Quantum Effects in Dispersive Qubit-Resonator System

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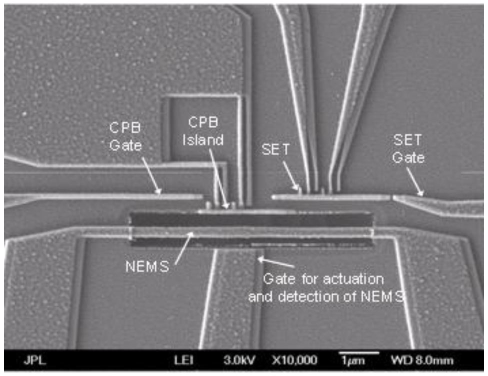
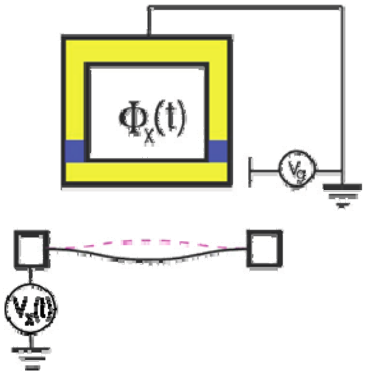
System

❖ Experiment:



Qubit-Nems
LaHaye et al.
(Caltech)

❖ Diagram:



- ❖ Study a dispersively coupled qubit-resonator system.
- ❖ Dispersive coupling arises when qubit freq. >> oscillator freq.
- ❖ Want to see quantum effects in oscillator:
  - ❖ Evidence of energy quantization
  - ❖ Entanglement

Method

- Hamiltonian of the system:

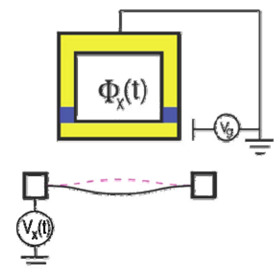
$$H = (\omega_M + \lambda\sigma_z)(a^\dagger a + \tfrac{1}{2}) + f(t)(a + a^\dagger) + \tfrac{1}{2}\Omega_{qb}\sigma_z + H_\gamma$$

- Dispersive coupling:

  qubit state changes frequency of oscillator.

  Each qubit energy eigenstates $|\uparrow\rangle, |\downarrow\rangle$

  leads to different oscillator frequencies.



- This type of coupling arises from Jaynes-Cumming

  $\hbar g(a^\dagger\sigma^- + a\sigma^+)$. Written in basis $|e\rangle|n\rangle$ & $|g\rangle|n+1\rangle$

  $g(n+1)^{1/2}\sigma_x$. In regime of $\omega_{qb} \gg \omega_M$

  perturbation $\frac{g^2}{\hbar\Delta}\sigma_z a^\dagger a$

Method

$$H = (\omega_M + \lambda\sigma_z)(a^\dagger a + \tfrac{1}{2}) + f(t)(a + a^\dagger) + \tfrac{1}{2}\omega_{qb}\sigma_z$$
$$+(a + a^\dagger)\sum_j \nu_j(b_j + b_j^\dagger) + \sum_j \omega_j(b_j^\dagger b_j + \tfrac{1}{2})$$

- Derive a standard master equation

$$\begin{aligned}\dot{\hat{\rho}} &= -i\,[H_0, \hat{\rho}] + \gamma(n_{\rm eq}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma n_{\rm eq}\mathcal{D}[\hat{a}^\dagger]\rho \\ &\quad +(\Gamma_\varphi/2)\mathcal{D}[\hat{\sigma}_z]\hat{\rho}\end{aligned}$$

$$\mathcal{D}[\hat{A}]\hat{\rho} = \hat{A}\hat{\rho}\hat{A}^\dagger - \left(\hat{A}^\dagger\hat{A}\hat{\rho} + \hat{\rho}\hat{A}^\dagger\hat{A}\right)/2. \qquad n_{eq} = \frac{1}{e^{\hbar\nu/k_bT}-1}$$

- Exact solution in terms of the displaced thermal distribution

$$\hat{\rho}_{eq}(T) = (1 - e^{-\omega/k_BT})\sum_{n=0}^{\infty} e^{-n\omega_M/(k_BT)}|n\rangle\langle n|$$

Method (cont'd)

- Exact solution in the qubit space:

$$\hat{\rho}_{\uparrow\uparrow}(t) = \frac{1}{2}\hat{D}[\alpha_{\uparrow}(t)]\cdot\hat{\rho}_{\rm eq}[T]\cdot\hat{D}^{\dagger}[\alpha_{\uparrow}(t)]$$

$$\hat{\rho}_{\downarrow\downarrow}(t) = \frac{1}{2}\hat{D}[\alpha_{\downarrow}(t)]\cdot\hat{\rho}_{\rm eq}[T]\cdot\hat{D}^{\dagger}[\alpha_{\downarrow}(t)]$$

$$\hat{\rho}_{\uparrow\downarrow}(t) = [\hat{\rho}_{\downarrow\uparrow}(t)]^{\dagger} = \frac{1}{2}e^{i\omega_{\rm qb}t}Y(t)\times$$

$$\hat{D}[\tilde{\alpha}_{\uparrow}(t)]\cdot\left(\hat{\rho}_{\rm eq}[T^{*}(t)]e^{-i\phi(t)(\hat{n}+\frac{1}{2})}\right)\cdot\hat{D}^{\dagger}[\tilde{\alpha}_{\downarrow}(t)]$$

$$\hat{\rho}_{\uparrow\uparrow}(t) = \langle\uparrow|\hat{\rho}(t)|\uparrow\rangle$$

$$\hat{D}[\alpha] = e^{(\alpha\hat{a}^{\dagger}-\alpha^{*}\hat{a})}$$

- Where we have defined:

$\alpha_{\uparrow}(t), \alpha_{\downarrow}(t)$: Average of $\langle a\rangle$ given that the qubit is $|\uparrow\rangle$, $|\downarrow\rangle$

$\rho_{eq}(T)$: Thermal oscillator density matrix at temperature *T*

$\tilde{\alpha}_{\uparrow}(t), \tilde{\alpha}_{\downarrow}(t)$: modified $\alpha_{\uparrow}(t), \alpha_{\downarrow}(t)$; reduces to $\alpha_{\uparrow}(t), \alpha_{\downarrow}(t)$ at *T=0*

$\rho_{eq}(T^{*})$: Thermal distribution with new temperature

$Y(t)$: factor describing oscillator–bath entanglement

$\phi(t)$: time–dependent phase factor $\propto\lambda$

Number Splitting

- Evidence of discrete energy level in mechanical system
- Qubit's off-diagonal density matrix element $\hat{\rho}_{\uparrow\downarrow}(t) = \langle\uparrow|\hat{\rho}(t)|\downarrow\rangle$
- Excite qubit from ground state via time dependent field ω_{rf}
- Can relate to absorption spectrum of qubit

$$\Gamma_{abs}(\omega_{rf}) = |A|^2 \int_{-\infty}^{\infty} dt e^{+i\omega_{rf}t} \langle \sigma_-(t)\sigma_+(0)\rangle$$

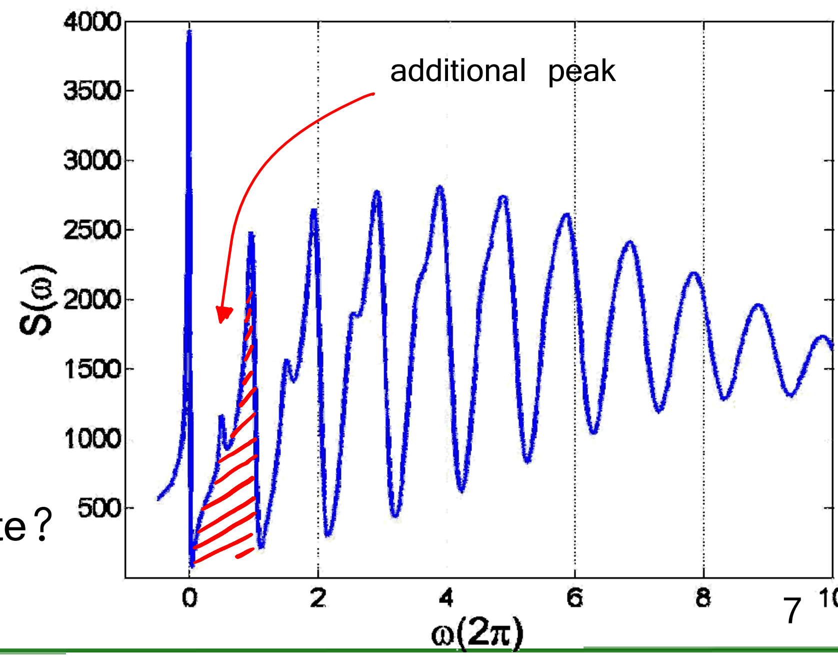
$$= 2|A|^2 \mathrm{Re} \left[\int_{0}^{\infty} dt e^{+i\omega_{rf}t} \rho_{\uparrow\downarrow}(t)\right] \quad \text{(approx)}$$

- With no force at zero temperature:

$$\rho_{\uparrow\downarrow}(t) = e^{-i\Omega_{qb}t} \sum_{n=0}^{\infty} P(n) e^{-2i\lambda\omega_M(n+\frac{1}{2})t} \rho_{\uparrow\downarrow}(0)$$

Number splitting in NEMS

- Need to consider with finite temperature AND finite drive,
- No longer $$\rho_{\uparrow\downarrow}(t) = e^{-i\Omega_{qb}t}\sum_{n=0}^{\infty} P(n) e^{-2i\lambda\omega_M(n+\frac{1}{2})t}\rho_{\uparrow\downarrow}(0)$$
- For an arbitrary detuning of the drive, there are too many peaks in the spectrum
- weight of ‘true’ peaks do not correspond to the initial number distribution of the mode…
- can we faithfully measure the mode number statistics if we don't already know its initial state?



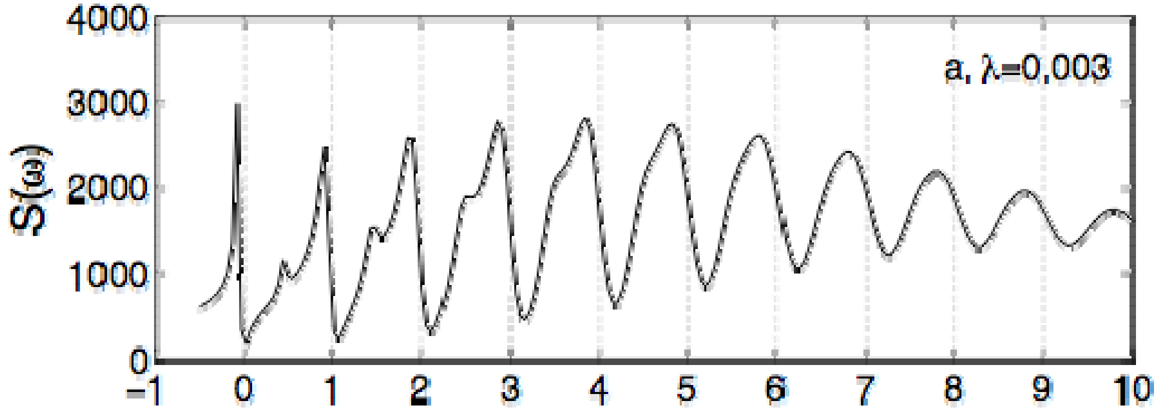
Analysis

- Suppose we start with someone unknown state of oscillator （finite temperature, finite driving force, arbitrary detuning） can still relate the absorption to $\rho_{\uparrow\downarrow}$?

YES, if:

- Keeping coupling off until t=0
- For t>0, use a large enough coupling

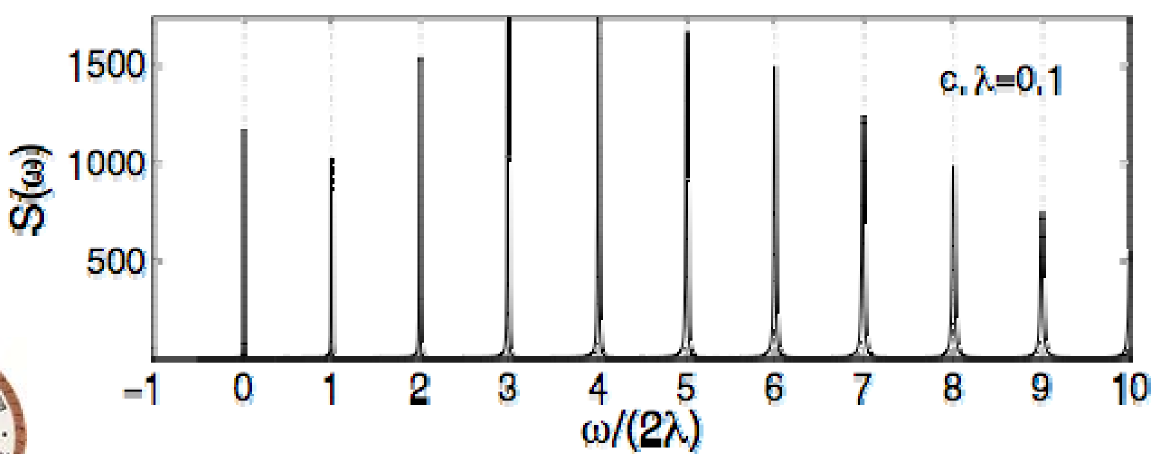
Dephasing Spectra



$$\bar{n}_{eq} = 1$$

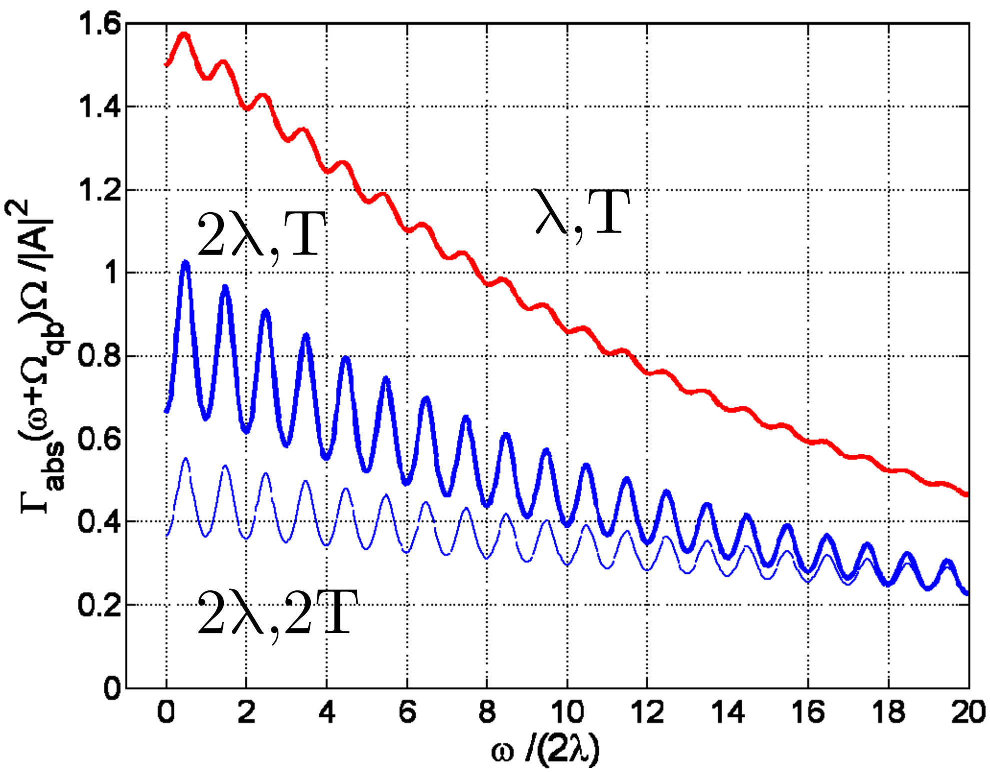
$$\gamma = 10^{-4}\omega_M$$

$$|\alpha_0|^2 = 9$$



Realistic Numbers

- use oscillator and coupling numbers similar to *Naik et al.*
  - $\omega_{osc} = 2\pi\ 21.9$ MHz
  - $\Omega_{qb} = 2\pi\ 2$ GHz
  - **Q** = 10^4
  - $\lambda = 0.71$ MHz
    (at 10 volts)
  - $\Gamma\phi = 1$ MHz
  - ***n***$_{eq}$ (at 15 mK): 13



Entanglement

- Consider two systems A & B
- The two systems is said to be entangled if:

$$\rho_T \neq \rho_A \otimes \rho_B \qquad \text{(not separable)}$$

- Quantifying the amount of entanglement

want a computable measure

Log negativity $E_N = \log_2(2\mathcal{N} + 1)$

$$\mathcal{N} = \frac{||\rho^T|| - 1}{2} = \sum_i |\mu_i|$$

μ_i negative eigenvalues of partially transposed ρ^T

Entanglement

$$
\begin{aligned}
\hat{\rho}_{\uparrow\uparrow}(t) &= \tfrac{1}{2}\hat{D}[\alpha_{\uparrow}(t)] \cdot \hat{\rho}_{\rm eq}[T] \cdot \hat{D}^{\dagger}[\alpha_{\uparrow}(t)] \\
\hat{\rho}_{\downarrow\downarrow}(t) &= \tfrac{1}{2}\hat{D}[\alpha_{\downarrow}(t)] \cdot \hat{\rho}_{\rm eq}[T] \cdot \hat{D}^{\dagger}[\alpha_{\downarrow}(t)] \\
\hat{\rho}_{\uparrow\downarrow}(t) &= [\hat{\rho}_{\downarrow\uparrow}(t)]^{\dagger} = \tfrac{1}{2}e^{i\omega_{\rm qb}t}Y(t) \times \\
&\quad \hat{D}[\tilde{\alpha}_{\uparrow}(t)] \cdot \left(\hat{\rho}_{\rm eq}[T^{*}(t)]e^{-i\phi(t)(\hat{n}+\frac{1}{2})}\right) \cdot \hat{D}^{\dagger}[\tilde{\alpha}_{\downarrow}(t)]
\end{aligned}
$$

- Result? – Analytical solution for T=0, non-zero dephasing

$$\mathcal{N} = -\frac{1}{4}(1 - Y - \sqrt{1 + Y^2 - 2Y\cos(2\theta)})$$

Purity

$$Y = \frac{\exp(-(2i\lambda)\int_0^t dt'(\alpha_{\uparrow}(t')\alpha_{\downarrow}^{*}(t')))}{\cos(\theta)}$$

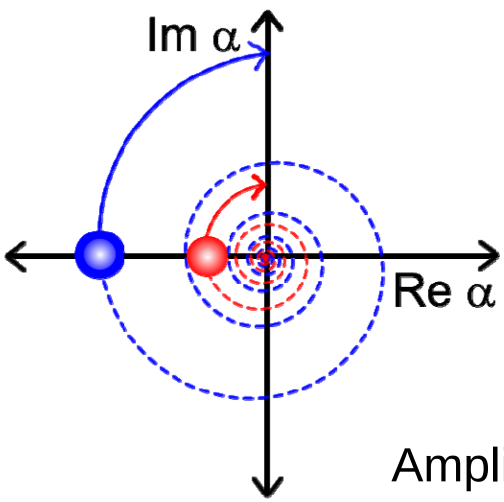
Overlap

$$\cos(\theta) = |\langle\alpha_{\uparrow}|\alpha_{\downarrow}\rangle|$$

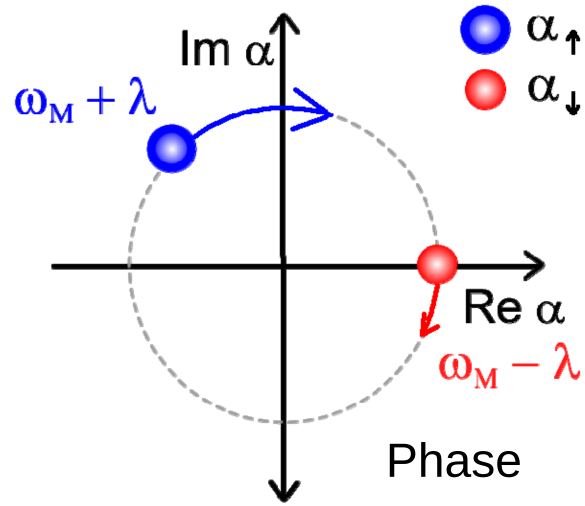
– Semi analytical calculation for finite temperature!

Entanglement

- At $t = 0$, prepare qubit in the state $|\uparrow\rangle$ + $|\downarrow\rangle$
- For $t > 0$, qubit-oscillator entanglement may develop.
- **NB: if $\langle a \rangle = 0$, then there is NEVER ANY ENTANGLEMENT !**
  - need to drive oscillator to get entanglement
  - two methods: – entangle qubit with oscillator **amplitude** i.e. $|a|$
    - entangle qubit with oscillator **phase**, i.e $arg(a)$



Amplitude



Phase

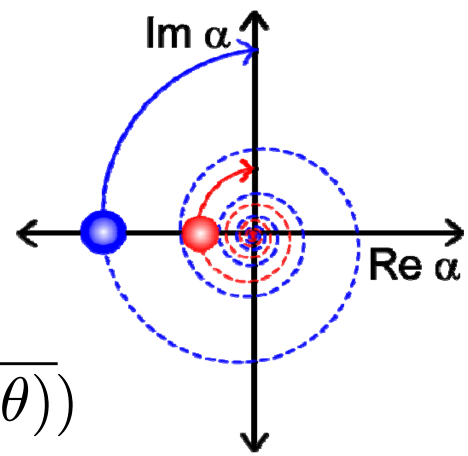
Amplitude Entanglement

- At t =0, prepare qubit in pure superposition state;

  drive the oscillator at $f(t) = \gamma\alpha_f \cos[(\omega_M + \lambda)t]$

  – only have resonance IF the qubit is up

  – will lead to a "cat" state



- Zero temperature: Simple analytic description

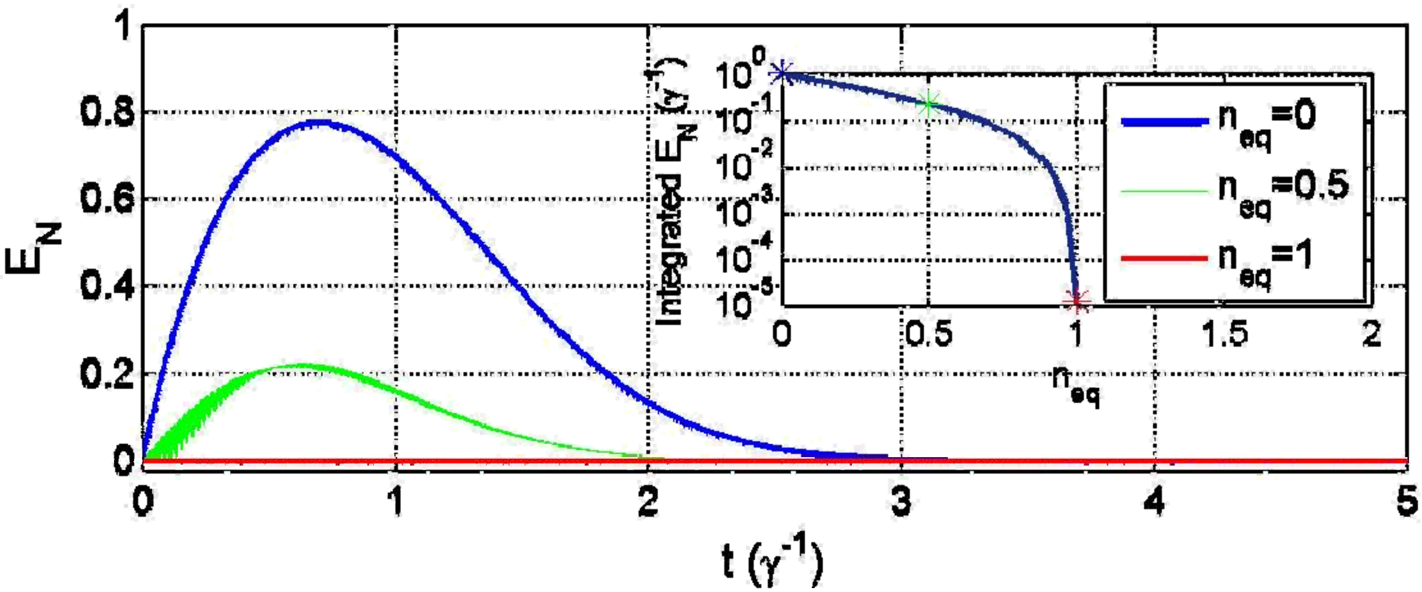
$$\mathcal{N} = -\frac{1}{4}\left(1 - Y - \sqrt{1 + Y^2 - 2Y\cos(2\theta)}\right)$$

$$\ln Y = \approx -\frac{\alpha_f^2 (t/\gamma)^3}{24} \qquad \cos(2\theta) \approx -\frac{\alpha_f^2 (t/\gamma)^2}{2} + \frac{\alpha_f^2 (t/\gamma)^3}{4}$$

- Finite temperature: Expand solution as matrix in basis of

  displaced Fock state

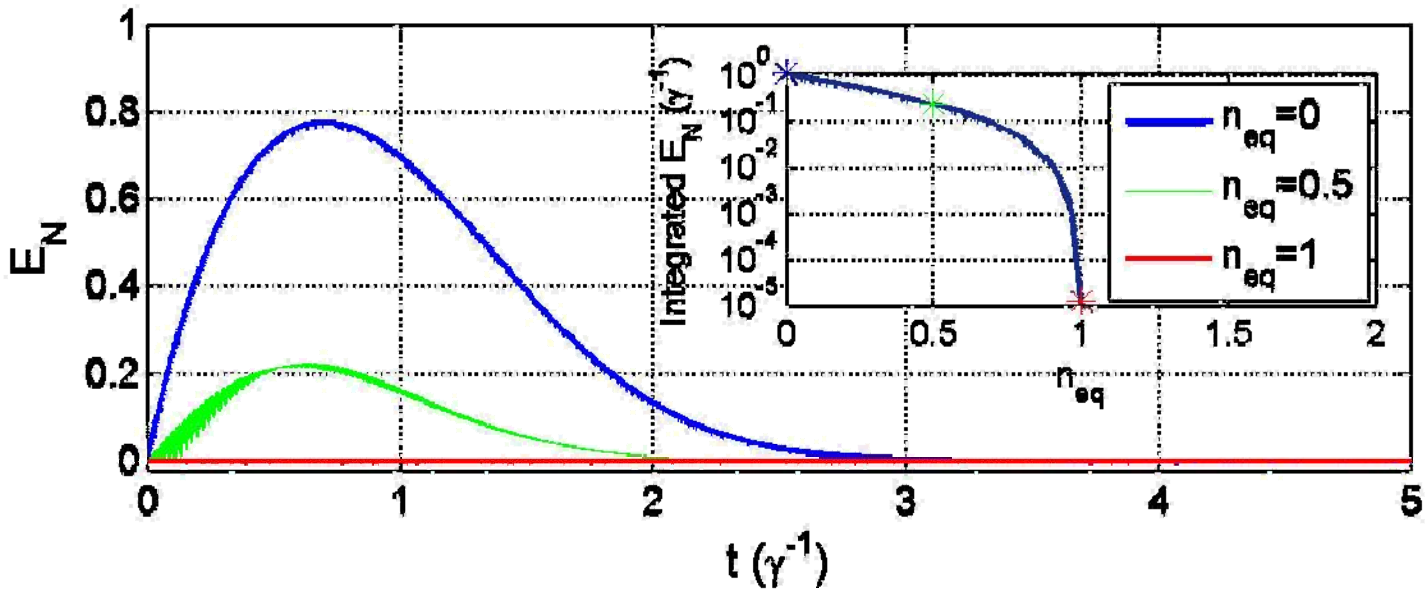
Amplitude Entanglement



$\alpha_f = 3.74$
$\lambda = 0.01\omega_M,$
$\gamma = 10^{-5}\omega_M$

- Find that entanglement is non-monotonic with time, has a maximum.
- Competition between two effects:
  - $|\alpha_\uparrow(t) - \alpha_\downarrow(t)|^2$ grows with time → increases $\mathcal{N}$
  - oscillator-qubit system gets entangled with bath → decreases $\mathcal{N}$

Amplitude Entanglement



$\alpha_f = 3.74$
$\lambda = 0.01\omega_M,$
$\gamma = 10^{-5}\omega_M$

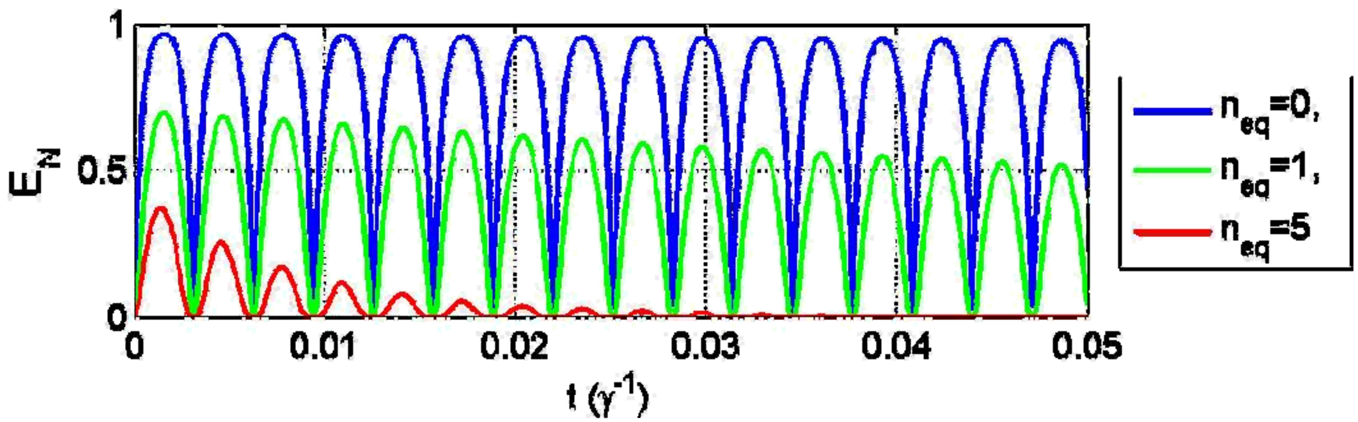
❖ Non-zero temperature? kills entanglement very quickly

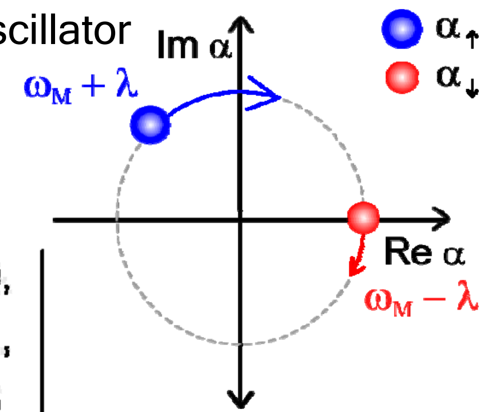
- easy for environment to distinguish two states of the superposition
- if oscillator has large $|a|$, creates many bath quanta;

  if oscillator $|a|$ small, few bath quanta created

Phase Entanglement

- At *t*=0 prepare qubit in pure superposition state, and prepare oscillator in a state where $\langle a \rangle \neq 0$.
  - As oscillator frequency set by qubit state, phase of oscillator (i.e. $\arg(a)$) will become entangled with qubit.



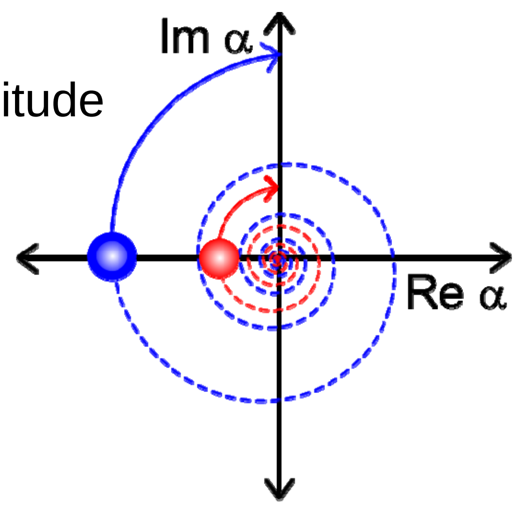


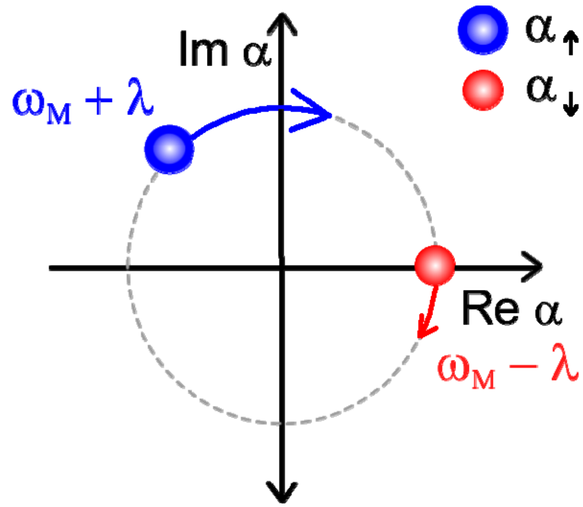
$\alpha_0 = 0.76$
$\lambda = 0.01\omega_M,$
$\gamma = 10^{-5}\omega_M$

Entanglement vanishes periodically with period $\frac{\pi}{\lambda}$

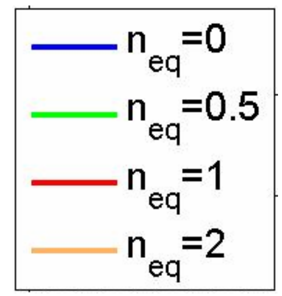
Phase vs Amplitude

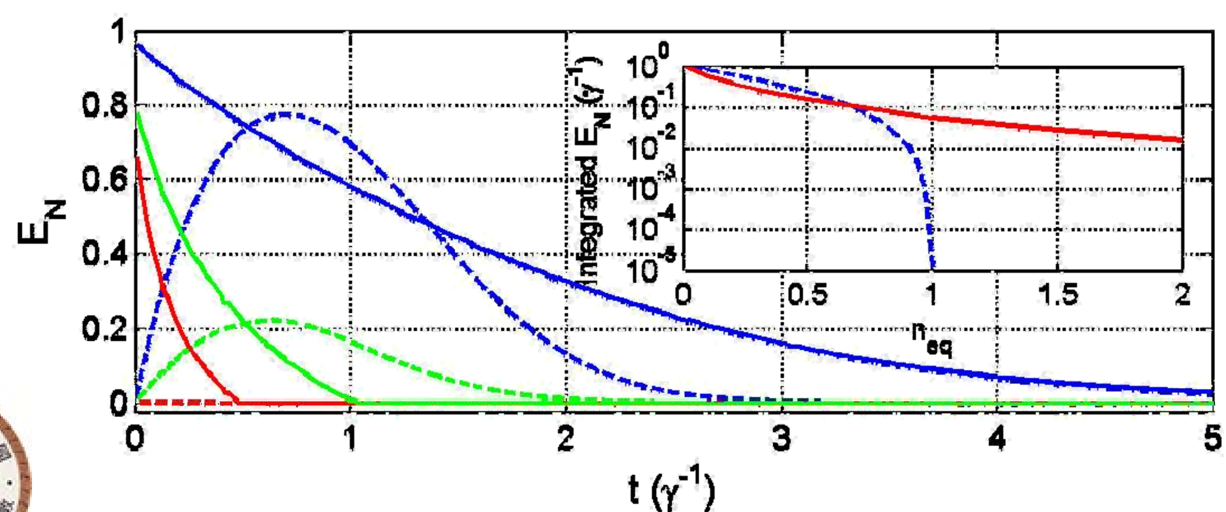
Amplitude





Phase





$$\alpha_0 = 0.76$$

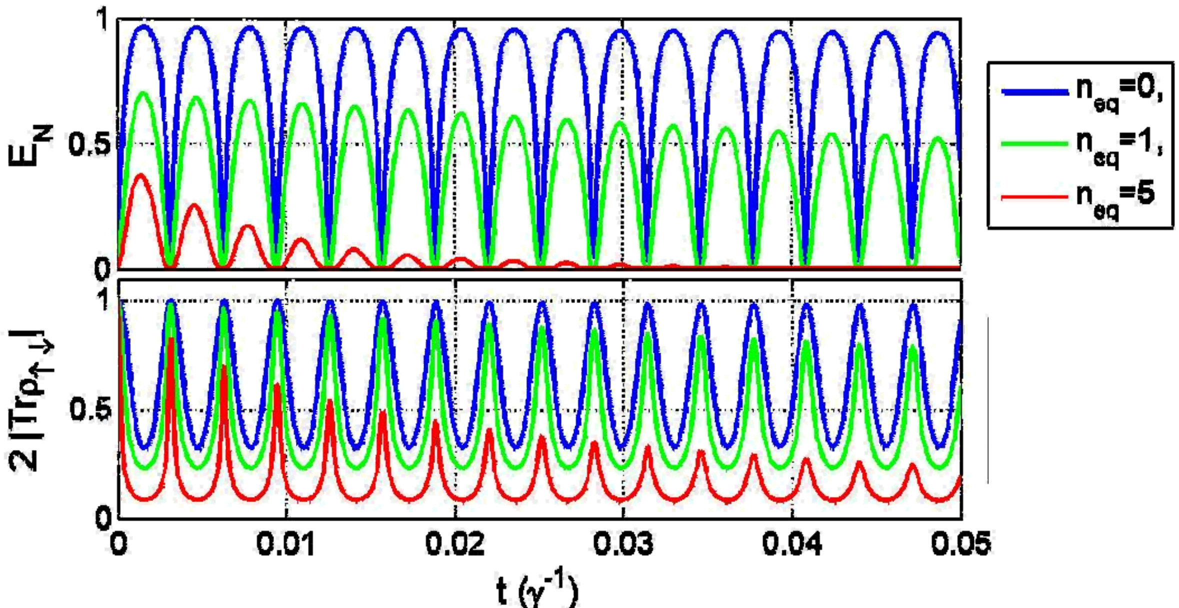
vs

$$\alpha_f = 3.74$$

Detecting entanglement

- In Phase Entanglement case:

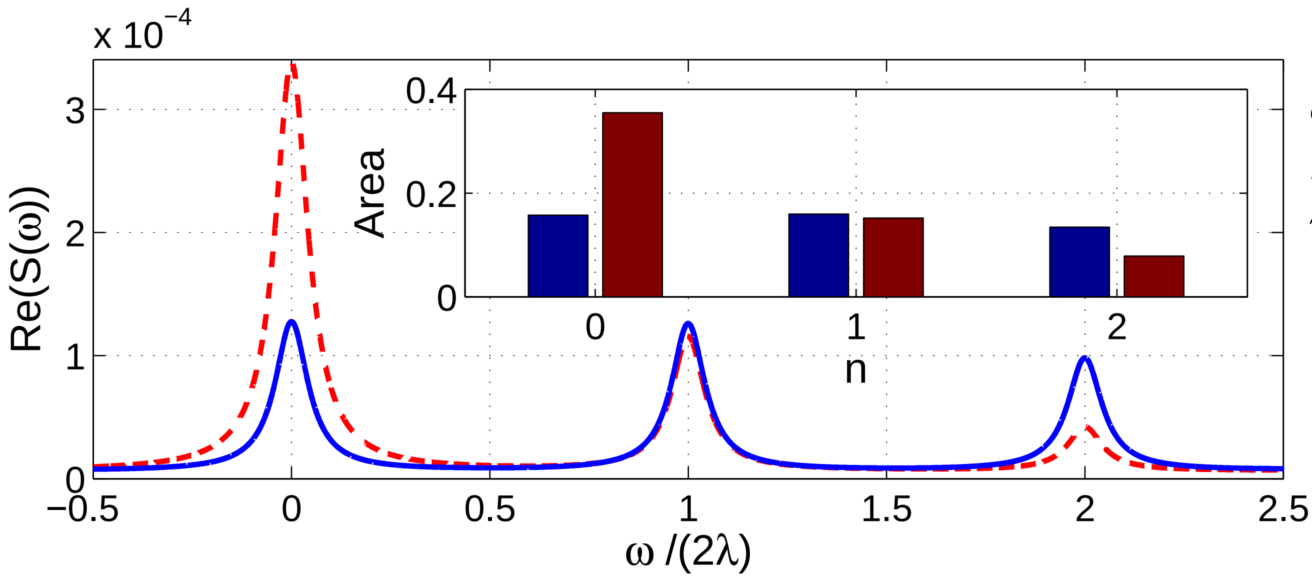
  Dephasing revivals (i.e. $\mathrm{Tr}[\rho_{\uparrow\downarrow}]$ not monotonic).



- $\mathrm{Tr}[\rho_{\uparrow\downarrow}]$ can be measured using standard Ramsey interference/ state tomography.

- Dephasing revivals **ARE NOT** proof of qubit-osc. entanglement

Entanglement Signature



$\alpha_0 = 0.76$
$\lambda = 0.01\omega_M,$
$\gamma = 10^{-5}\omega_M$

initial thermal state
$n_{\rm eq} = 0.5$
with $\langle a(0)\rangle = 0$
Follow a Boltzmann distribution
no entanglement

same temperature
With $\langle a \rangle$ initially nonzero
peaks DO NOT follow
Boltzmann distribution
entanglement

Conclusion

- Learn about dispersively-coupled qubit-oscillator system
- A theory to see evidence of energy quantization that includes arbitrary temperature and driving of the mode
- We have studied the entanglement dynamics.
  Two types: Amplitude & Phase entanglement
- Unambiguous way to detect entanglement in the system
- Use the frequency spectrum of $\rho_{\uparrow\downarrow}(t)$ ($S(\omega)$)
  as a "fingerprint" to deduce entanglement.