m}$$

From Faraday's law

$$\Rightarrow D_{//} = \varepsilon_0 E_{//} + P_{//} = \frac{a_d}{a} D_d + \frac{a_m}{a} D_m$$

$$= \left(\frac{a_d}{a} \varepsilon_d + \frac{a_m}{a} \varepsilon_m \right) E_{//} = \varepsilon_{//} E_{//}$$

$$\Rightarrow \boxed{\varepsilon_{//} = \frac{a_d}{a} \varepsilon_d + \frac{a_m}{a} \varepsilon_m}$$

From "div D = 0"

B. Normal field

$$E_{\perp} = \frac{V}{a} = \frac{V_d}{a} + \frac{V_m}{a} = \frac{E_d a_d}{a} + \frac{E_m a_m}{a}, \quad \boxed{D_{\perp} = D_d = D_m}$$

$$\frac{E_{\perp}}{D_{\perp}} = \frac{a_d}{a} \frac{E_d}{D_d} + \frac{a_m}{a} \frac{E_m}{D_m} \Rightarrow \boxed{\frac{1}{\varepsilon_{\perp}} = \frac{a_d}{a} \frac{1}{\varepsilon_d} + \frac{a_m}{a} \frac{1}{\varepsilon_m}}$$

Isotropic Media

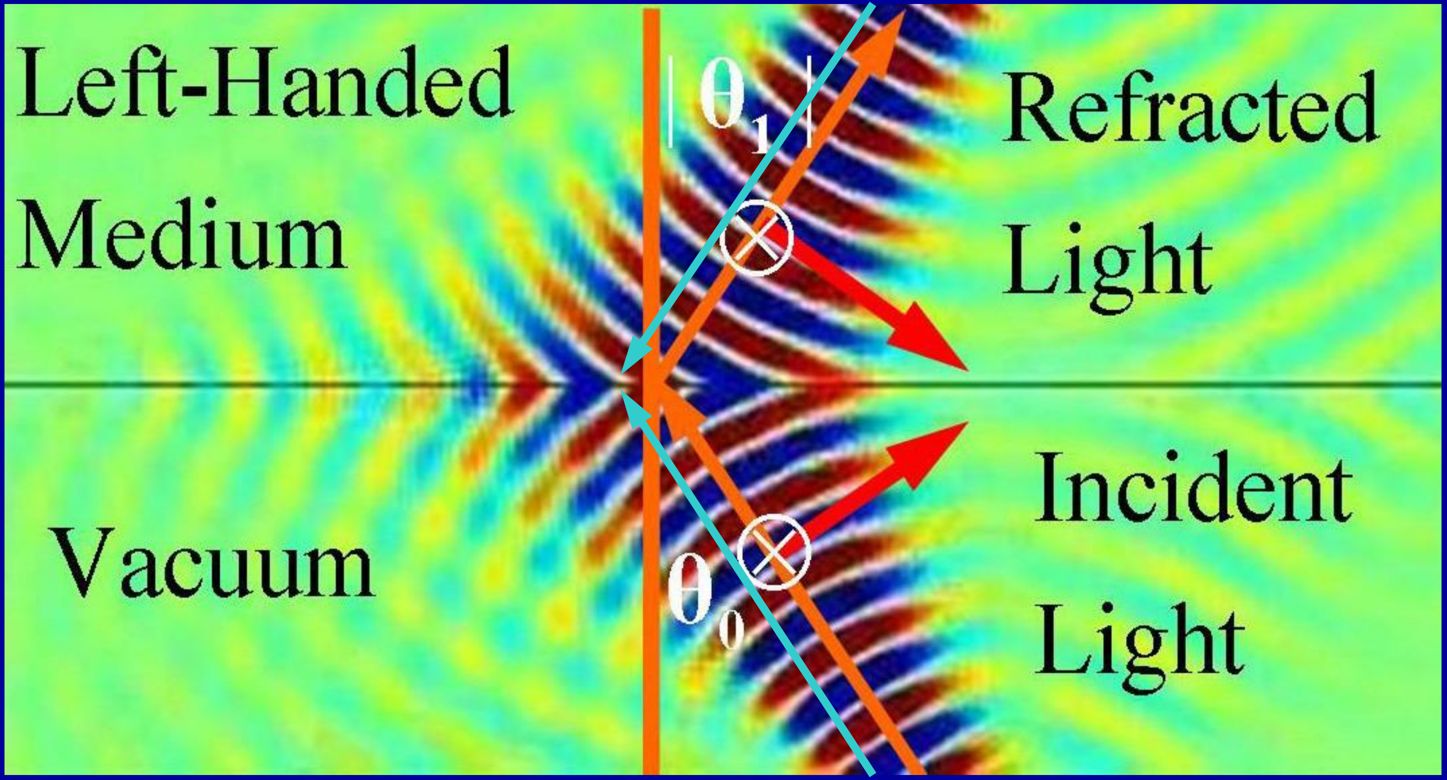
Negative Refraction in Left-Handed Medium ($\varepsilon<0$, $\mu<0$): The TM wave case

Boundary Conditions I

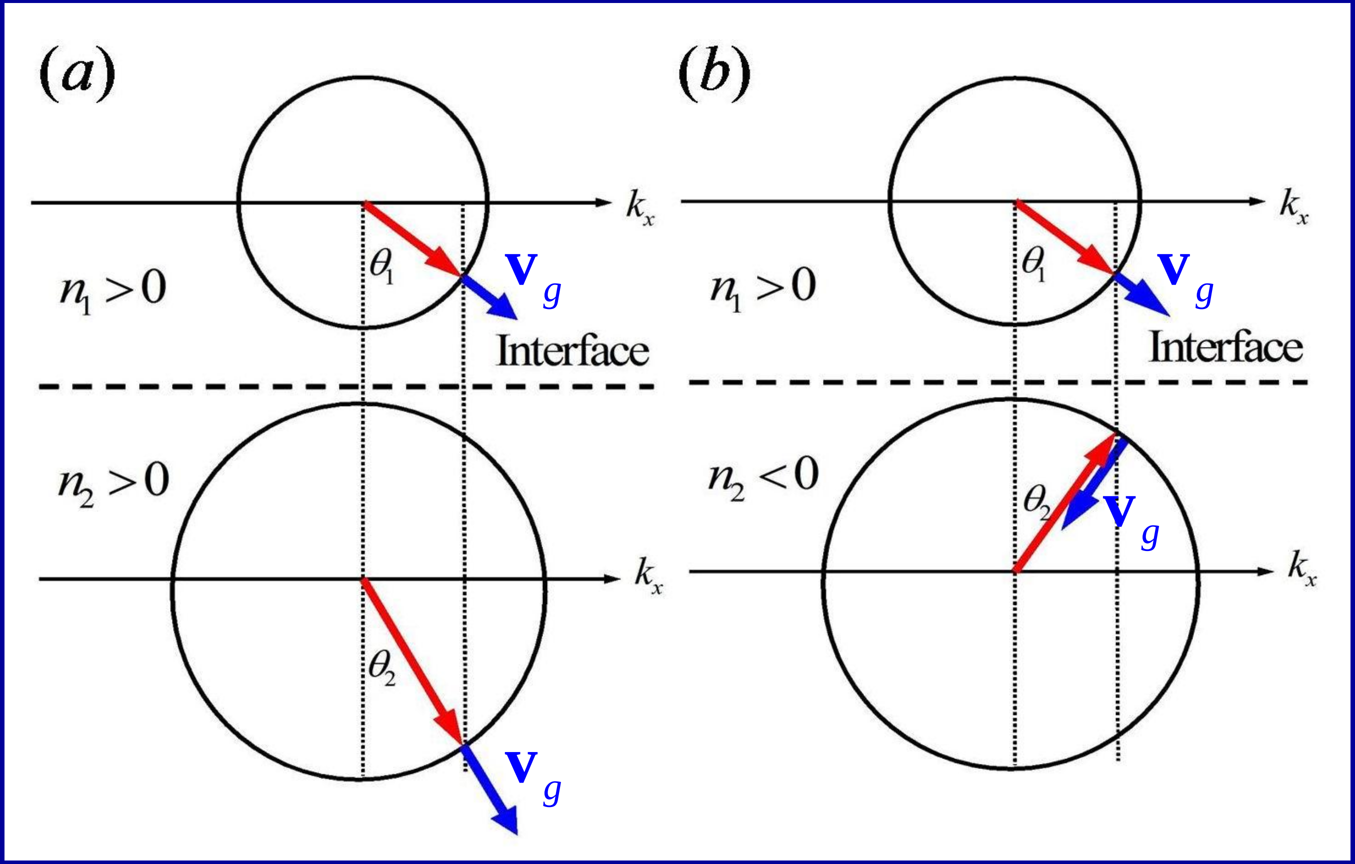
continuity of E_t, H_t, B_n, D_n

Boundary Conditions II

continuity of S_n, k_t

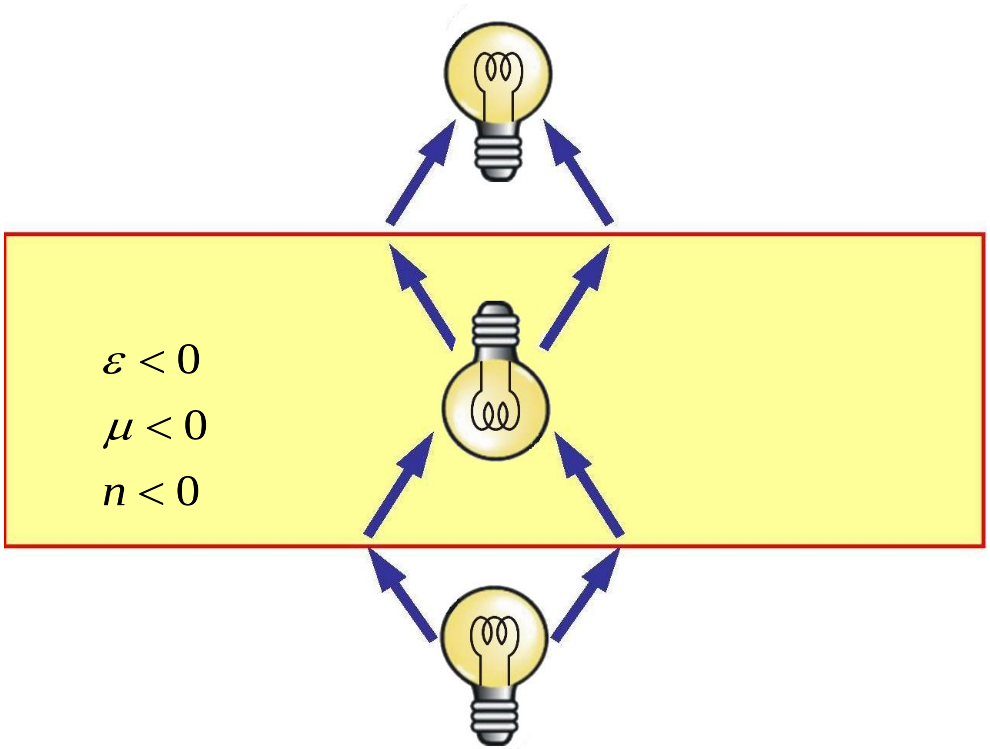


Snell's Law



$$k_{1x} = k_{2x} \quad \text{or} \quad \frac{\omega}{c} n_1 \sin\theta_1 = \frac{\omega}{c} n_2 \sin\theta_2$$

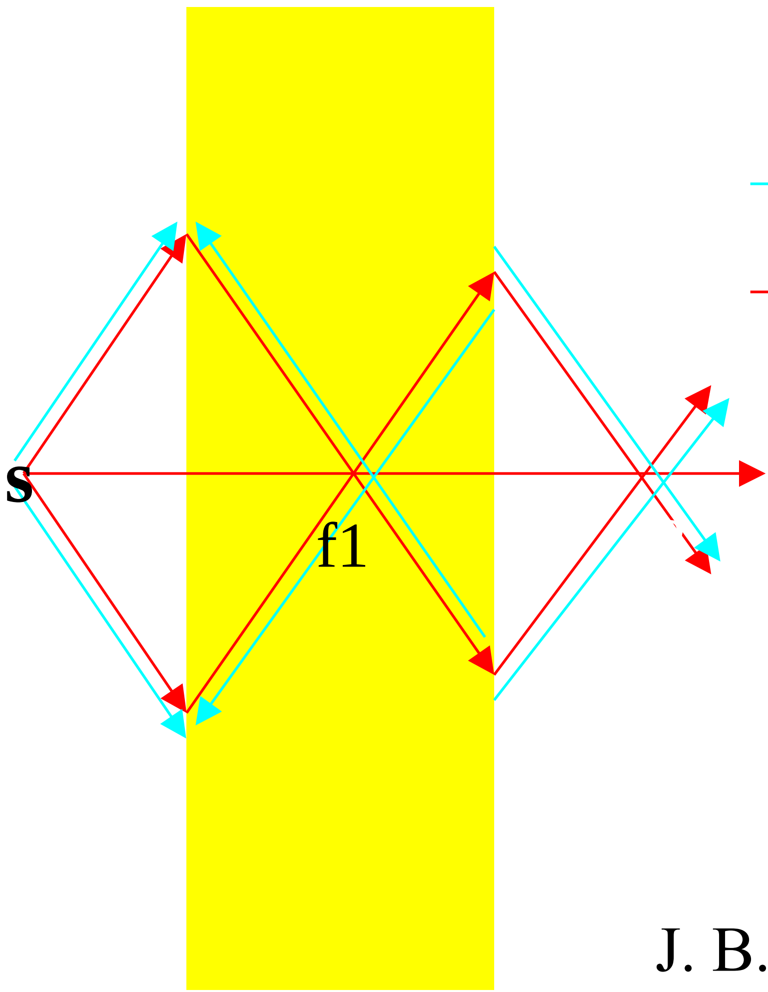
Veselago's Left Handed Slab Lens



The phase increasing outside can be cancelled by the phase decreasing inside the slab

V. G. Veselago, Sov. Phys. Usp. **10**, 509 (1968)

Pendry’s idea of Perfect Lens



$$\varepsilon = \mu = n = -1$$

$\vec{k}$ vector (phase velocity)

Poynting vector (energy flow)

Veselago lens of $\varepsilon = \mu = -1$

+ propagating waves

+ evancesent waves

Phase and Amplitude Compensation

J. B. Pendry, “*Negative Refraction Makes a Perfect Lens*”, Phys. Rev. Lett. **80**, 3966 (2000)

Mechanism of Subwavelength Focusing

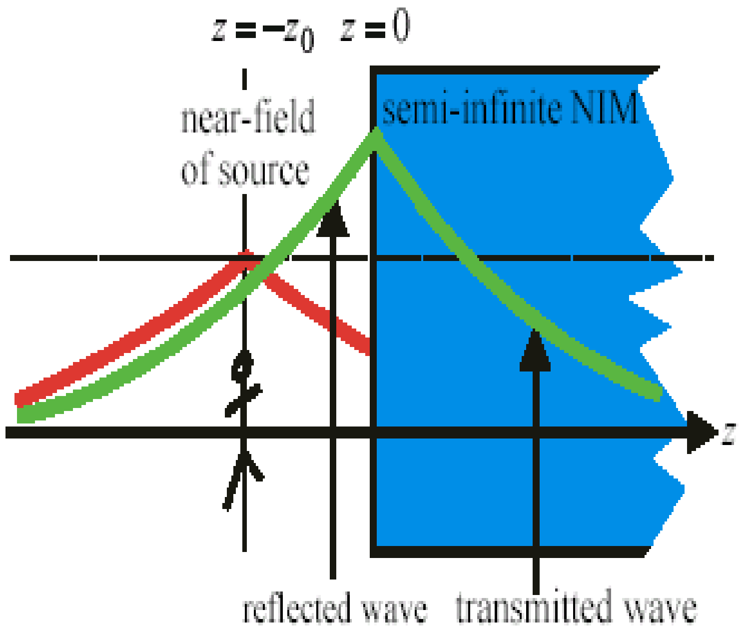


Figure 1. An source of electromagnetic fields excite surface plasmons at the vacuum/NIM interface.

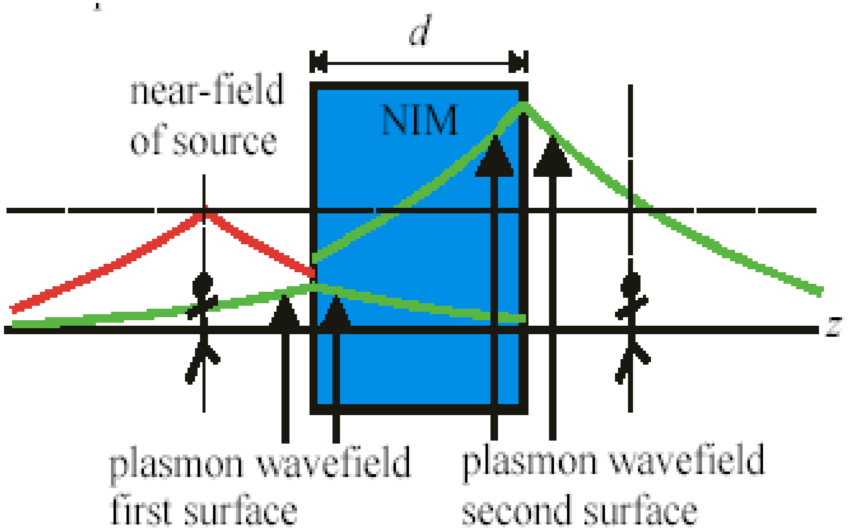
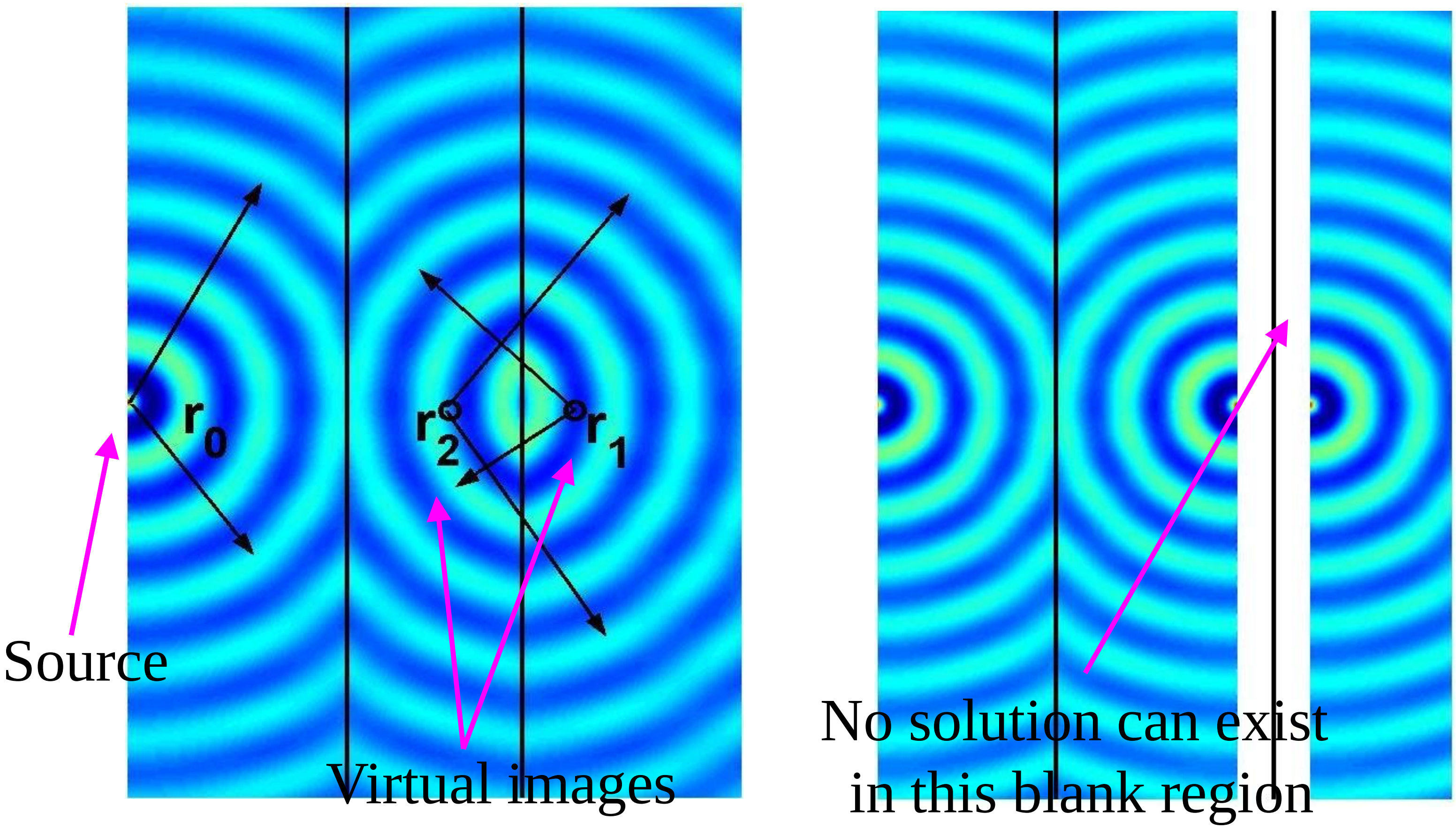


Figure 2. As for figure 1 but with a finite slab of NIM. In the lossless limit of $\delta \to 0$ the first surface plasmon has zero amplitude and only the second surface is excited, refocusing the image at a distance $2d$ from the source.

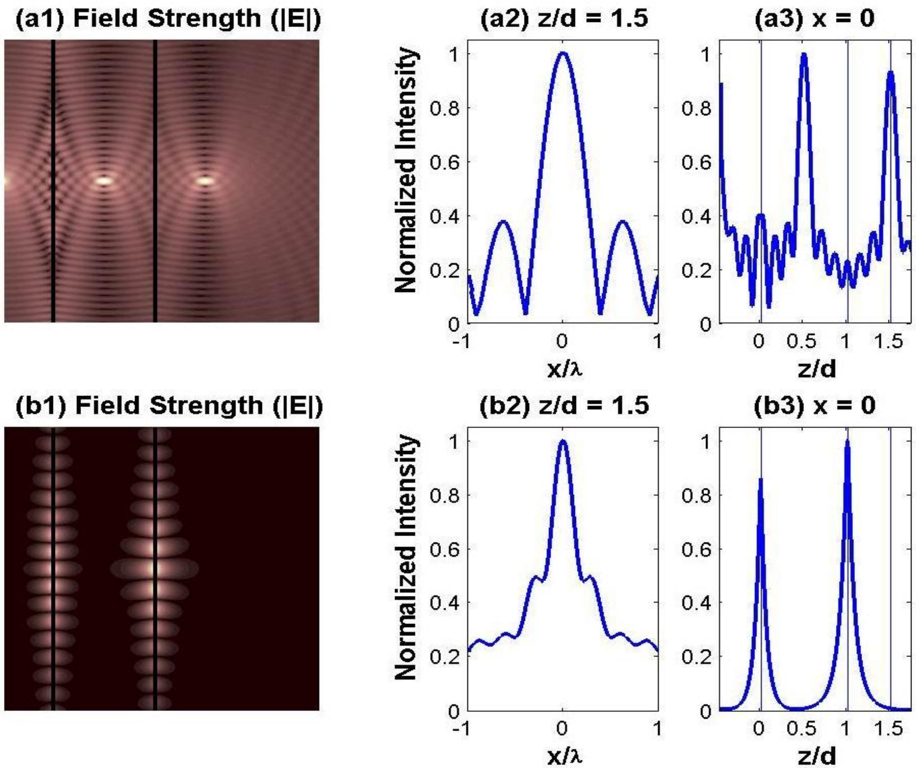
“Perfect” is unphysical
“Super” is achievable



$d < |z_0|$

$d > |z_0|$

“Imperfect” Superlens for Subwavelength Imaging (superlensing effect)



$$\text{Case 1: } \Delta x \approx \lambda$$

$$z_0 = -1, d = 2,$$

$$\lambda = 2\pi / k = 0.3,$$

$$\varepsilon = \mu = -1.0 + 0.001i$$

$$\text{Case 2: } \Delta x \ll \lambda$$

$$z_0 = -1, d = 2,$$

$$\lambda = 2\pi / k = 2,$$

$$\varepsilon = \mu = -1.00 + 0.000001i$$

Uncertainty Principle vs. Subwavelength Focusing

This decaying behavior can be easily explained by the uncertainty principle. According to this principle, we must have the relation $\Delta x \, \Delta k_x \geq 1$, here Δx represents the width of the image, and Δk_x represents the fluctuation of k_x. A subwavelength image is mainly formed by summing over the Fourier components of those $|k_x| \gg \omega/c$ terms. Since $k_x^2 + k_z^2 = \omega^2 / c^2$, these components must have imaginary k_z's. This leads to the decaying profile of the field strength.

References for understanding the ‘Subtleties’ of the ‘Perfect Lens’

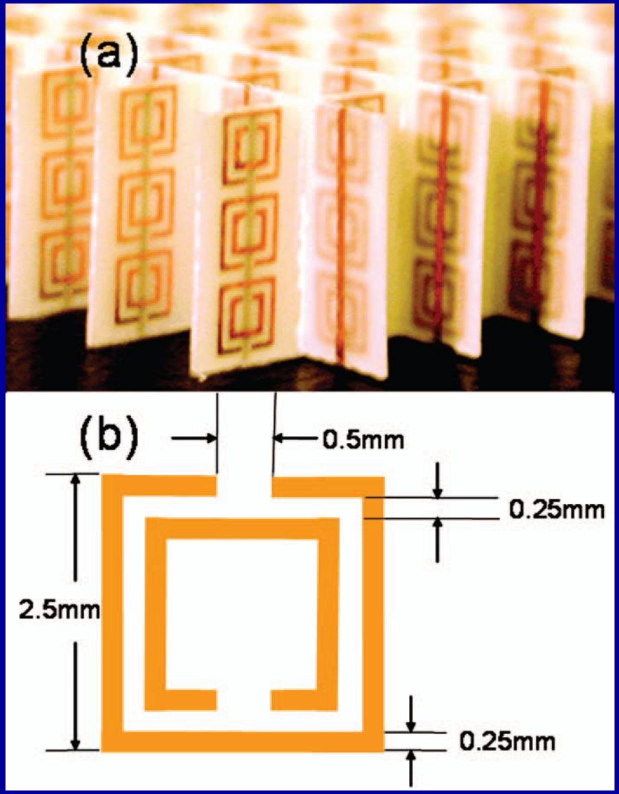
- *Negative Refraction Makes a Perfect Lens*
  J. B. Pendry, Phys. Rev. Lett. 85, 3966 (2000)
- *Left-Handed Materials Do Not Make a Perfect Lens*
  N. Garcia et al., Phys. Rev. Lett. 88, 207403 (2002)
- *Perfect lenses made with left-handed materials: Alice’s mirror?*
  Daniel Maystre and Stefan Enoch, J. Opt. Soc. Am. A, 21, 122 (2004)
- *Universal Features of the Time Evolution of Evanescent Modes in a Left-Handed Perfect Lens*
  G. Gomez-Santos, Phys. Rev. Lett. 90, 077401 (2003)
- *Analysis on the imaging properties of a left-handed material slab Pi-Gang Luan, Hung-Da Chien, Chii-Chang Chen, Chi-Shung Tang*, arXiv:physics/0311122;
  Proper boundary conditions for the imaging problem of a negative-refraction slab lens, WSEAS transactions on electronics, Vol. 1, 236 (2004)

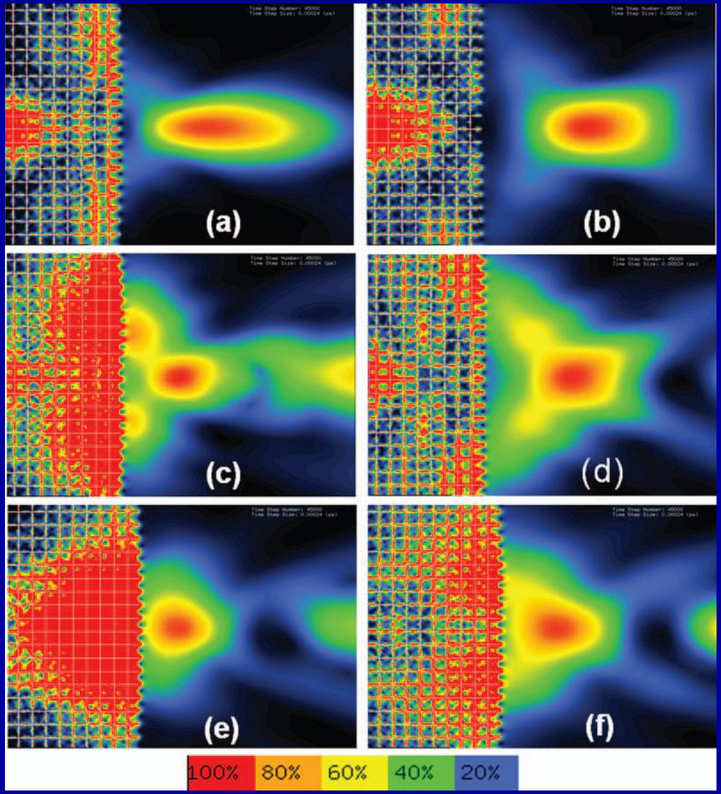
Is the working wavelength much longer than the lattice constant?


Typical value: λ/a = 5~7

FDTD simulations of SRR-Wire based metamaterial slab lens

Simulation:
SGI Altix parallel computer using 32-48 processors and require about
20 h of processor time



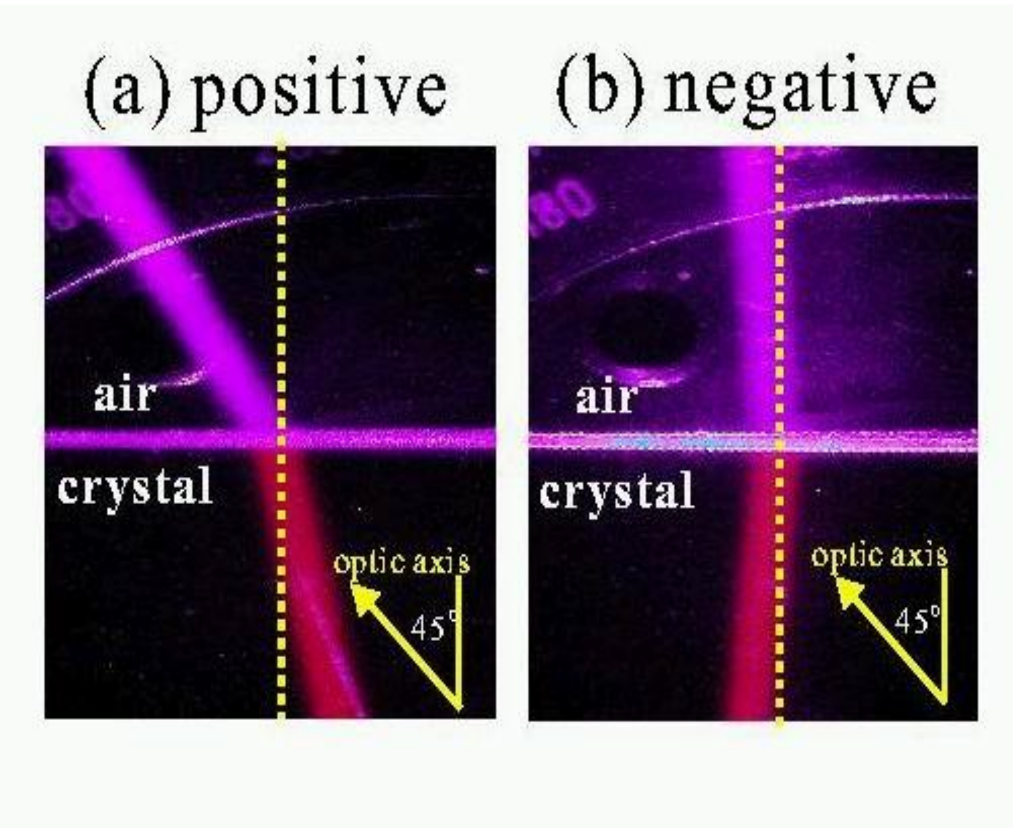


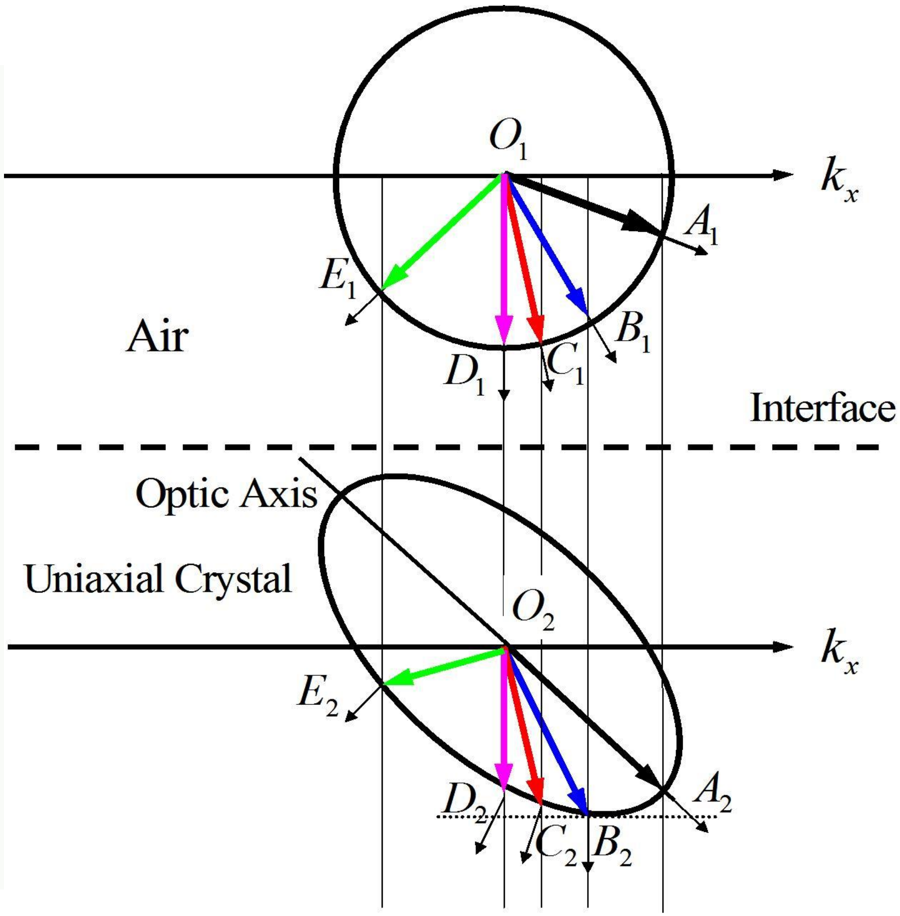
$\lambda \sim 4.5\ a$

Appl. Phys. Lett. **91**, 154102 (2007)

Anisotropic Media

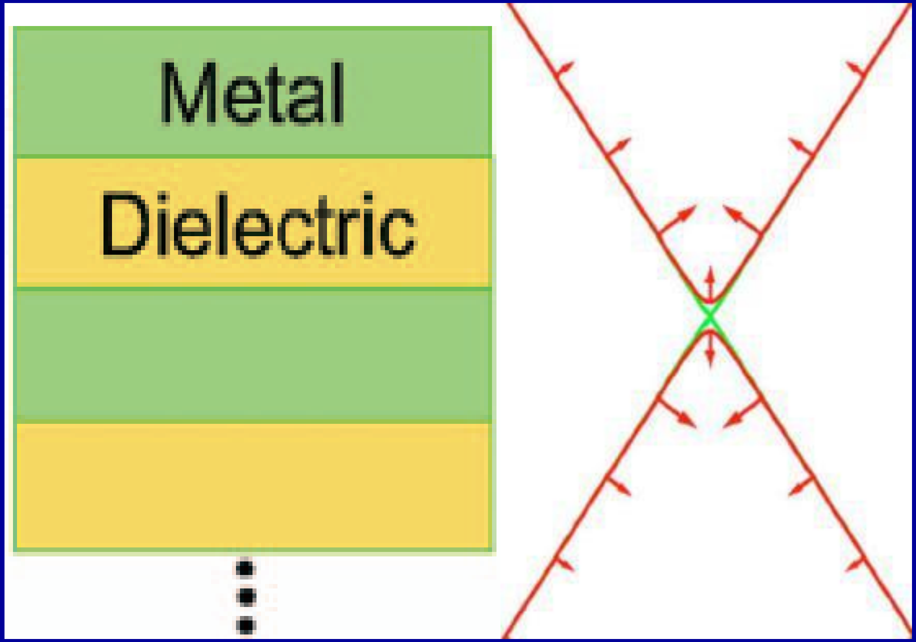
Negative Refraction in Calcite (Yau *et.al*.)

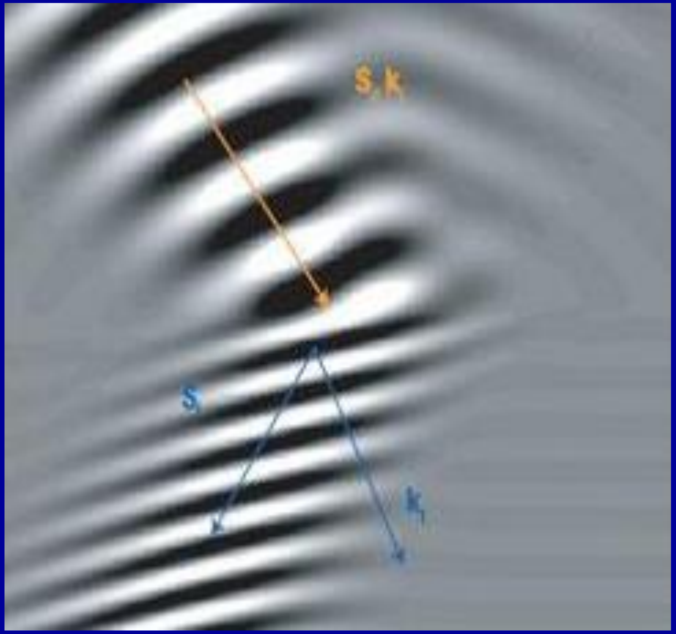




http://arxiv.org/abs/cond-mat/0312125

Indefinite media and Hyperbolic Dispersion



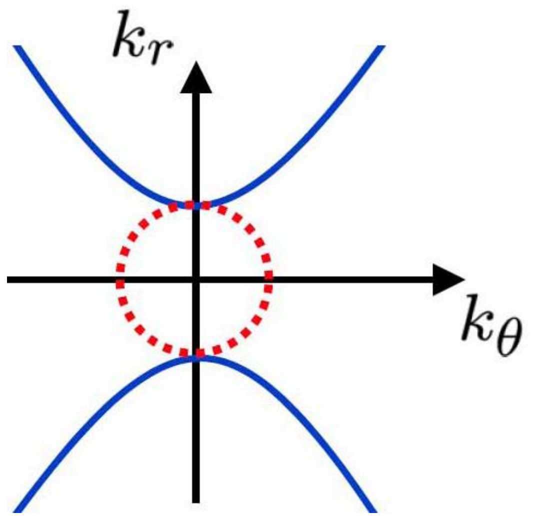


Hyperbolic dispersion relation

If $\varepsilon_{\parallel} > 0,\ \varepsilon_{\perp} < 0$ $\rightarrow$ $\dfrac{k_{\perp}^{2}}{\varepsilon_{\parallel}} - \dfrac{k_{\parallel}^{2}}{|\varepsilon_{\perp}|} = \dfrac{\omega^{2}}{c^{2}}$

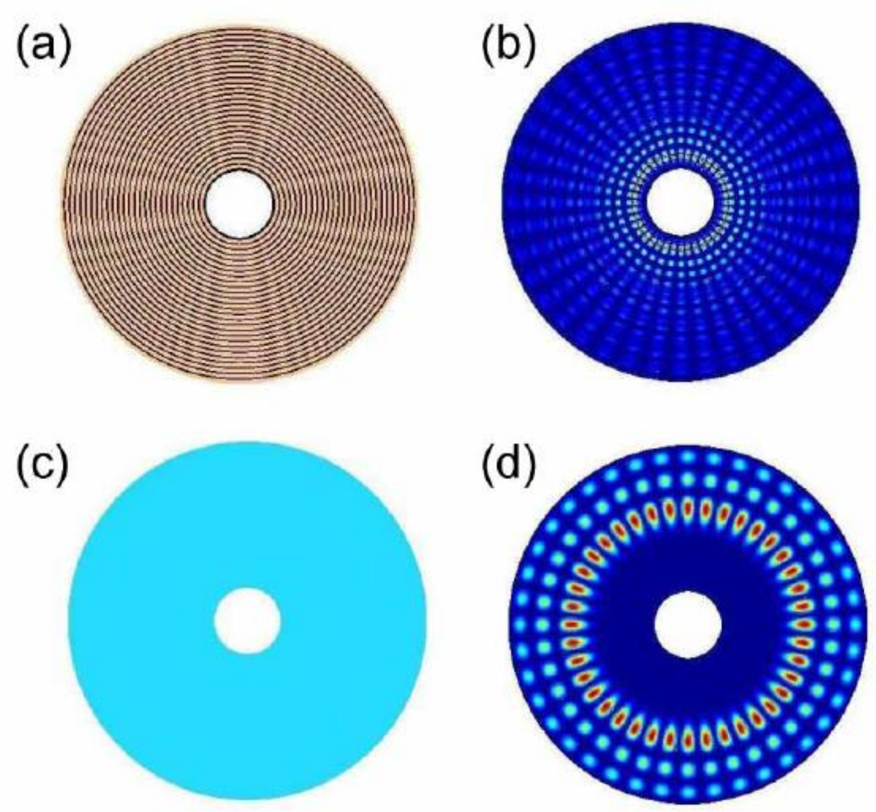
Hyperlens

Far-field imaging beyond the diffraction limit



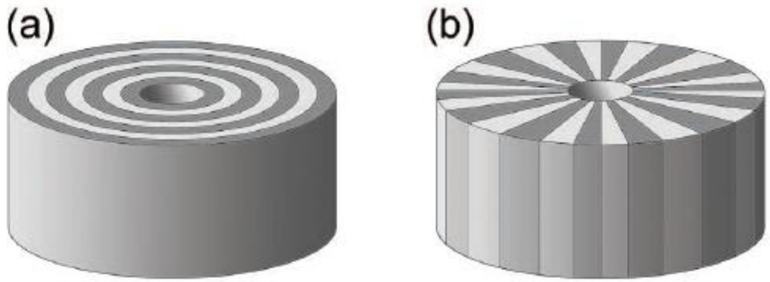
$$\frac{k_r^2}{\varepsilon_\theta} - \frac{k_\theta^2}{|\varepsilon_r|} = \frac{\omega^2}{c^2}$$

$$\varepsilon_\theta > 0, \quad \varepsilon_r < 0$$

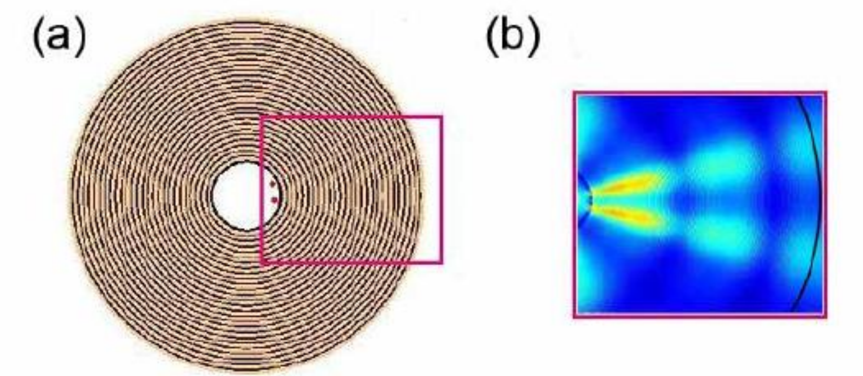


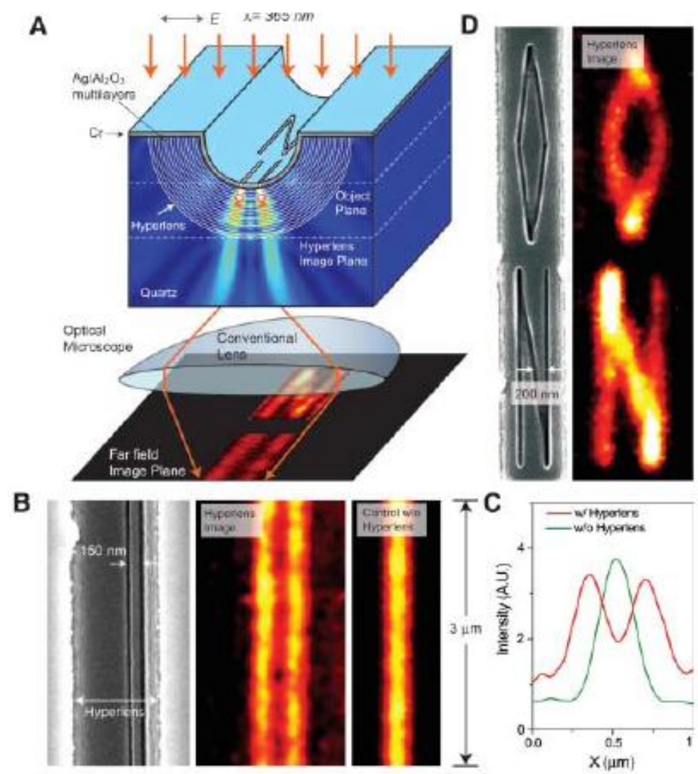
OPTICS EXPRESS 14, 8247 (2006)

Effective medium: layered structures



$$\varepsilon_\theta = \frac{\varepsilon_m + \varepsilon_d}{2}, \quad \varepsilon_r^{-1} = \frac{1}{2}\left(\varepsilon_m^{-1} + \varepsilon_d^{-1}\right)$$





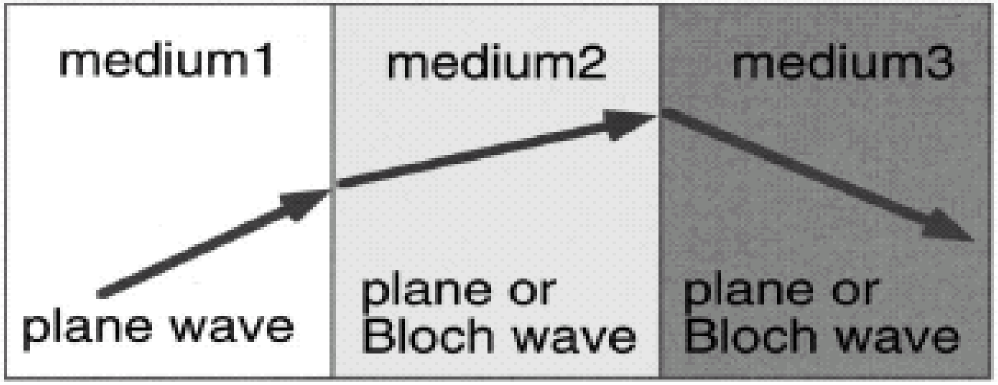
SCIENCE 315, 1686 (2007)

Inhomogeneous Media

Theory of light propagation in strongly modulated photonic crystals: Refractionlike behavior in the vicinity of the photonic band gap

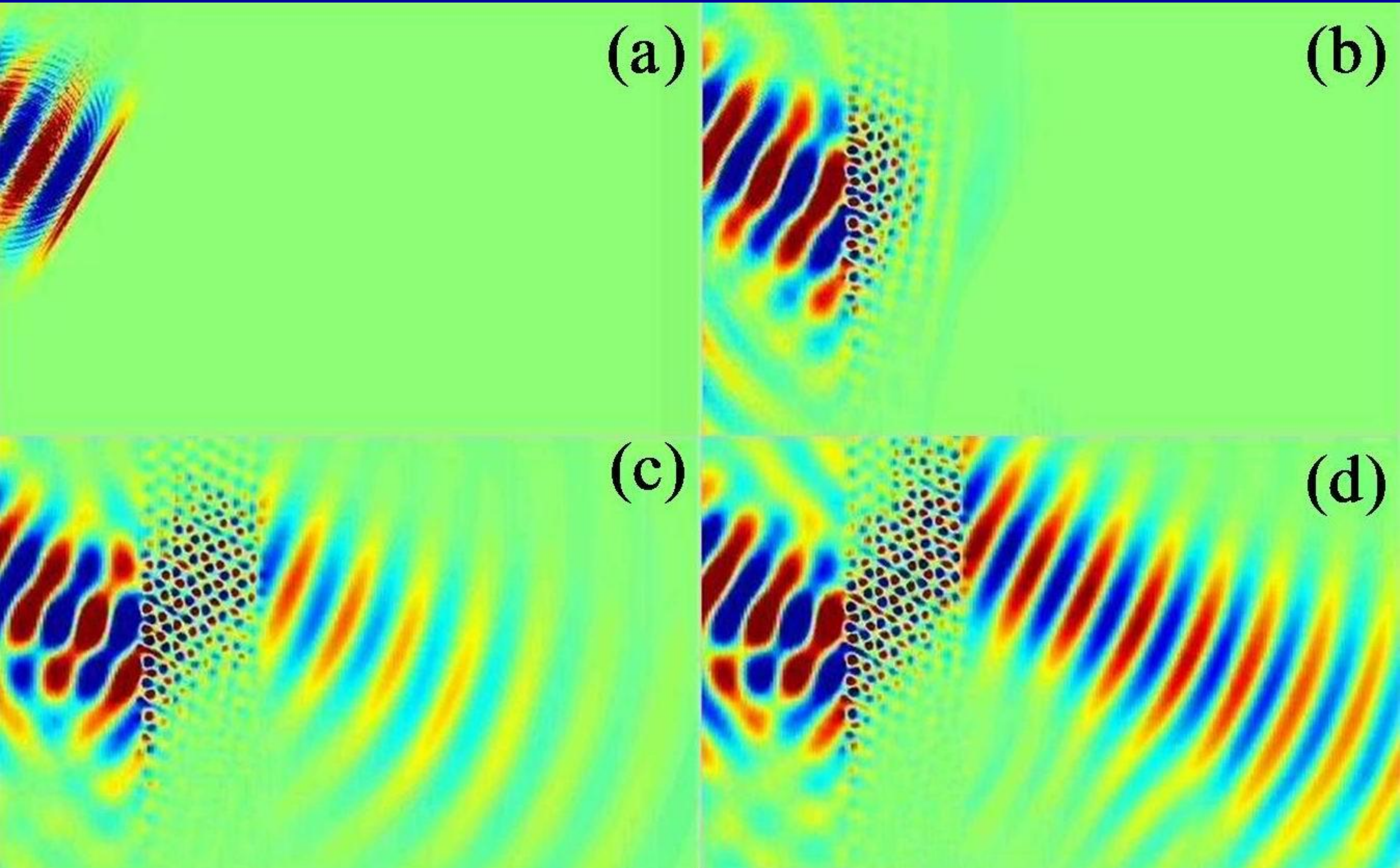
M. Notomi*
NTT Basic Research Laboratories, 3-1 Morinosato-Wakamiya, Atsugi, Kanagawa 243-0198, Japan
(Received 7 April 2000)

Although light propagation in weakly modulated photonic crystals is basically similar to propagation in a diffraction grating in which conventional refractive index loses its meaning, we demonstrate that light propagation in strongly modulated two-dimensional (2D)/3D photonic crystals becomes refractionlike in the vicinity of the photonic bandgap. Such a crystal behaves as a material having an effective refractive index controllable by the band structure. This situation is analogous to the effective-mass approximation in electron-band theory. By utilizing this phenomenon, negatively refractive material can be realized, which has interesting optical properties such as mirror-image refraction.



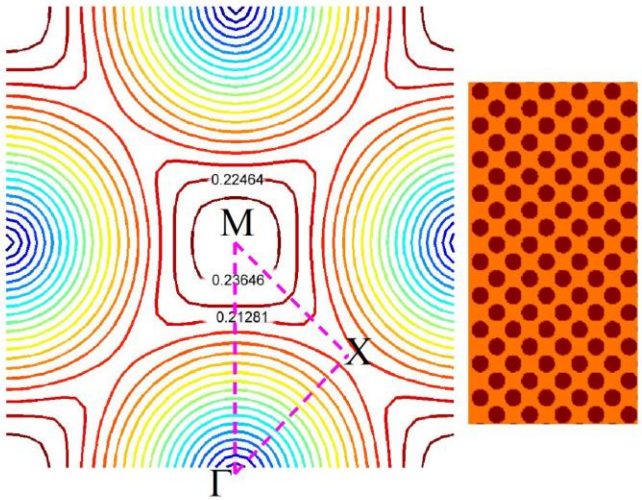
M. Notomi, Phys. Rev. B **62**, 10 696 (2000).

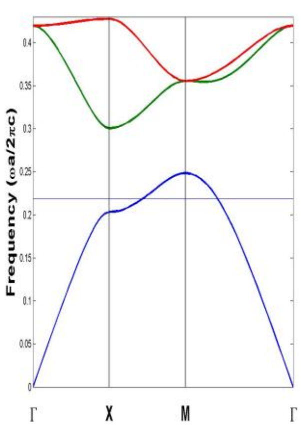
Negative Refraction by PC

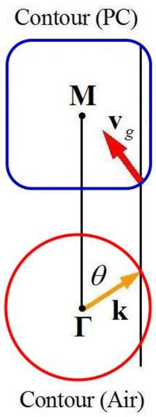


Snell's Law—The Generalized Form

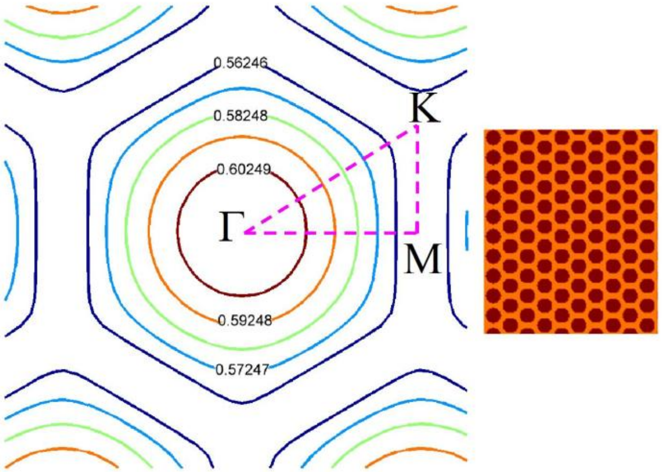
Square Lattice

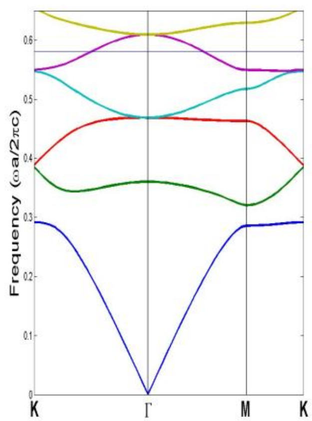


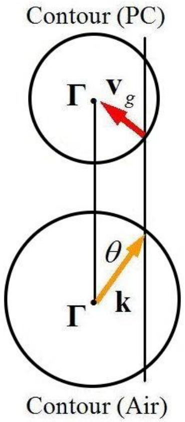




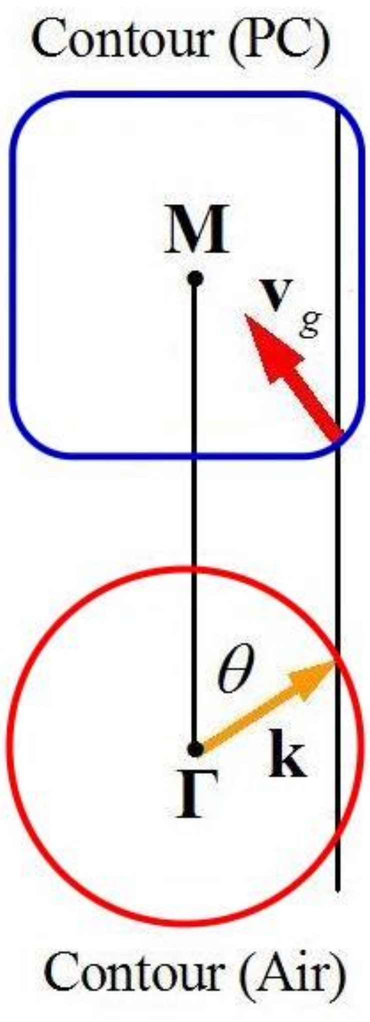
Triangular Lattice

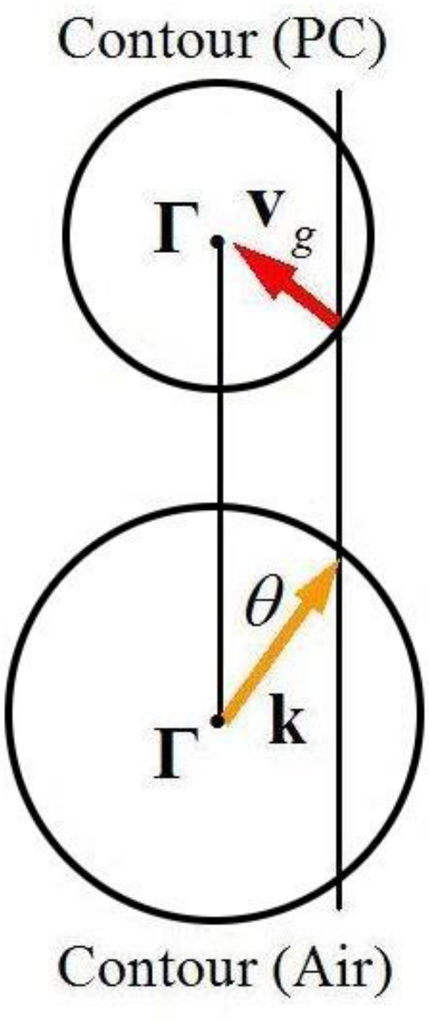




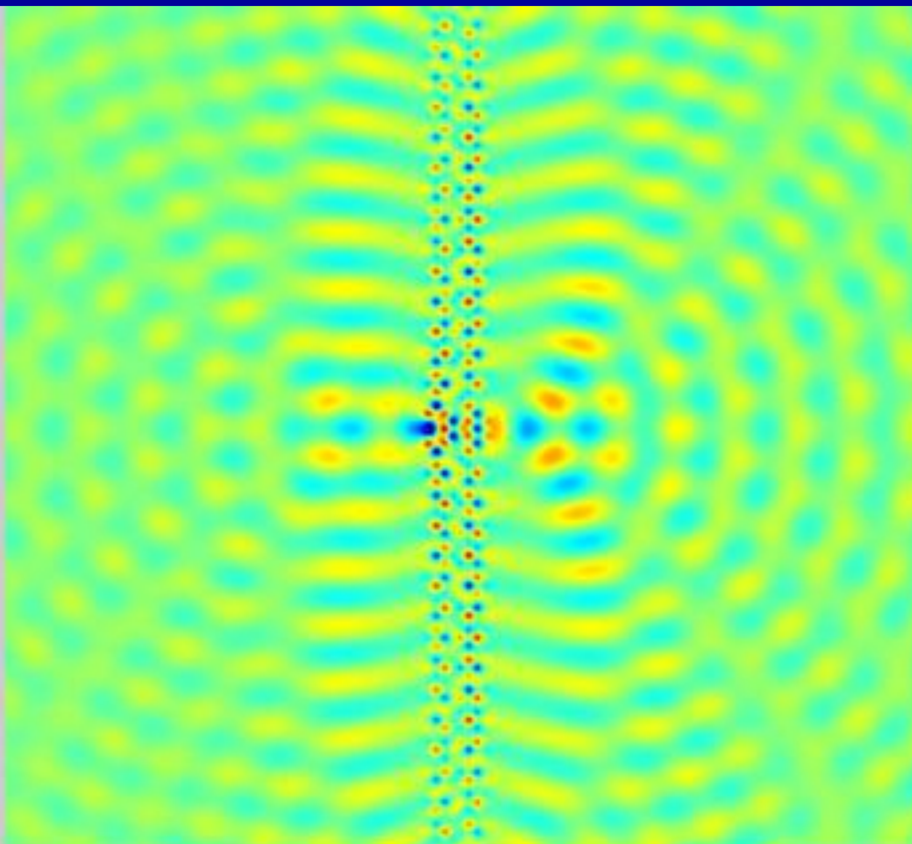


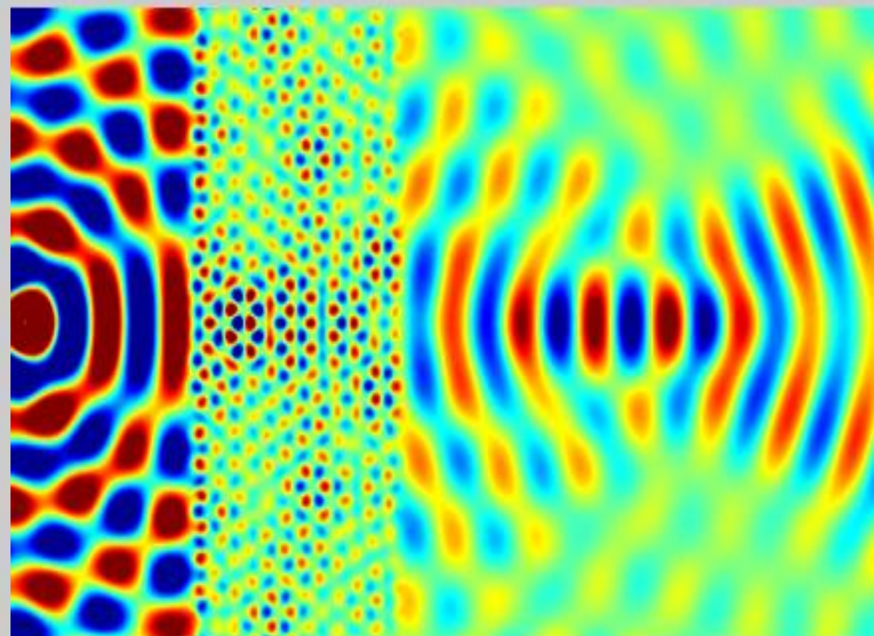
Constant Frequency Curve
Square Lattice v.s. Triangular Lattice

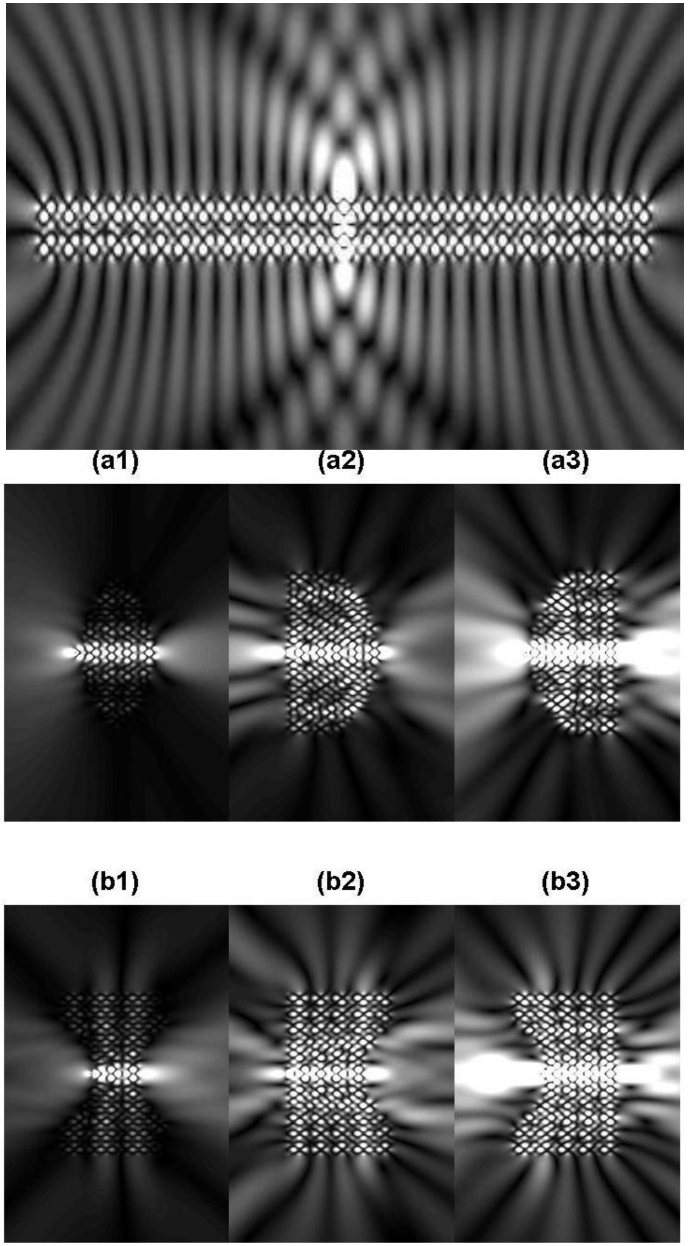




Superlens and Subwavelength Imaging





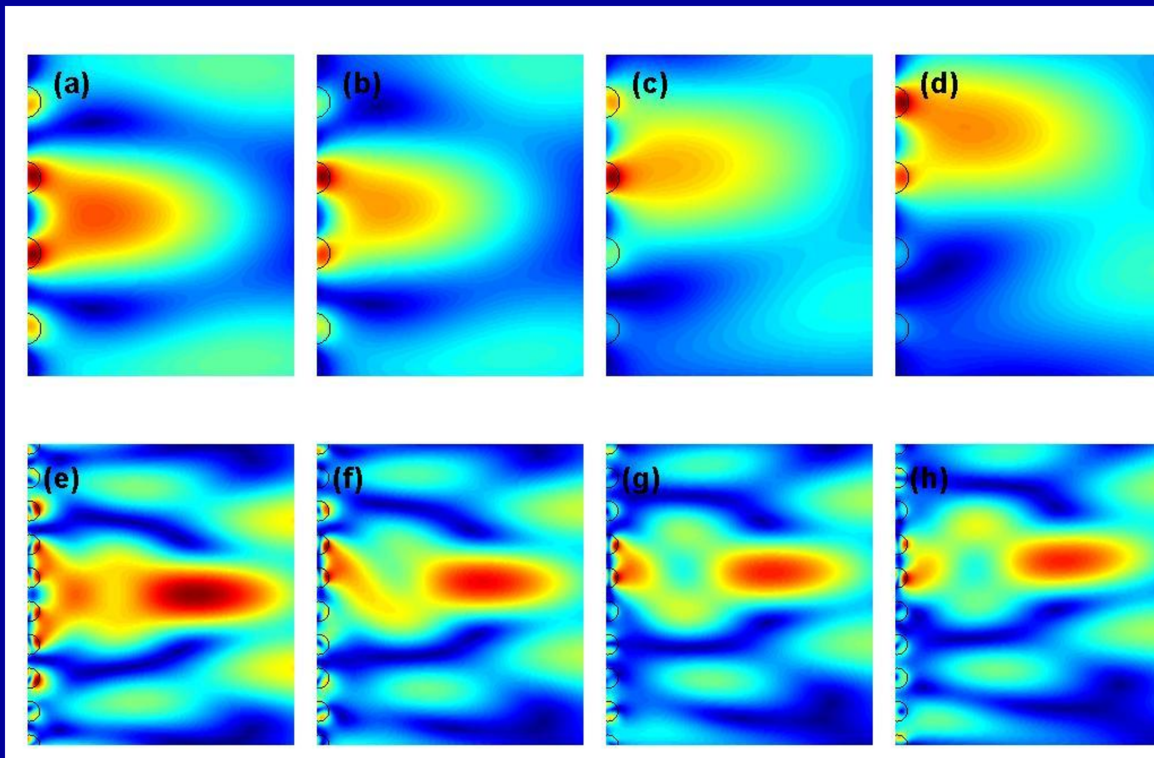


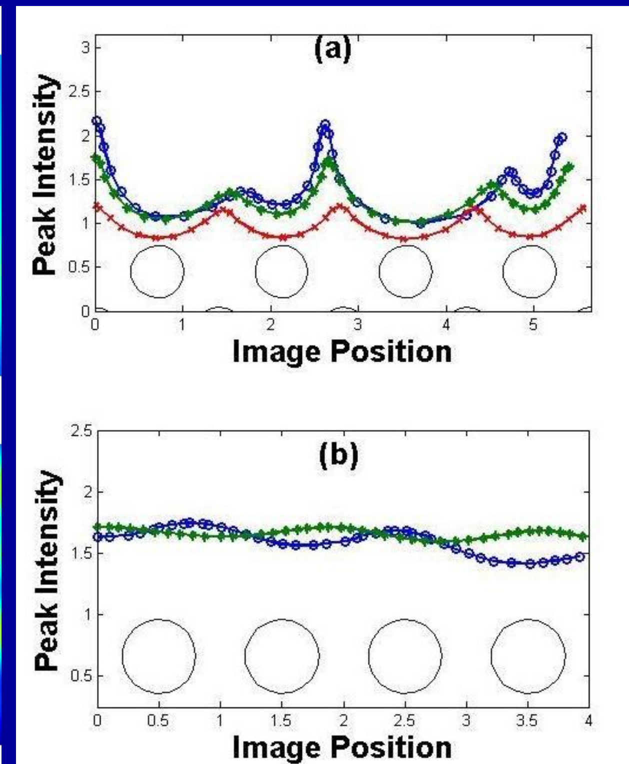
Negative Refraction or Evanescent Wave Coupling ?

Pi-Gang Luan and Kao-Der Chang, "*Superlensing effect without obvious negative refraction* ", J. Nanophotonics, 1, 013518 (2007)

Evanescent wave can be transmitted from source to image via evanescent wave coupling effect (using a stack of dielectric gratings)

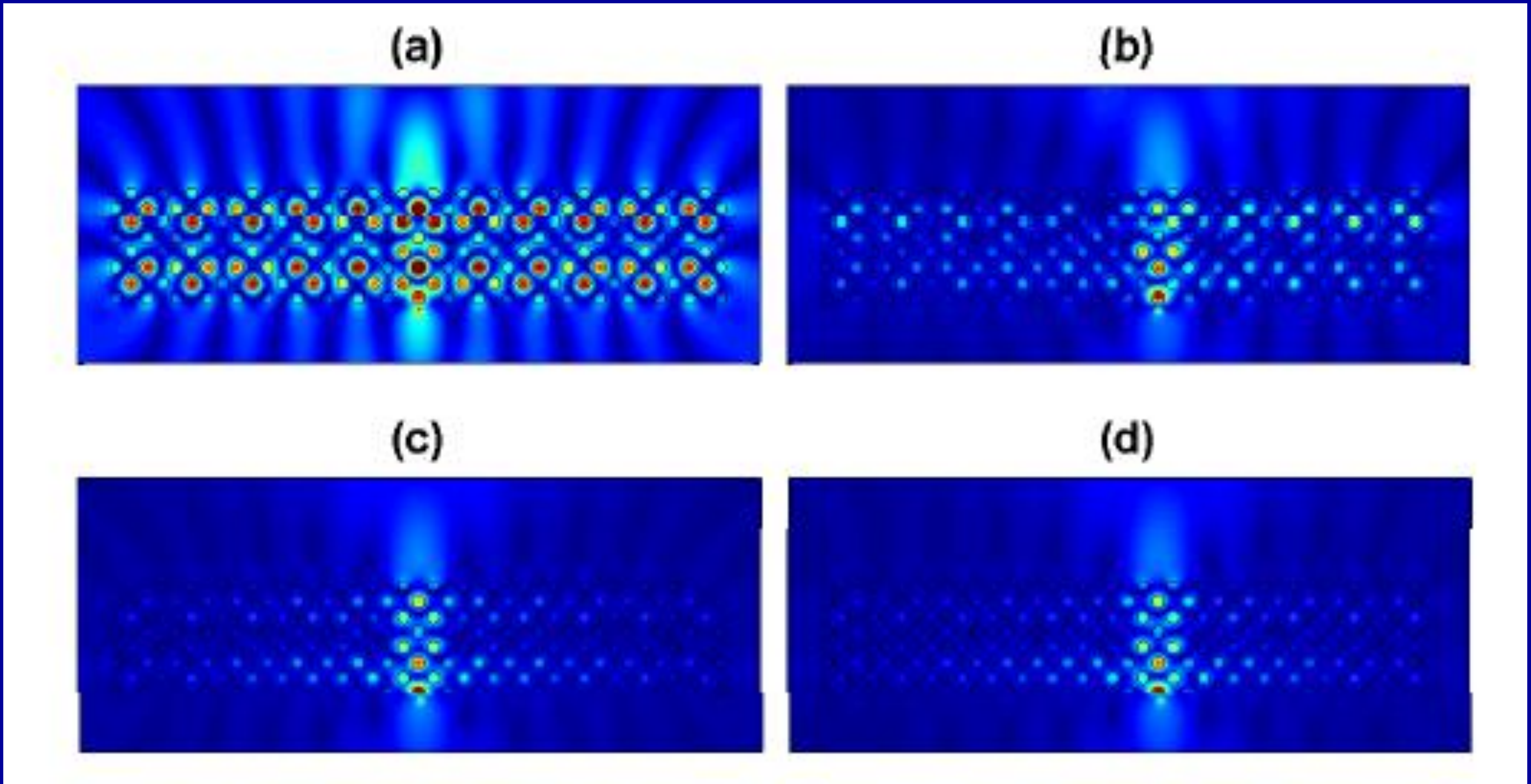
Is PhC Superlens Flat Lens ? (I)





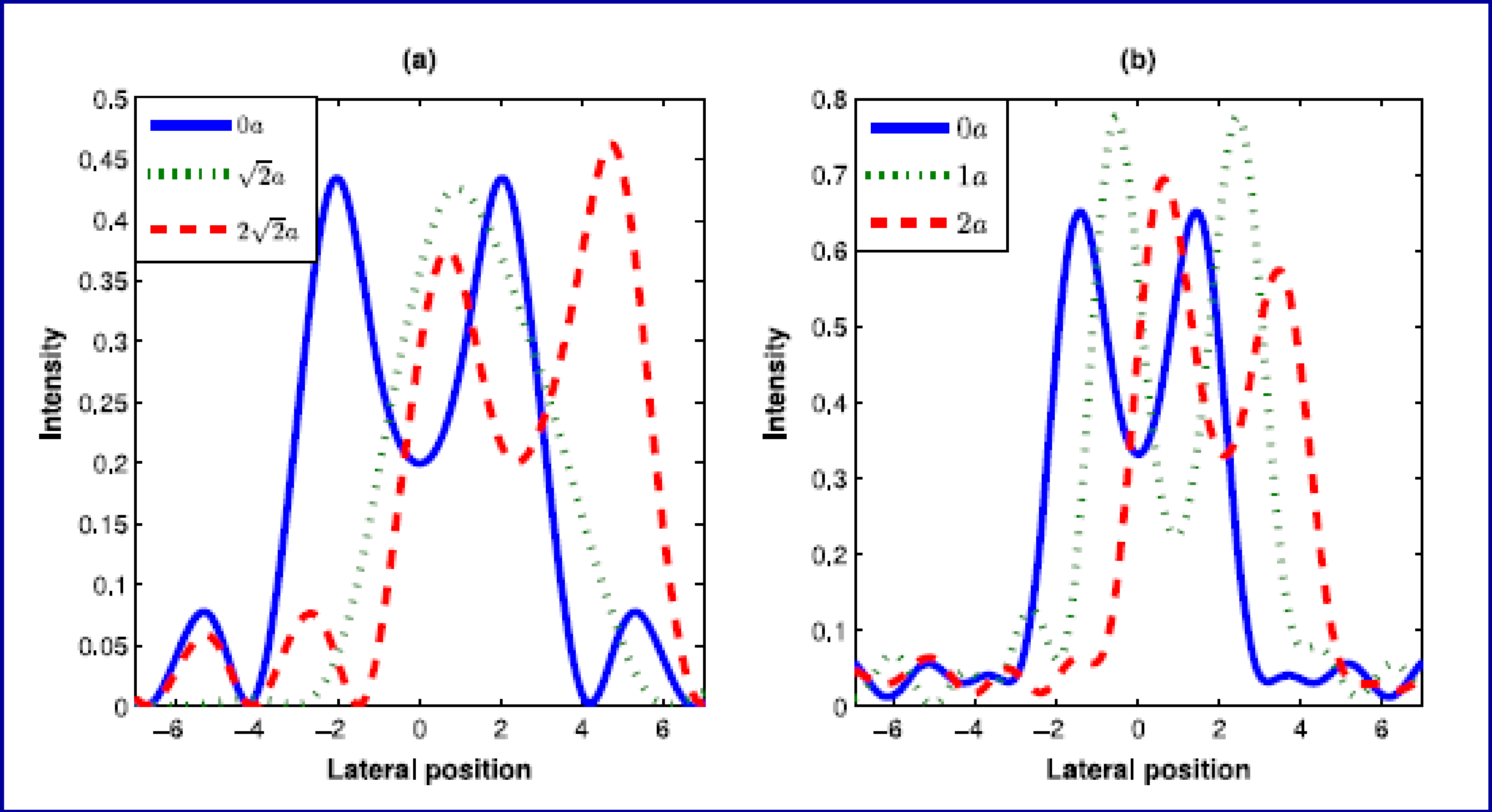
J. Phys.: Condens. Matter **23** (2011) 035301

Is PhC Superlens Flat Lens ? (II)



J. Phys.: Condens. Matter **23** (2011) 035301

Is PhC Superlens Flat Lens ? (III)



J. Phys.: Condens. Matter **23** (2011) 035301

Wave Propagation in Periodic Structures — *Electric Filters and Crystal Lattices*

"Waves always behave in a similar way, whether they are longitudinal or transverse, elastic or electric.

Scientists of the last (19th) century always kept this idea in mind."

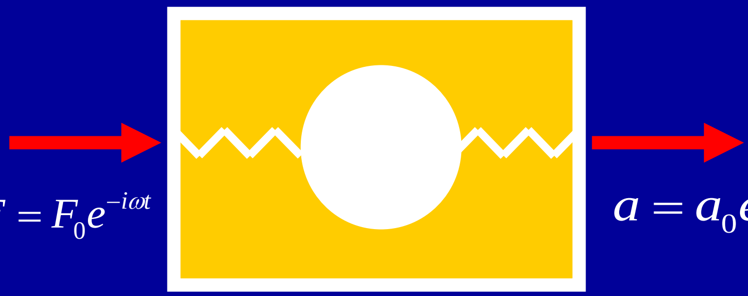
--- ***L. Brillouin***

Acoustic Metamaterials

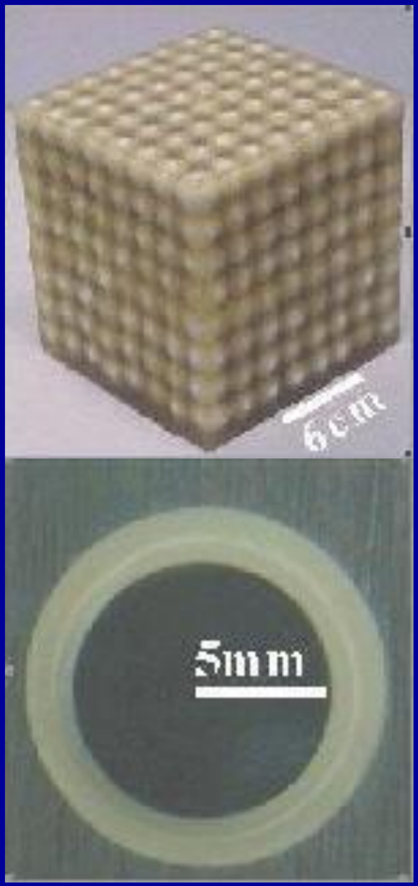
Locally Resonant Sonic Material

Negative Dynamic Mass leads to Polariton-like band gap

$F = F_0 e^{-i\omega t}$



$a = a_0 e^{-i\omega t}$



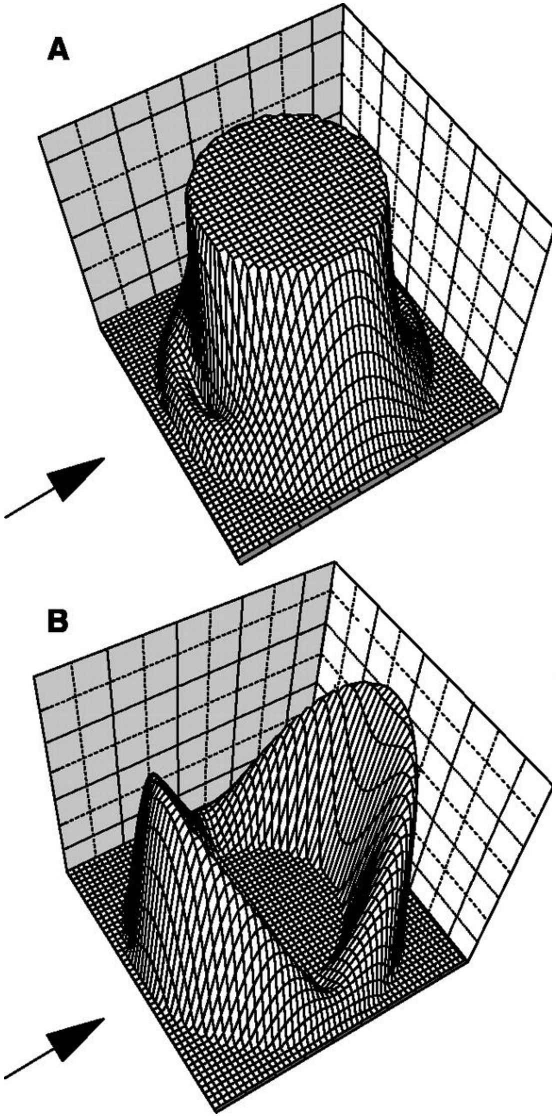
$m_{box} = M, \quad m_{ball} = m,$ Displacement: $x, \quad y$

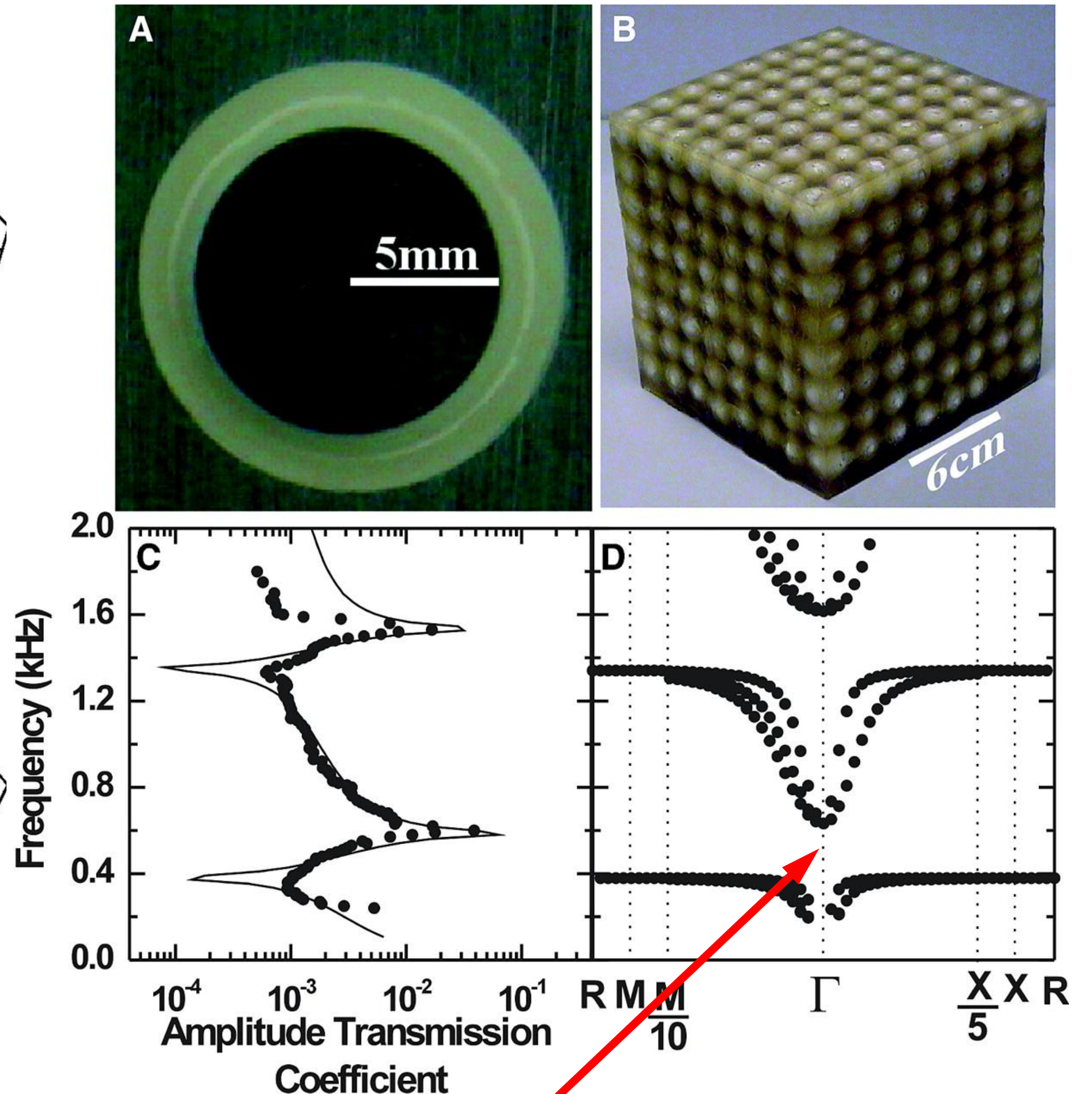
elastic constant: $K, \quad K$

$M\ddot{x} = F - 2K(x - y), \quad m\ddot{y} = -2K(y - x)$

$$m_{eff} = \frac{F}{a} = M\left(1 - \frac{f\omega_0^2}{\omega^2 - \omega_0^2}\right), \quad f = \frac{m}{M}, \quad \omega_0 = \sqrt{\frac{2K}{m}}$$

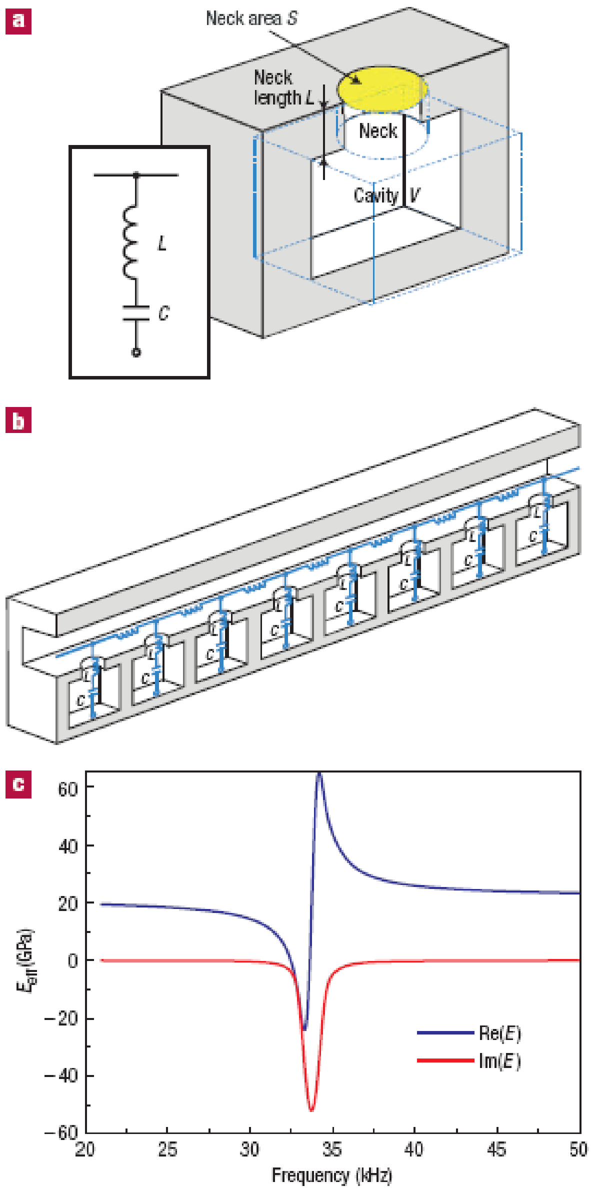
Ping Sheng *et. al.*, Science **289**, 1734 (2000)
Physica B: Condensed Matter 394, 256 (2007)





Polariton-like gap

Ultrasonic metamaterials with negative modulus



$$E_{eff}^{-1}(\omega) = E_0^{-1}\left[1 - \frac{F\omega_0^2}{\omega^2 - \omega_0^2 + i\Gamma\omega}\right]$$

Helmholtz Resonator

The force exerted on a column of air with cross sectional area A by a pressure change dP is given by

$$F = A\,dP = A\frac{dP}{dV}\Delta V = AV\frac{dP}{dV}\frac{Ax}{V}, \qquad (1)$$

where V is the volume of air and x is the vertical offset. For an ideal gas, the adiabatic bulk modulus is

$$K_S = -V\frac{dP}{dV} = \gamma P_0, \qquad (2)$$

where γ is the heat capacity ratio and P_0 is the undisturbed pressure.

Plugging in gives

$$F = ma = -\frac{A^2\gamma P_0}{V}x = \rho AL\frac{d^2x}{dt^2}, \qquad (3)$$

where a is the acceleration, ρ is the density of gas, and L is the vertical offset. Rearranging gives

$$\frac{d^2x}{dt^2} + \frac{A\gamma P_0}{V\rho L}x = 0, \qquad (4)$$

so the angular frequency of the oscillator is given by

$$\omega_0 = \sqrt{\frac{A\gamma P_0}{V\rho L}} = v\sqrt{\frac{A}{VL}}, \qquad (5)$$

where v is the velocity of sound in the medium.

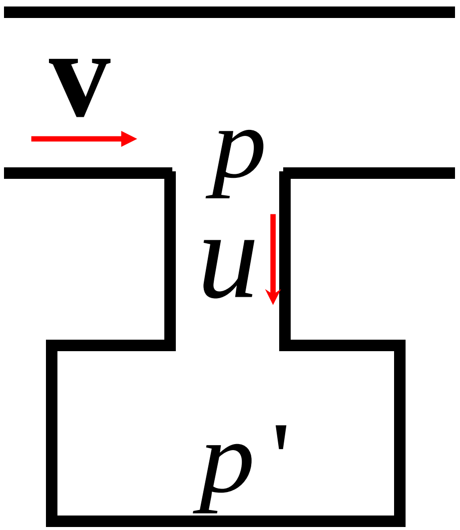
Xiang Zhang *et. al.*,
Nature Materials, 5, 452 (2006)

1D HR array metamaterial

$$-\nabla p = \rho \frac{\partial \mathbf{v}}{\partial t}, \quad -\nabla \cdot \mathbf{v} = \frac{1}{B}\frac{\partial p}{\partial t} + \frac{\sigma_{sh}}{A} u$$

$$M \frac{du}{dt} = \left(p - p' \right) S - \beta u, \quad \frac{dp'}{dt} = \frac{BS}{V} u$$

$$\Rightarrow M \frac{du}{dt} + \beta u + \frac{BS}{V} x = pS \quad \text{(Driven Oscillation)}$$



For monochromatic wave

$$-\nabla \cdot \mathbf{v} = \frac{-i\omega p}{B} + \frac{\sigma_{sh}}{A} \left(\frac{Sp}{-iM\omega + \beta + \dfrac{BS}{-i\omega V}} \right) \equiv \frac{-i\omega p}{B_{eff}}$$

$$\Rightarrow B_{eff}^{-1}(\omega) = B^{-1} \left(1 - \frac{F\omega_0^2}{\omega^2 - \omega_0^2 + i\Gamma\omega} \right)$$

Transformation Optics and Invisibility Cloak

Principle of Invisibility cloak (I)

所謂隱形斗篷，就是用適當電磁材料所製成的一個「殼」，而藏在這個殼裡面的物體不但不會被外界看到，連這個殼本身都不能被外界看到。入射到隱形斗篷的光線不會被散射，而是會沿著這個殼繞過。這些光線於繞過斗篷後會沿著入射前的軌跡傳播，並且不產生殼的陰影。

在這個過程中，由於光線不會穿透進入中心的空腔，所以藏在空腔裡的物體不會與外界有電磁交互作用。

John B. Pendry 等人已於 2006 年的 Science 雜誌上發表了一個符合 Maxwell 方程式的隱形斗篷理論。根據此一理論，在連續與平滑的任意坐標變換之下，Maxwell 方程式的形式可以維持不變，代價是介電常數與磁導率的表達式可能會變得很複雜——一般而言它們會變成非等向的 (anisotropic) 與非均勻的 (inhomogeneous)。更精確一點說，它們都是所謂的張量密度 (tensor densities)。

Principle of Invisibility cloak (II)

　對於這個座標變換可以採用以下兩種觀點做解釋。

　第一種：空間、電磁場、介質都沒有變，只是座標系的選擇被改變了(就像把笛卡兒座標換成球座標)，因此介質相對於座標系的表現 (representation) 改變了。第二種：空間、介質與電磁場都改變了，但坐標系沒變。

　隱形斗篷的設計關鍵，即在於先採用第一種觀點做座標變換，再將變換結果用第二種觀點解釋。由於在變換之後的介質與原來的介質之間存在著一對一的對應關係，因此光線在轉換後介質 (transformed medium) 的軌跡也只不過是轉換前那個軌跡的座標變換。根據這個想法，可以選擇真空做為轉換前的介質空間(真空對光線不散射，也不製造陰影)，而埋 (embedded) 在其中的一顆真空球作為座標變換的操作範圍，將此球轉換為一個殼，並採用第二種觀點去解釋，就完成了一個隱形斗篷的設計了。

Principle of Invisibility cloak (III)

杜克大學（Duke University）的科學家已經製造出了一個可在微波頻段下幾乎隱形 (還是有部份的散射) 的二維隱形斗篷。為了在實驗上較容易製作，此斗篷所採用的介電與導磁參數並非直接根據座標變換所得的理論參數，而是將它們換成了另一組現階段在實驗上就做得到的參數，稱為約化參數 (reduced parameters)。

光線在這個以「裂環共振器」(split-ring resonator, SRR) 的環狀周期陣列設計成的約化參數斗篷中 (殼區) 的傳播軌跡會與理想斗篷的一致，但約化參數斗篷外表面與真空的阻抗不會完全匹配，因此還是會有一些散射，並不是百分之百隱形的。上述這些關於隱形斗篷的理論與實驗研究解放了科學家的想像力，激起了一股隱形斗篷與超材料的研究熱潮。一個更好的二維斗篷或是三維斗篷，甚至是光學隱形斗篷的實現，似乎都是很有希望的。

Principle of Invisibility cloak (IV)

此小節對二維環狀隱形斗篷的參數設計做一介紹。按以下轉換公式，將一個半徑為 b 的圓盤轉換為一個內半徑為 a，外半徑為 b 的圓環 ($r<a$ 及 $r>b$ 的範圍都不需轉換) [11]：

$$r' = \left(\frac{b-a}{b}\right) r + a, \quad \phi' = \phi, \quad z' = z$$

轉換前後對應的線元 (line elements) 為 $(dr, rd\phi, dz)$ 與 $(dr', r'd\phi', dz')$；兩者透過以下的 Jacobian 矩陣連繫起來:

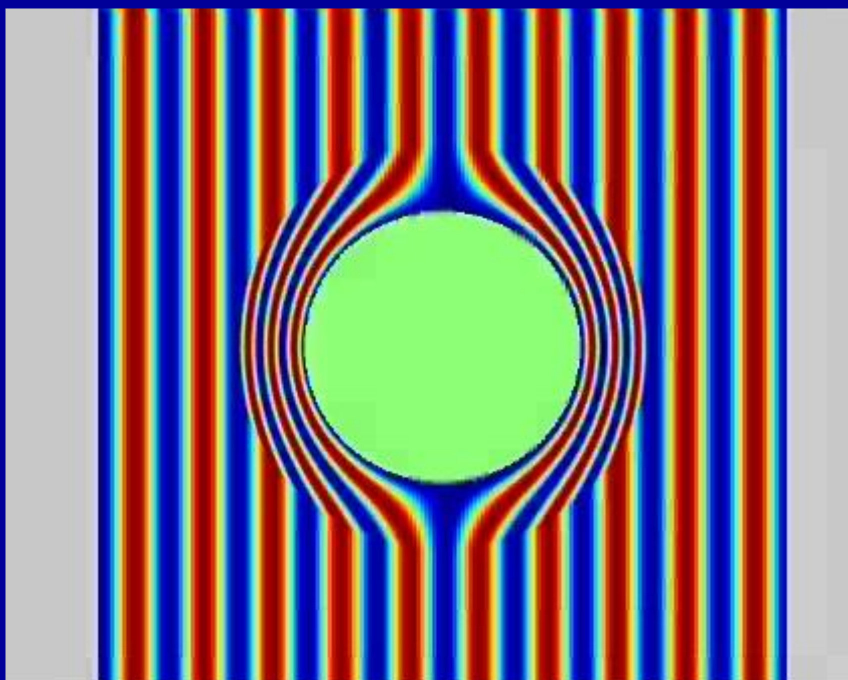
$$J = \begin{pmatrix} \frac{b-a}{b} & 0 & 0 \\ 0 & \frac{b-a}{b}\left(\frac{r'}{r'-a}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} = J^T, \quad \det J = \left(\frac{b-a}{b}\right)^2 \left(\frac{r'}{r'-a}\right)$$

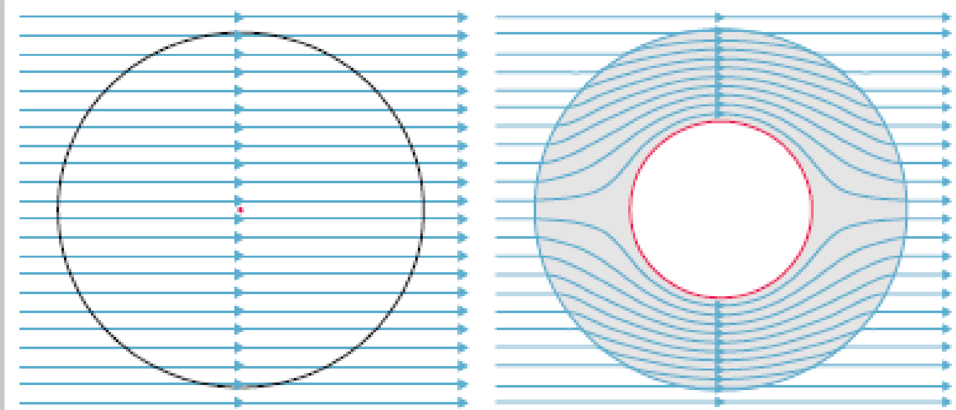
利用介電常數與磁導率的座標轉換公式 (假定兩者有相同的主軸) $\varepsilon' = \dfrac{J\varepsilon J^T}{\det J}$, $\mu' = \dfrac{J\mu J^T}{\det J}$，並代入 $\varepsilon = \mu = I$ (真空)，就得到以下的設計參數：

$$\varepsilon_r = \mu_r = \frac{r-a}{r}, \quad \varepsilon_\phi = \mu_\phi = \frac{r}{r-a}, \quad \varepsilon_z = \mu_z = \left(\frac{b}{b-a}\right)^2 \frac{r-a}{r}$$

此處已將轉換後介質各符號的撇號去除。

Invisibility Cloak utilizes *Anisotropic and Inhomogeneous* Metamaterials



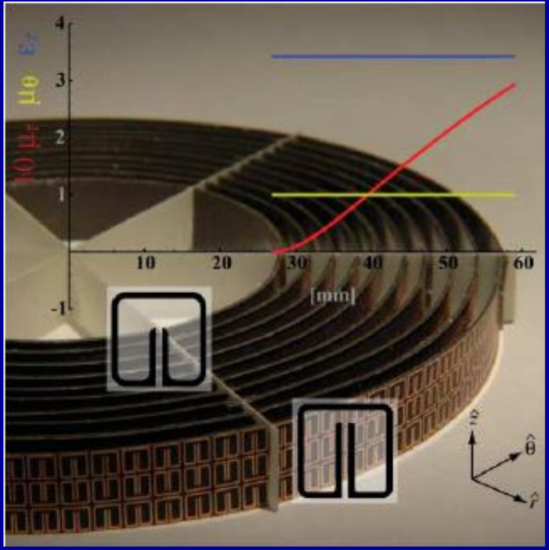


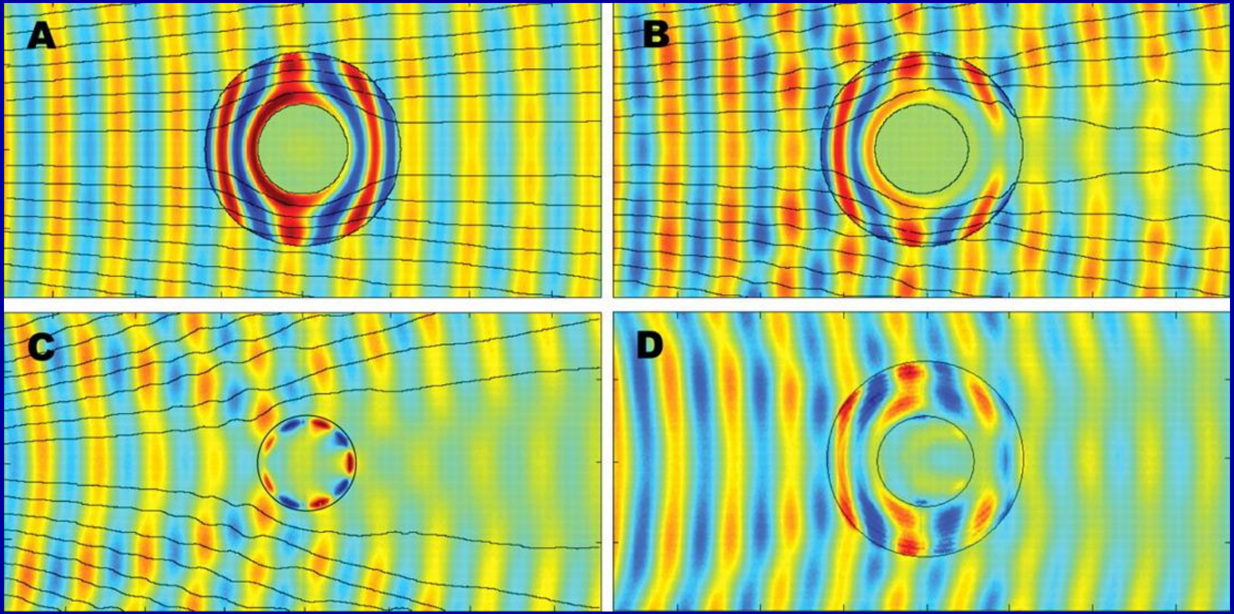
Hidden realms. (Left) Blue lines indicate the flow of electromagnetic energy through a region of space. **(Right)** The shaded region has been subject to a coordinate transformation. The center of the region denoted by the red dot at left is transformed to the red circle. The region inside the red circle is cloaked and cut off electromagnetically from the region outside. Outside the shaded area, the flows are identical in both panels, implying electromagnetic nondetectability of objects inside the cloaked region.

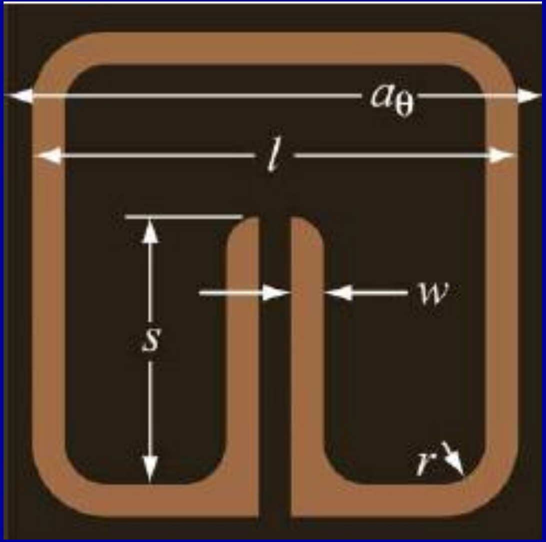
Science 312, 1777 (2006) Science 312, 1780 (2006)
PRE 74, 036621 (2006) New J. Phys. 8, 247 (2006)
Phys. World, Sep. 30 (2006)

Science 313, 1399 (2006)

Experimental Realization







Ideal theoretical Parameters

$$\varepsilon_r = \mu_r = \frac{r-a}{r}, \quad \varepsilon_\phi = \mu_\phi = \frac{r}{r-a}$$

$$\varepsilon_z = \mu_z = \left(\frac{b}{b-a}\right)^2 \frac{r-a}{r}$$

Parameters for real experiment

$$\mu_r = \left(\frac{r-a}{r}\right)^2, \quad \mu_\phi = 1, \varepsilon_z = \left(\frac{b}{b-a}\right)^2$$

Science 314, 977 (2006)

Cloaking of Matter Waves

A cloaking of matter wave can be realized at given energy by designing the potential and effective mass of the matter waves in the cloaking region

$$-\frac{\hbar^2}{2}\vec{\nabla}\cdot(\hat{m}^{*-1}\vec{\nabla}\psi)+V\psi=E\psi,$$

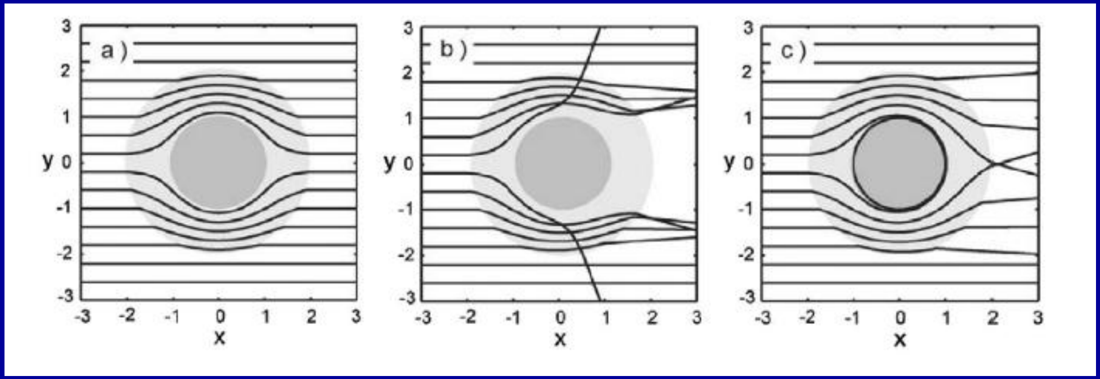
$$\vec{u}=\hat{m}^{-1}\vec{\nabla}\psi,\qquad -\frac{\hbar^2}{2m_0}\vec{\nabla}\cdot\vec{u}=(E-V)\psi.$$

$$\vec{\nabla}_{\vec{x}}\psi=\hat{h}^{-1}\vec{\nabla}_{\vec{q}}\psi,\qquad \vec{\nabla}_x\cdot\vec{u}=\frac{1}{|\det(\hat{h})|}\vec{\nabla}_{\vec{q}}\cdot\vec{v},$$

$$-\frac{\hbar^2}{2m_0}\nabla_q\cdot\vec{v}=\det|\hat{h}|(E-V)\psi$$

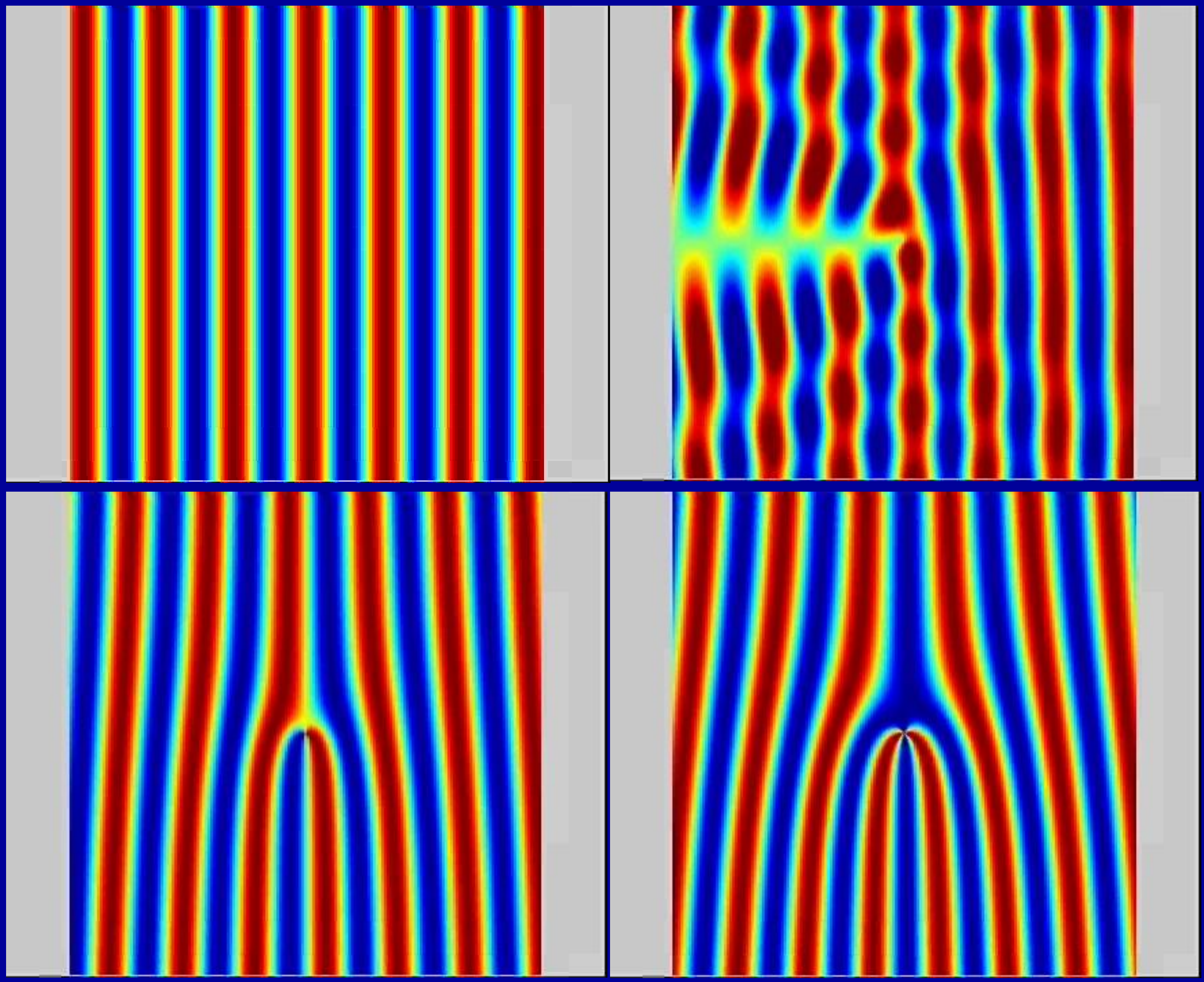
$$\vec{v}=\det|\hat{h}|(\hat{h}\,\hat{m}\,\hat{h})^{-1}\vec{\nabla}_q\psi.$$

$$\hat{m}'=\frac{\hat{h}\,\hat{m}\,\hat{h}}{\det|\hat{h}|},\qquad V'=E+|\det(\hat{h})|(V-E)$$

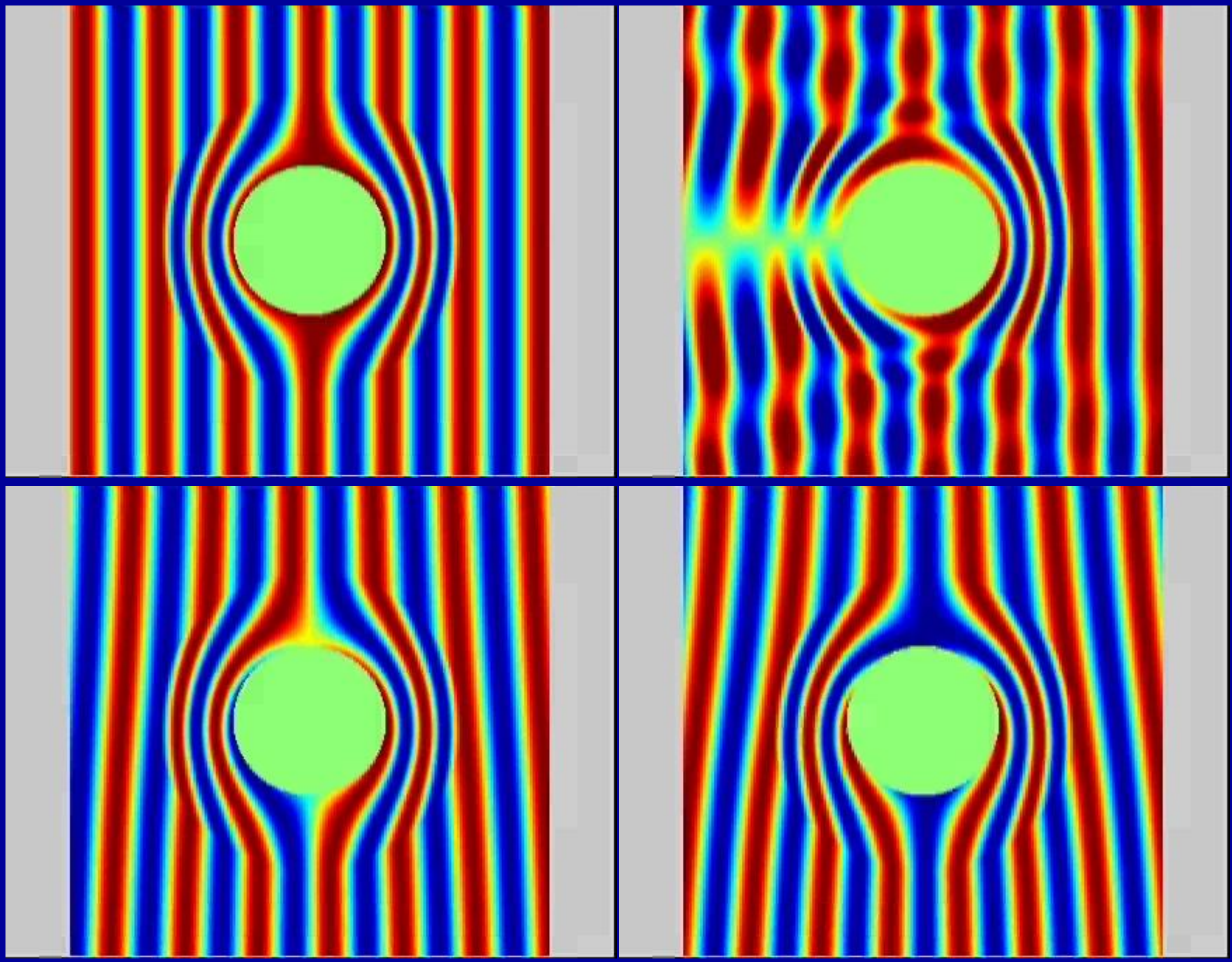


Shuang Zhang, Dentcho A. Genov, Cheng Sun, and Xiang Zhang
Phys. Rev. Lett. 100, 123002 (2008)

Aharonov-Bohm effect



Cloaking Matters under Aharonov-Bohm effect



D. H. Lin & P. G. Luan, PRA 79, 051605(R) (2009)

Conclusion

- Most proposed metamaterials consist of periodically arranged resonators or resonator pairs.
- A single resonator couples almost only to its nearest neighbors.
- If the spatial dispersion of the effective medium can be neglected, then its local properties are not influenced under arbitrary bending. Examples: hyperlens, invisibility cloak,…
- Photonic crystals (PhC) and metamaterials (MtM) are both periodic structures, however, spatial dispersion effect cannot be neglected in PhC. This might be the main difference between PhC and MtM.
- In order to solve the more and more challenging problems encountered in metamaterial research, in the future we need to develop a different education/discipline system to train up more "Sciengineers".

Thank you for your attention!

Metamaterial