# *Electromagnetic Field Energy in a Metamaterial Medium Consisting of Metallic Wires and Split-ring Resonators*

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# History and motivation

- Brillouin discussed the energy density of EM waves in a dispersive medium with neglecting absorption. (adiabatic process was assumed ). The same formulas can also be found in Landau and Lifshitz.
- Loudon discussed the case of dispersive medium (permittivity only) with finite loss. The behavior of the microscopic electric dipoles determines the form of permittivity and energy density macroscopically.
- Ruppin, Tretyakov, and Boardman & Marinov derived the energy density formula for EM waves in a dispersive metamaterials. Ruppin used a simplified model for the magnetic dipoles (a pair of positive and negative monopoles). In the more realistic situation, the magnetic dipoles are current loops (like SRRs). Tretyakov studied this case and derived the formula using an equivalent circuit model (EC approach). Boardman & Marinov derived the energy density formula using the electrodynamical approach (ED approach). B&M's results are different from Tretyakov's. Which one is correct? Or both are incorrect?
- In fact,  we find that the form of the power loss determines the form of the energy density.

# Electromagnetic energy density in nondispersive media

Poynting vector : $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

From Faraday's Law and Ampere's Law, we can derive

$$-\nabla \cdot \mathbf{S} = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \mathbf{J}$$

Using the relations: $\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}$, $\mathbf{B} = \mu_0 \mu \mathbf{H}$, we obtain

$$-\nabla \cdot \mathbf{S} = \frac{\partial W}{\partial t} + \mathbf{E} \cdot \mathbf{J} \qquad \left( -\nabla \cdot \mathbf{S} = \frac{\partial W}{\partial t} \text{ if } \mathbf{J} = \mathbf{0} \right)$$

Energy density : $W = \frac{1}{2}\left( \varepsilon_0 \varepsilon E^2 + \mu_0 \mu H^2 \right)$

For harmonic waves, **E** and **H** become complex vectors,

and $\overline{\mathbf{S}}$ and $\overline{\mathrm{W}}$ denote the time-averaged values:

$$\overline{\mathbf{S}} = \frac{1}{2} \mathrm{Re}\left( \mathbf{E} \times \mathbf{H}^* \right), \quad \overline{W} = \frac{1}{4}\left( \varepsilon_0 \varepsilon \, |E|^2 + \mu_0 \mu \, |H|^2 \right)$$

$\nabla \cdot \overline{\mathbf{S}} = 0$ (if no loss)

# EM energy density in dispersive media with neglecting absorption (Brillouin)

For harmonic waves

$\mathbf{D} = \varepsilon_0 \varepsilon(\omega)\mathbf{E}, \quad \mathbf{B} = \mu_0 \mu(\omega)\mathbf{H}$

We still have

$$\bar{\mathbf{S}} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E}\times\mathbf{H}^*\right),$$

$$\bar{W} = \frac{1}{4}\left(\varepsilon_0 \frac{d}{d\omega}\left(\omega\varepsilon\left(\omega\right)\right)|\,\mathbf{E}\,|^2 + \mu_0 \frac{d}{d\omega}\left(\omega\mu\left(\omega\right)\right)|\,\mathbf{H}\,|^2\right)$$

–Brillouin, Landau, and Jackson

Only for the monochromatic or adiabatic case.

# EM energy density in dispersive media with finite absorption (I)

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0 \mathbf{H}$$

$$\mathbf{P} = N\mathbf{p}, \quad \mathbf{p} = q\mathbf{r}, \quad m\left(\ddot{\mathbf{r}} + \Gamma\dot{\mathbf{r}} + \omega_r^2 \mathbf{r}\right) = q\mathbf{E}$$

$$\Rightarrow \ddot{\mathbf{P}} + \Gamma\dot{\mathbf{P}} + \omega_r^2 \mathbf{P} = Nq^2\mathbf{E}/m = \varepsilon_0 \omega_p^2 \mathbf{E}, \quad \omega_p^2 \equiv Nq^2/m\varepsilon_0$$

$$\mathbf{E} = \left(\ddot{\mathbf{P}} + \Gamma\dot{\mathbf{P}} + \omega_0^2 \mathbf{P}\right)/\omega_p^2 \varepsilon_0$$

$$-\nabla \cdot \mathbf{S} = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\varepsilon_0 E^2 + \mu_0 H^2\right)\right] + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t}$$

$$= \frac{\partial W_0}{\partial t} + \frac{1}{\omega_p^2 \varepsilon_0}\left(\ddot{\mathbf{P}} + \omega_0^2 \mathbf{P}\right) \cdot \dot{\mathbf{P}} + \frac{\Gamma}{\omega_p^2 \varepsilon_0}\dot{\mathbf{P}}^2$$

$$= \frac{\partial W_0}{\partial t} + \frac{\partial}{\partial t}\left[\frac{1}{2\omega_p^2 \varepsilon_0}\left(\dot{\mathbf{P}}^2 + \omega_r^2 \mathbf{P}\right)\right] + \frac{\Gamma}{\omega_p^2 \varepsilon_0}\dot{\mathbf{P}}^2 = \frac{\partial W}{\partial t} + \frac{\Gamma}{\omega_p^2 \varepsilon_0}\dot{\mathbf{P}}^2$$

Energy density: $W = W_0 + \frac{1}{2\omega_p^2 \varepsilon_0}\left(\dot{\mathbf{P}}^2 + \omega_r^2 \mathbf{P}\right)$, where $W_0 = \frac{1}{2}\left(\varepsilon_0 E^2 + \mu_0 H^2\right)$

# EM energy density in dispersive media with finite absorption (II)

For harmonic waves (electric dispersion only)

$$\mathbf{D} = \varepsilon_0 \varepsilon(\omega)\mathbf{E}, \quad \varepsilon(\omega) = \left(1+\chi(\omega)\right) = \left(1+\frac{\omega_p^2}{\omega_r^2 - \omega\left(\omega + i\Gamma\right)}\right)$$

$$\bar{W} = \frac{1}{4}\left(\varepsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2\right) + \frac{1}{4\omega_p^2\varepsilon_0}\left(\omega^2+\omega_r^2\right)|\mathbf{P}|^2$$

$$= \frac{\varepsilon_0}{4}\left(1+\frac{\left(\omega^2+\omega_r^2\right)\omega_p^2}{\left(\omega^2-\omega_r^2\right)^2+\Gamma_e\omega^2}\right)|\mathbf{E}|^2 + \frac{\mu_0}{4}|\mathbf{H}|^2$$

$$\to \frac{1}{4}\left(\varepsilon_0 \frac{d}{d\omega}\left(\omega\varepsilon\left(\omega\right)\right)|\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2\right) \quad \text{if} \quad \Gamma \to 0,$$

Power loss: $P_{loss} = \dfrac{\omega^2\Gamma_e}{2\omega_p^2\varepsilon_0}|\mathbf{P}|^2$

Loudon's result (R. Loudon 1970): $W = \dfrac{\varepsilon_0}{2}\left(n^2 + \dfrac{2\omega n\kappa}{\Gamma_e}\right)|\mathbf{E}|^2$ can also be obtained

by using the relations: $\varepsilon_0|\varepsilon(\omega)||\mathbf{E}|^2 = \mu_0|\mathbf{H}|^2$ and $\varepsilon(\omega)=\left(n+i\kappa\right)^2$

# Effective permittivity and permeability

$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \varepsilon \mathbf{E}$

$\mathbf{P} = \text{dipole density (electric)} = N\mathbf{p}$

$\mathbf{B} = \mu_0 \left( \mathbf{H} + \mathbf{M} \right) = \mu\mu_0 \mathbf{H}$

$\mathbf{M} = \text{magnetic dipole density} = N\mathbf{m}$

$V$: large microscopically, small macroscopically

Usually we use resonance effect to make $\mathbf{M}$ large

When $\mu_0 \mathbf{M} > \mathbf{B}$, we have $\mathbf{B} \cdot \mathbf{H} < 0$ and $\mu < 0$

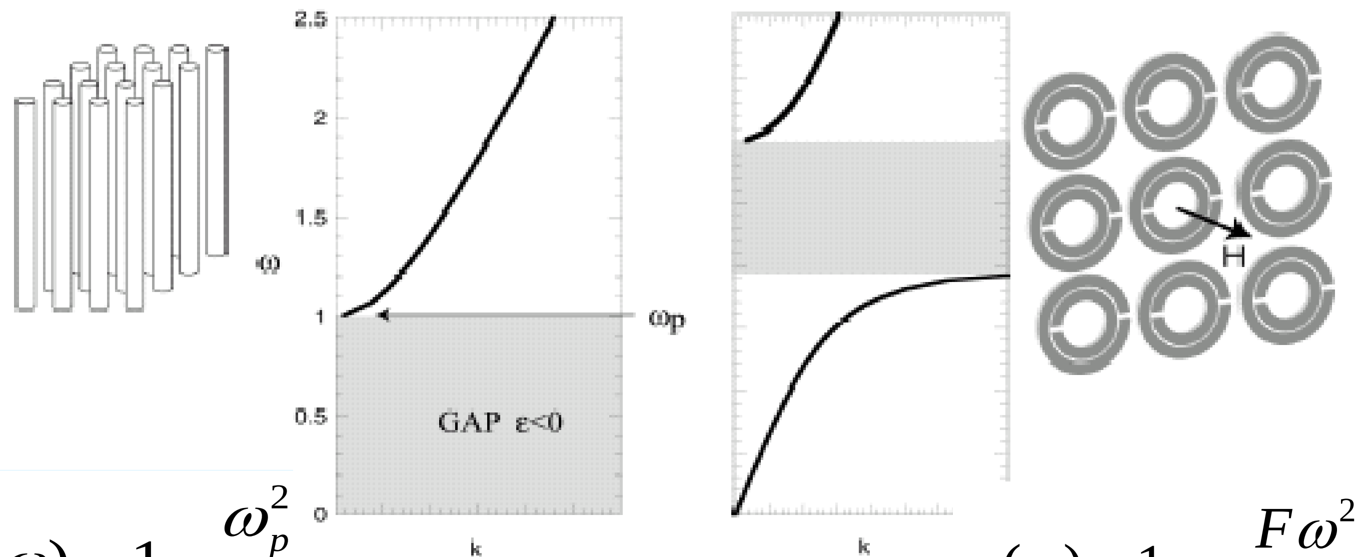
# Metamaterials consisting of wires and SRRs

D. R. Smith *et. al*., Physics Today, 17, **May** (2000).

Phys. Rev. Lett. **84**, 4184 (2000) ; Science, **292**, 77 (2001)

# The **Building Blocks** of LHM

## Electric Dipoles + Magnetic Dipoles




$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\mu(\omega) = 1 - \frac{F\omega^2}{\omega^2 - \omega_0^2}$$

## A SRR is a LC circuit !

# EM energy density in metamaterials with finite absorption (Ruppin 2002) (I)

The electric part is the same as that of Loudon's. The 'mass' appears in the equation is the "effective mass" of the charge carrier. The in calculating the effective mass, the magnetic effect has been included (Pendry 1996).

For the magnetic part, Ruppin chose the following simplified form of permeability

$$\mu(\omega) = 1 + \chi_m(\omega) = 1 - \frac{F\omega_0^2}{\omega^2 - \omega_0^2 + i\Gamma_h \omega},$$

which corresponds to the equaion of motion

$$\ddot{\mathbf{M}} + \Gamma_h \dot{\mathbf{M}} + \omega_0^2 \mathbf{M} = F\omega_0^2 \mathbf{H}.$$

This leads to: $-\nabla \cdot \mathbf{S} = \dfrac{\partial W}{\partial t} + P_{loss},$ where

$$W = \frac{\varepsilon_0}{2} E^2 + \frac{\mu_0}{2} H^2 + \frac{1}{2\omega_p^2 \varepsilon_0}\left(\dot{\mathbf{P}}^2 + \omega_r^2 \mathbf{P}^2\right) + \frac{\mu_0}{2F\omega_0^2}\left(\dot{\mathbf{M}}^2 + \omega_0^2 \mathbf{M}^2\right),$$

$$P_{loss} = \frac{\Gamma_e \dot{\mathbf{P}}^2}{\omega_p^2 \varepsilon_0} + \frac{\mu_0 \Gamma_h \dot{\mathbf{M}}^2}{F\omega_0^2}$$

(the Joule Heat caused by the currents in the wires and SRRs)

# EM energy density in metamaterials with finite absorption (Ruppin 2002) (II)

For harmonic waves, the time averaged energy density is

$$\bar{W} = \frac{1}{4}\left(\varepsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2\right) + \frac{1}{4\omega_p^2 \varepsilon_0}\left(\omega^2 + \omega_r^2\right)|\mathbf{P}|^2 + \frac{\mu_0}{4F\omega_0^2}\left(\omega^2 + \omega_0^2\right)|\mathbf{M}|^2$$

$$= \frac{\varepsilon_0}{4}\left(\varepsilon' + \frac{2\omega\varepsilon''}{\Gamma_e}\right)|\mathbf{E}|^2 + \frac{\mu_0}{4}\left(\mu' + \frac{2\omega\mu''}{\Gamma_h}\right)|\mathbf{H}|^2, \text{ where } \varepsilon = \varepsilon' + i\varepsilon'',\ \ \mu = \mu' + i\mu''$$

$$\to \frac{1}{4}\left(\varepsilon_0 \frac{d}{d\omega}\left(\omega\varepsilon(\omega)\right)|\mathbf{E}|^2 + \mu_0 \frac{d}{d\omega}\left(\omega\mu(\omega)\right)|\mathbf{H}|^2\right) \text{ if } \ \Gamma_e \text{ and } \Gamma_h \to 0,$$

The time averaged power loss is

$$P_{loss} = \frac{\omega^2 \Gamma_e}{2\omega_p^2 \varepsilon_0}|\mathbf{P}|^2 + \frac{\mu_0 \Gamma_h}{2F\omega_0^2}|\mathbf{M}|^2$$

# EM energy density in metamaterials with finite absorption (Ruppin 2002) (III)

Remarks: the simplified permeability implies that the magnetic dipoles have a similar origin like that of the electric dipoles ($\pm$monopole pairs ?), but this is not ture.

The magnetic dipoles are in fact originated from the oscillating currents in the SRRs ($m = IS,\ m$: magnetic dipole, $I:$ current, $S:$ area encircled by the ring)

The unsimplified permeability is $\mu(\omega) = 1 - \dfrac{F\omega^2}{\omega^2 - \omega_0^2 + i\Gamma_h\omega}$,

which corresponds to the equaion of motion $\ddot{\mathbf{M}} + \Gamma_h \dot{\mathbf{M}} + \omega_0^2 \mathbf{M} = -F\ddot{\mathbf{H}}$

This equation can be derived by the considering current in a SRR under the influence of an applied magnetic field. The effect of the depolaring field has been included (Pendry *et*. *al*. 1999, Kong *et*. *al*. 2006).

# Equivalent circuit (EC) method (Tretyakov 2005) (I)

For the wire-SRR sistem, the effective permittivity and permeability are (define $j=-i$)

$$\varepsilon(\omega)=1-\frac{\omega_p^2}{\omega(\omega-j\nu)}, \qquad \mu(\omega)=1-\frac{F\omega^2}{\omega^2-\omega_0^2-j\omega\gamma}$$

For the electric part, inserting a piece of the metamaterial into a parallel-plate capacitor, the adimittance of the capacitor becomes

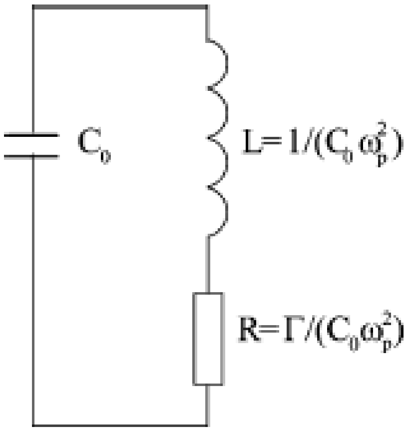
$$Y=j\omega C=j\omega C_0\varepsilon(\omega)=j\omega C_0+\frac{1}{j\omega L+R}, \text{ here } L=\frac{1}{\omega_p^2 C_0}, \quad R=\frac{\nu}{\omega_p^2 C_0}$$

The time averaged energy stored in the equivalent circuit is

$$\bar{W}_e S d=\frac{1}{4}\left(C_0|V_C|^2+L|I_L|^2\right)=\frac{1}{4}C_0|V_C|^2\left(1+\frac{L}{C_0\left(\omega^2L^2+R^2\right)}\right)$$

$$=\frac{1}{4}\frac{\varepsilon_0 S}{d}|E|^2 d^2\left(1+\frac{L}{C_0\left(\omega^2L^2+R^2\right)}\right), \quad C_0=\frac{\varepsilon_0 S}{d}, \quad V_C=Ed$$

$$\Rightarrow \bar{W}_e=\frac{\varepsilon_0}{4}\left(1+\frac{\omega_p^2}{\omega^2+\nu^2}\right)|E|^2$$



# Equivalent circuit (EC) method (Tretyakov 2005) (II)

Similarly, for the magnetic part, inserting a piece of the metamaterial into a solenoid with inductance $L_0$, the inductance becomes

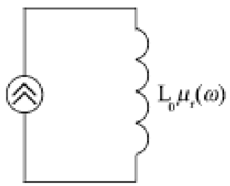
$$Z(\omega)=j\omega L_0\mu(\omega)=j\omega L_0+\frac{j\omega^3 FL_0}{\omega_0^2-\omega^2+j\omega\gamma}=j\omega L_0+\frac{j\omega^3 M^2/L}{\dfrac{1}{LC}-\omega^2+j\omega\dfrac{R}{L}},\ \text{here}\ \omega_0^2=\frac{1}{LC},\quad \gamma=\frac{R}{L},\quad FL_0=\frac{M^2}{L}.$$

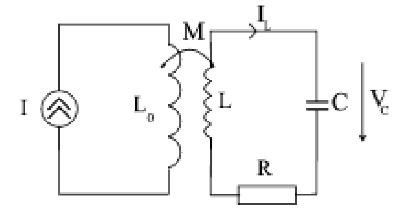
The time averaged energy stored in the circuit is

$$\bar{W}_b S=\frac{1}{4}\left(L_0|I|^2+L|I_L|^2+C|V_C|^2\right)+\frac{1}{2}\mathrm{Re}\left(MI^*I_L\right)=\frac{1}{4}L_0|I|^2\left(1+\frac{F\omega^2\left(3\omega_0^2-\omega^2\right)}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\gamma^2}\right)$$

$$=\frac{1}{4}\mu_0 n^2 S\frac{|H|^2}{n^2}\left(1+\frac{F\omega^2\left(3\omega_0^2-\omega^2\right)}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\gamma^2}\right),\quad L_0=\mu_0 n^2 S,\quad I=H/n$$

$$\Rightarrow \bar{W}_b=\frac{\mu_0}{4}\left(1+\frac{F\omega^2\left(3\omega_0^2-\omega^2\right)}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\gamma^2}\right)|H|^2$$




Remark:

Tretyakov had forgot to included the term $\frac{1}{2}\mathrm{Re}\left(MI^*I_L\right)$

and thus he obtained an incorrect result: $\bar{W}_b^{Tretyakov}=\dfrac{\mu_0}{4}\left(1+\dfrac{F\omega^2\left(\omega_0^2+\omega^2\right)}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\gamma^2}\right)|H|^2$.

This incorrect result does not reduce to Landau's formula $\dfrac{\mu_0}{4}\dfrac{d\left(\omega\mu(\omega)\right)}{d\omega}|H|^2$

but the correct one does when $\gamma\rightarrow 0$.

# Electrodynamic (ED) approach (Boardman & Marinov 2006) (I)

One can derive the following result from Maxwell's equations.

$$-\nabla\cdot\mathbf{S}=\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}=\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\varepsilon_0E^2+\mu_0H^2\right)\right]+\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t}+\mu_0\mathbf{H}\cdot\frac{\partial\mathbf{M}}{\partial t}$$

The equations of motion for the polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ are given by

$$\ddot{\mathbf{P}}+\nu\dot{\mathbf{P}}=\varepsilon_0\omega_p^2\mathbf{E}\quad(\text{the wire medium})$$

$$\ddot{\mathbf{M}}+\Gamma_h\dot{\mathbf{M}}+\omega_0^2\mathbf{M}=-F\ddot{\mathbf{H}}\quad(\text{the SRR medium})$$

Using the method similar to that of Loudon's, one can derive the result

$-\nabla\cdot\mathbf{S}=\frac{\partial}{\partial t}\left(W_e+W_b\right)+P_L$. The electric energy density $W_e$ is

$$W_e=\frac{\varepsilon_0\mathbf{E}^2}{2}+\frac{\dot{\mathbf{P}}^2}{2\omega_p^2\varepsilon_0},$$ and the magnetic energy density $W_b$ is

$$W_b=\frac{\mu_0\left(1-F\right)\mathbf{H}^2}{2}+\frac{\mu_0}{2\omega_0^2F}\left[\left(\dot{\mathbf{M}}+F\dot{\mathbf{H}}\right)^2+\omega_0^2\left(\mathbf{M}+F\mathbf{H}\right)^2\right].$$

$$P_L=\frac{\nu\dot{\mathbf{P}}^2}{\omega_p^2\varepsilon_0}+\frac{\gamma\mu_0\left(\dot{\mathbf{M}}+F\dot{\mathbf{H}}\right)\cdot\dot{\mathbf{M}}}{\omega_0^2F}$$ is the power loss.

# Electrodynamic (ED) approach (Boardman & Marinov 2006) (II)

For harmonic waves, the time-averaged Poynting vector $\bar{\mathbf{S}}$ and power loss $\bar{P}_L$ satisfies

$$-\nabla\cdot\bar{\mathbf{S}} = \bar{P}_L = \frac{\mu_0}{4}\left[\frac{F\omega^4\gamma}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\gamma^2}\right]|\,\mathbf{H}\,|^2$$

The time averaged electric energy density $\bar{W}_e$ is

$$\bar{W}_e = \frac{\varepsilon_0}{4}\left(1+\frac{\omega_p^2}{\omega^2+\nu^2}\right)|\,\mathbf{E}\,|^2,$$

and the time-averaged magnetic energy density $\bar{W}_b$ is

$$\bar{W}_b = \frac{\mu_0}{4}\left(1+F\,\frac{\omega^2\left[\omega_0^2\left(3\omega_0^2-\omega^2\right)+\omega^2\gamma^2\right]}{\omega_0^2\left[\left(\omega_0^2-\omega^2\right)^2+\omega^2\gamma^2\right]}\right)|\,\mathbf{H}\,|^2$$

The time averaged energy density

$$\bar{W} = \bar{W}_e+\bar{W}_b \rightarrow \frac{1}{4}\left(\varepsilon_0\frac{d}{d\omega}\big(\omega\varepsilon(\omega)\big)|\,\mathbf{E}\,|^2+\mu_0\frac{d}{d\omega}\big(\omega\mu(\omega)\big)|\,\mathbf{H}\,|^2\right)\ \text{ if }\quad \nu \text{ and } \gamma\rightarrow 0.$$

# General remarks on Tretyakov's EC approach and Boardman & Marinov's ED approach

1. Tretyakov's result does not reduce to Landau's formula when turn off the loss effect, thus the formula must be incorrect. However, this has nothing to do with the EC approach. The incorrectness happens because Tretyakov forgot to consider the energy contribution of the mutual inductance term $MII_L$ in his calculation.
2. When turn off the loss, Boardman & Marinov's magnetic energy density formula does reduce to the Landau's formula. When comparing with Tretyakov's correct magnetic energy density (harmonic wave), Boardman's formula contains an unphysical term, corresponding to $\frac{1}{2}C\,|V_R\,|^2$.
3. There is ambiguity in Boardman's ED approach. For example, if we add a term $f$ into the $W_b$ formula, we must subsract $\partial f / \partial t$ from $P_L$ at the same time. This transformation has no effect on the time-averaged power loss $P_L$.
4. In a metamaterial system consisted of the wires and SRRs, the only origin of the power loss should be the Joule heat of the conducting currents in the wires and the SRRs. Consquently, the power loss related to the magnetic field should be propotional to $\mathbf{M}^2$. This requirement will fix the form of the magnetic energy density.

# Deriving the magnetic energy density formula using ED approach (Luan 2007) (I)

Maxwell's equations lead to:

$$-\nabla\cdot\mathbf{S}=\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}=\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\varepsilon_0E^2+\mu_0H^2\right)\right]+\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t}+\mu_0\mathbf{H}\cdot\frac{\partial\mathbf{M}}{\partial t}$$

Using $\ddot{\mathbf{P}}+\nu\dot{\mathbf{P}}=\varepsilon_0\omega_p^2\mathbf{E},$ one can find that: $W_e=\dfrac{\varepsilon_0\mathbf{E}^2}{2}+\dfrac{\dot{\mathbf{P}}^2}{2\omega_p^2\varepsilon_0},$

and the electric part of the power loss is: $P_{loss}^{e}=\dfrac{\nu\dot{\mathbf{P}}^2}{\omega_p^2\varepsilon_0}$

Integrating the equation $\ddot{\mathbf{M}}+\gamma\dot{\mathbf{M}}+\omega_0^2\mathbf{M}=-F\ddot{\mathbf{H}}$

and subtituting it into the $\mu_0\mathbf{H}\cdot\dfrac{\partial\mathbf{M}}{\partial t}$ term, we have

$$\mu_0\mathbf{H}\cdot\frac{\partial\mathbf{M}}{\partial t}=\frac{\partial}{\partial t}\left(\mu_0\mathbf{H}\cdot\mathbf{M}\right)-\mu_0\mathbf{M}\cdot\frac{\partial\mathbf{H}}{\partial t}=\frac{\partial}{\partial t}\left(\mu_0\mathbf{H}\cdot\mathbf{M}\right)+\frac{\mu_0}{F}\mathbf{M}\cdot\left(\dot{\mathbf{M}}+\gamma\mathbf{M}+\omega_0^2\int\mathbf{M}dt\right)$$

$$=\frac{\partial}{\partial t}\left[\mu_0\mathbf{H}\cdot\mathbf{M}+\frac{\mu_0}{2F}\mathbf{M}^2+\frac{\mu_0\omega_0^2}{2F}\left(\int\mathbf{M}dt\right)^2\right]+\frac{\gamma\mu_0}{F}\mathbf{M}^2$$

# Deriving the magnetic energy density formula using ED approach (Luan 2007) (II)

Thus we have the magnetic energy density

$$W_b = \frac{\mu_0 \mathbf{H}^2}{2} + \mu_0 \mathbf{H} \cdot \mathbf{M} + \frac{\mu_0}{2F} \mathbf{M}^2 + \frac{\mu_0 \omega_0^2}{2F} \left( \int \mathbf{M} dt \right)^2,$$

and the magnetic part of the power loss: $P_{loss}^b = \frac{\gamma \mu_0}{F} \mathbf{M}^2$ (pure Joule heat !)

Substituting the relation: $\omega_0^2 \int \mathbf{M} dt = -\left( \dot{\mathbf{M}} + F \dot{\mathbf{H}} + \gamma \mathbf{M} \right)$ into $W_b$,

we can also obtain :

$$W_b = \frac{\mu_0 (1 - F) \mathbf{H}^2}{2} + \frac{\mu_0}{2 \omega_0^2 F} \left[ \left( \dot{\mathbf{M}} + F \dot{\mathbf{H}} + \gamma \mathbf{M} \right)^2 + \omega_0^2 \left( \mathbf{M} + F \mathbf{H} \right)^2 \right].$$

$P_{loss} = \frac{\nu \dot{\mathbf{P}}^2}{\omega_p^2 \varepsilon_0} + \frac{\gamma \mu_0 \mathbf{M}^2}{F}$ is the total power loss.

# Deriving the magnetic energy density formula using ED approach (Luan 2007) (III)

Remarks:

1. My result of $W_b$ is different from Boardman's in the $\left(\dot{\mathbf{M}}+F\dot{\mathbf{H}}+\gamma\mathbf{M}\right)^2$ term, which is $\left(\dot{\mathbf{M}}+F\dot{\mathbf{H}}\right)^2$ in Boardmn's paper.

2. The difference modifies the form of $P_{loss}$, changing it from the misterious form

$$\frac{\nu\dot{\mathbf{P}}^2}{\omega_p^2\varepsilon_0}+\frac{\gamma\mu_0\left(\dot{\mathbf{M}}+F\dot{\mathbf{H}}\right)\cdot\dot{\mathbf{M}}}{\omega_0^2 F}$$ to

the pure Joule heat form $\dfrac{\nu\dot{\mathbf{P}}^2}{\omega_p^2\varepsilon_0}+\dfrac{\gamma\mu_0\mathbf{M}^2}{F}$. Joule heat should be the only possible origin of the power loss.

3. However, this modification to $P_{loss}$ will not change the time-averaged power loss $\bar{P}_{loss}$, because

$$\left\langle\frac{\partial}{\partial t}\left[\left(\dot{\mathbf{M}}+F\dot{\mathbf{H}}+\gamma\mathbf{M}\right)^2-\left(\dot{\mathbf{M}}+F\dot{\mathbf{H}}\right)^2\right]\right\rangle_t=0 \text{ for harmonic field.}$$

# Deriving the magnetic energy density formula using ED approach (Luan 2007) (IV)

4. A carefully analysis shows that the $\frac{\mu_0}{2\omega_0^2 F}\left(\dot{\mathbf{M}} + F\dot{\mathbf{H}} + \gamma\mathbf{M}\right)^2 = \frac{\mu_0\omega_0^2}{2F}\left(\int \mathbf{M}dt\right)^2$ term corresponds to the electric potential energy stored in the capicitor part of the SRRs (A SRR is a $LC$ circuit).

5. For harmonic fields, the time averaged magnetic energy density is

$$\bar{W}_b = \frac{\mu_0}{4}\left(1 + \frac{F\omega^2\left(3\omega_0^2 - \omega^2\right)}{\left(\omega^2 - \omega_0^2\right)^2 + \omega^2\gamma^2}\right)|H|^2,$$

which is the correct result (the mutual inductance energy is included) obtained using the EC method.

6. When we turn off the loss effect, the energy density indeed reduce to Landau's result $\frac{1}{4}\left(\varepsilon_0 \frac{d}{d\omega}\left(\omega\varepsilon\left(\omega\right)\right)|\mathbf{E}|^2 + \mu_0 \frac{d}{d\omega}\left(\omega\mu\left(\omega\right)\right)|\mathbf{H}|^2\right)$.

# Conclusion

- For a metamaterial medium consisting of wires and SRRs, the only origin of the power loss is the Joule heat (I^2 R terms) in these conducting elements.
- The correct from of the power loss determines the correct form of the energy density.
- The method can also be applied to the metamaterial medium consisting of other kinds of conducting elements (eq. ER+MR).

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Thanks for your attention !