Black Hole Entropy in Loop Quantum Gravity

Mar. 5, 2009 @ NTHU

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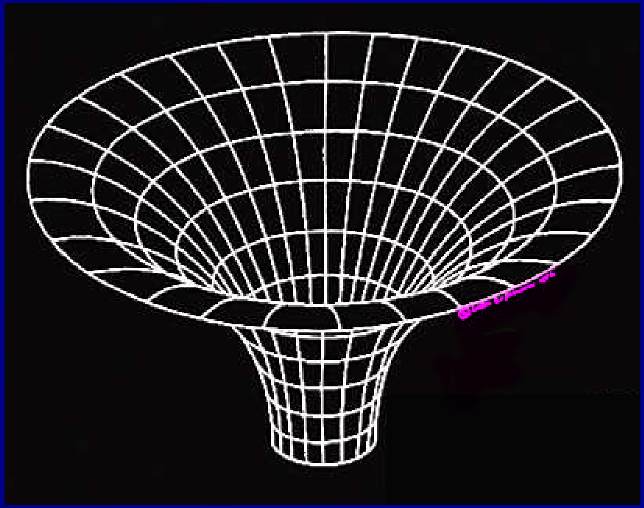
Spacetime, geometry and gravity

Puzzles in general relativity

- Black holes

$$S = \frac{A}{4G}$$



Black hole and singularity theorem

- Schwarzschild [1916]:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{1}{1 - \frac{2GM}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

  - $r = 2GM$: horizon; $r = 0$: singularity.

- **Pattern Singularity Theorem**:
  If a spacetime of sufficient differentiability satisfies
  - a condition on the curvature
  - a causality condition
  - and an appropriate initial and/or boundary condition

  then there are null or timelike inextensible incomplete geodesics.

  $\Rightarrow$ Singularities are unavoidable in GR.

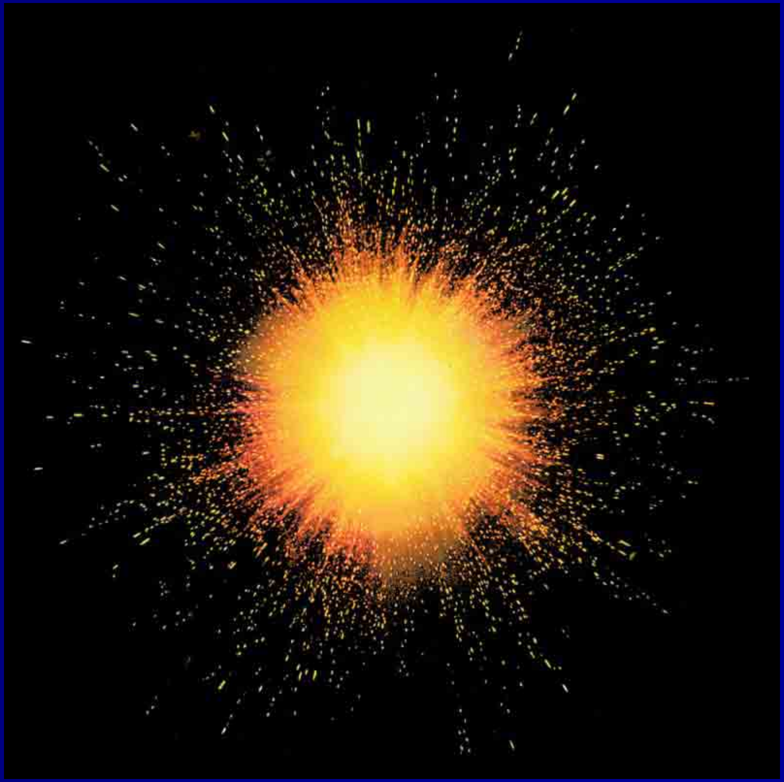
- GR can not be complete! It predicts its own breakdown.

Spacetime, geometry and gravity

- Cosmology

  Big bang

  Singularity



Black hole thermodynamics

- Hawking (1972): the area of the event horizon of a black hole cannot decrease.
- Bekenstein (1973): associate an entropy to a black hole

$$S_{BH} = kA$$

- Hawking (1975): black hole temperature $T = \frac{1}{8\pi M}$,

$$S_{BH} = \frac{1}{4}A$$

- What are the microscopic degrees of freedom responsible for this entropy?
- What are the higher order corrections to the Benkenstein-Hawking entropy formula?

PHYSICAL REVIEW D VOLUME 7, NUMBER 8 15 APRIL 1973

Black Holes and Entropy*

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(Received 2 November 1972)

There are a number of similarities between black-hole physics and thermodynamics. Most striking is the similarity in the behaviors of black-hole area and of entropy: Both quantities tend to increase irreversibly. In this paper we make this similarity the basis of a thermodynamic approach to black-hole physics. After a brief review of the elements of the theory of information, we discuss black-hole physics from the point of view of information theory. We show that it is natural to introduce the concept of black-hole entropy as the measure of information about a black-hole interior which is inaccessible to an exterior observer. Considerations of simplicity and consistency, and dimensional arguments indicate that the black-hole entropy is equal to the ratio of the black-hole area to the square of the Planck length times a dimensionless constant of order unity. A different approach making use of the specific properties of Kerr black holes and of concepts from information theory leads to the same conclusion, and suggests a definite value for the constant. The physical content of the concept of black-hole entropy derives from the following generalized version of the second law: When common entropy goes down a black hole, the common entropy in the black-hole exterior plus the black-hole entropy never decreases. The validity of this version of the second law is supported by an argument from information theory as well as by several examples.

Quantum statistical mechanics

- In quantum statistical mechanics the mean value of some dynamical variable $f(q)$ can be expressed in the form

$$< f >= \frac{1}{Z} \sum_E \int \phi_E^*(q) f(q) \phi_E(q) e^{-\beta E}\, dq$$

where $\phi_E(q)$ is the stationary state eigenfunction with $H\phi_E = E\phi_E, \beta = (1/T)$ is the inverse temperature and $Z(\beta)$ is the partition function.

- The quantum mechanical kernel giving the probability amplitude for the system to go from the state q at time $t = 0$ to the state q' at time t is given by

$$K(q', t; q, 0) = \sum_E \phi_E^*(q') \phi_E(q) e^{-itE}$$

- The thermal average can be obtained by

$$< f >= \frac{1}{Z} \int dq\, K(q, -i\beta; q, 0) f(q)$$

with the following:

(i) Analytically continuation to imaginary time with $it = \tau$.

(ii) Periodicity in the imaginary time τ with period β.

Horizon and Temperature

- Spacetimes with horizons possess a natural analytic continuation from Minkowski signature to the Euclidean signature with $t \to \tau = it$.
- If the metric is periodic in τ, then one can associate a natural notion of a temperature to such spacetimes.
  - Consider a metric of the form

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - dL_\perp^2$$

    where $dL_\perp^2$: transverse 2-dimensional metric, $f(r)$ has a simple zero at $r = r_H$.
  - Near $r = r_H$, $f(r) \approx B(r - r_H)$ where $B \equiv f'(r_H)$.
  - Since $g_{00} \approx (1 + 2\phi_N)$ in the Newtonian limit, the surface gravity

$$\kappa = |\phi_N'(r_H)| = \frac{1}{2}|g_{00}'(r_H)| = \frac{1}{2}|f'(r_H)| = \frac{1}{2}|B|$$

  - Shifting to the coordinate $\xi \equiv [2\kappa^{-1}(r - r_H)]^{1/2}$ the metric near the horizon becomes

$$ds^2 \approx \kappa^2\xi^2dt^2 - d\xi^2 - dL_\perp^2$$

Horizon and Temperature

- The Euclidean continuation $t \to \tau = it$ now leads to the metric

$$-ds^2 \approx \xi^2 d(\kappa\tau)^2 + d\xi^2 + dL_\perp^2$$

which is essentially the metric in the polar coordinates in the $\tau - \xi$ plane.

- For this metric to be well defined near the origin, $\kappa\tau$ should behave like an angular coordinate θ with periodicity 2π.
- Therefore, we require all well defined physical quantities defined in this spacetime to have a periodicity in τ with the period $(2\pi/|\kappa|)$.
- Thus, all metrics of the form with a simple zero for $f(r)$ leads to a horizon with temperature $T = |\kappa|/2\pi = |f'(r_H)|/4\pi$.
  - In the case of de Sitter spacetime, this gives $T = (H/2\pi)$ where H is the Hubble constant.
  - For the Schwarzschild metric, this gives $T = (1/8\pi M)$ where M is the mass of the black hole.

Horizon and Entropy

- The partition function for this set of metrics $\mathcal{S}$ is given by the path integral sum

$$Z(\beta) = \sum_{g \epsilon \mathcal{S}} \exp(-A_E(g)) = \sum_{g \epsilon \mathcal{S}} \exp\left(-\frac{1}{16\pi}\int_0^\beta d\tau \int d^3x \sqrt{g_E} R_E[f(r)]\right)$$

where Einstein action has been continued in the Euclidean sector and we have imposed the periodicity in τ with period $\beta = 4\pi/|B|$.

- Using $R = \nabla_r^2 f - (2/r^2)(d/dr)\left[r(1-f)\right]$ valid for metrics of the particular form,

$$-A_E = \frac{\beta}{4}\int_a^b dr\,\left[-[r^2 f']' + 2[r(1-f)]'\right] = \frac{\beta}{4}[a^2 B - 2a] + Q[f(b), f'(b)]$$

where we have used the conditions $[f(a) = 0, f'(a) = B]$.

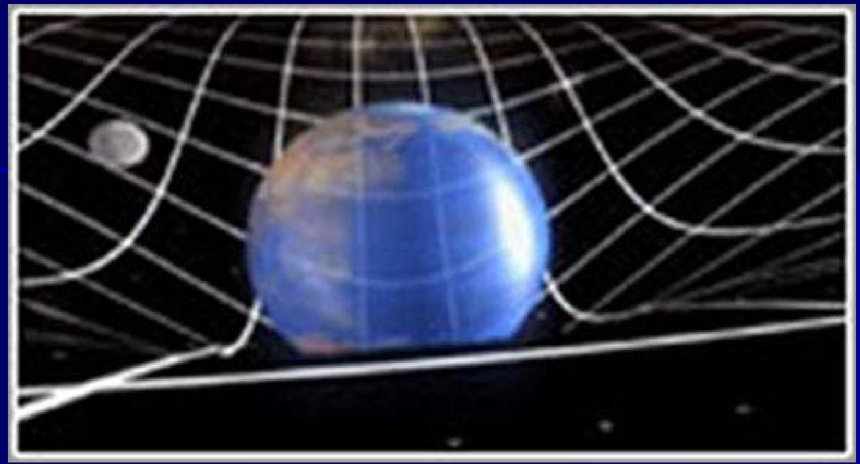
- Using $\beta = 4\pi/B$ the final result can be written as:

$$Z(\beta) = Z_0 \exp\left[\frac{1}{4}(4\pi a^2) - \beta(\frac{a}{2})\right] \propto \exp\left[S(a) - \beta E(a)\right]$$

with the identifications for the entropy and energy being given by:

$$S = \frac{1}{4}(4\pi a^2) = \frac{1}{4}A_{\text{horizon}}; \quad E = \frac{1}{2}a = \left(\frac{A_{\text{horizon}}}{16\pi}\right)^{1/2}$$

Einstein field equation： A theory of space, time and Matter



Spacetime is curved due to the gravitational action of matter

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu}$$

Geometry of space time

Quantization

Quantum Gravity？

Matter

Quantization

QM,QED,QCD…

Sketch of canonical quantization

- Pick a Poisson algebra of classical quantities.
- Represent these quantities as quantum operators acting on a space of quantum states.
- Implement any constraint you may have as a quantum operator equation and solve for the physical states.
- Construct an inner product on physical states.
- Develop a semiclassical approximation and compute expectation values of physical quantities.

Brief history of loop quantum gravity

- 1920, Einstein-Cartan Theory

  Metric ⟶ Connection

  1960, Roger Penrose
  Spin networks

- 1986，Abhay Ashtekar
  Complex new variables

- 1990，Carlo Rovelli, Lee Smolin
  Loop representation, Spin networks，quantum geometry

Dr. Lee Smolin

Canonical analysis in ADM variable

- Einstein-Hilbert action [in metric variables]

$$I[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(\mathcal{R} - 2\Lambda)$$

- ADM Decomposition: introduce a foliation of spacetime $\mathcal{M} = \Sigma \times \mathrm{R}$
  - $g_{\mu\nu} \longrightarrow q_{ab}$, N_a : shift function, N: laspe function.
  - $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2(dx^0)^2 + q_{ab}(dx^a + N^a dx^0)(dx^b + N^b dx^0)$

$$g_{\mu\nu} = \begin{pmatrix} q_{ab}N^aN^b - N^2 & q_{ab}N^a \\ q_{ab}N^b & q_{ab} \end{pmatrix}, g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^a/N^2 \\ N^b/N^2 & q^{ab} - N^aN^b/N^2 \end{pmatrix}$$

- After performing the Legendra transformation:

$$I[q_{ab}, \pi^{ab}, N_a, N] = \frac{1}{16\pi}\int dt \int_\Sigma d^3x[\pi^{ab}\dot{q}_{ab} - \mathcal{H}]$$

  - $\pi^{ab} = -\frac{\sqrt{q}}{16\pi G}(K^{ab} - Kq^{ab})$: momenta canonically conjugate to q_{ab},
$K_{ab} = \frac{1}{2N}(-\partial_0 q_{ab} + \nabla_a N_b + \nabla_b N_a)$: extrinsic curvature.

Canonical analysis in ADM variable

$$\mathcal{H}(q_{ab}, \pi^{ab}, N_a, N) = N^a H_a(q_{ab}, \pi^{ab}) + N H(q_{ab}, \pi^{ab})$$

- Super-momentum constraint: $H_a(q_{ab}, \pi^{ab}) = -\frac{2}{16\pi G}\nabla_b \pi^b{}_a \quad (= 0)$
- Super-Hamiltonian constraint:

$$\begin{aligned} H(q_{ab}, \pi^{ab}) &= \frac{8\pi G}{\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})\pi^{ab}\pi^{cd} - \frac{\sqrt{q}}{16\pi G}(R(q) - 2\Lambda) \\ &= \frac{\sqrt{q}}{16\pi G}[K^{ab}K_{ab} - K^2 - R(q) + 2\Lambda] \quad (= 0) \end{aligned}$$

- Degrees of freedom of GR in 4D:

  6 pairs (q_{ab}, π^{ab}) subject to 4 constraints = 2 FIELD d.o.f.

- The Poisson brackets are

$$\begin{aligned} \{\pi^{ab}(x), q_{cd}(y)\} &= 16\pi \delta^a_{(c}\delta^b_{d)}\delta(x, y), \\ \{q_{ab}(x), q_{cd}(y)\} &= \{\pi^{ab}(x), \pi^{cd}(y)\} = 0 \end{aligned}$$

- Phase space variables: (q_{ab}, π^{cd})

Canonical Quantization of GR

- Does not require background spacetime (background independence)
- Can be used for strong and weak GR fields.
- Conjugate variables:

$$\{q_{ab}(\vec{x}), \pi^{cd}(\vec{y})\}_{P.B.} = \frac{1}{2}({\delta_a}^c {\delta_b}^d + {\delta_b}^c {\delta_a}^d)\delta^3(\vec{x} - \vec{y})$$

- Canonical Quantization :

$$\{\ ,\ \}_{P.B.} \longrightarrow \frac{1}{i\hbar}[\ ,\ ]; \quad q_{ab} \longrightarrow \hat{q}_{ab}, \quad \pi^{ab} \longrightarrow \hat{\pi}^{ab}$$

- Metric representation: Wavefunction $\Psi[q_{ab}]$
  - $\hat{q}_{ab}\Psi[q_{ab}] = q_{ab}\Psi[q_{ab}]$; $\hat{\pi}^{ab}\Psi[q_{ab}] = \frac{1}{i\hbar}\frac{\delta}{\delta q_{ab}}\Psi[q_{ab}]$
- Constraints (First Class) (Dirac Quantization):

$$\hat{H}_a(\hat{q}_{ab}, \hat{\pi}^{ab})\Psi[q_{ab}] = \hat{H}_a(\hat{q}_{ab}, \tfrac{1}{i\hbar}\tfrac{\delta}{\delta q_{ab}})\Psi[q_{ab}] = 0$$

$\Leftrightarrow$ $\Psi[q'_{ab}] = \Psi[q_{ab}]$ if q_{ab} is related to q'_{ab} by a 3-dimensional diffeomorphism

Canonical Quantization of GR

$\Leftrightarrow \Psi[\mathcal{G}]$. 3-geometry $\mathcal{G} \in$ SUPERSPACE:

Space of all 3-geometries (equivalence class of 3-metrics) $q'_{ab} \sim q_{ab}$ iff they are related by 3-dim. general coordinate transformation.

- Constraint Algebras (Classical):

(Definition: $H_a[N^a] \equiv \int_\Sigma N^a(\vec{x})H_a(\vec{x})d^3x \qquad \Sigma =$ Cauchy surface)

• Dirac Algebra (explicitly with (q_{ab}, π^{ab}) conjugate pair and Einstein's theory)

$$\begin{aligned}
\{H_a[N^a], H_b[M^b]\}_{P.B.} &= -H_a[(\mathcal{L}_{\vec{N}} M)^a] \\
\{H_a[N^a], H[M]\}_{P.B.} &= -H[(\mathcal{L}_{\vec{N}} M)] \\
\{H[N], H[M]\}_{P.B.} &= -H_a[(q^{ab}(N\partial_b M - M\partial_b N))]
\end{aligned}$$

- Quantum super-Hamiltonian Constraint: Wheeler-DeWitt Equation

$$"[G_{abcd}\frac{\delta}{\delta q_{ab}}\frac{\delta}{\delta q_{cd}} + \sqrt{q}(R(q) - 2\Lambda)]"\,\Psi[\mathcal{G}] = 0$$

Canonical Quantization of GR

Supermetric $G_{abcd} = \frac{8\pi G}{\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})$.

Symbolically,

$$[\frac{\delta^2}{\delta \mathcal{G}^2} + (R(q) - 2\Lambda)]\Psi[\mathcal{G}] = 0$$

- Technical issues:

  Ordering, Regularization, Anomalies, Explicit Solutions, of Wheeler-DeWitt Equation.

- Important conceptual issues: Where/what is physical "time" in Quantum Gravity?

  • Note: x^0 is not "time". Theory is reparametrization invariant. H does not generate "time" translation: $\exp(\frac{-ix^0 H}{\hbar})\Psi[\mathcal{G}] = \Psi[\mathcal{G}]$.

- B. S. DeWitt [Phys. Rev. **160**, 1113 (1967)]:

The triad formulation

- To use a triad (a set of 3 1-forms at each point in Σ)

$$q_{ab} = e_a^i e_b^j \delta_{ij}$$

  - Densitized triad: $E_i^a = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e_b^j e_c^k$
  - Additional 3 (Gauss) constraints: $G_i(E_j^a, K_a^j) = \epsilon_{ijk}E^{aj}K_a^k = 0$

- With new variables, the action of GR becomes

$$I[E_j^a, K_a^j, N_a, N, N^j] = \frac{1}{8\pi}\int dt \int_\Sigma d^3x[E_i^a \dot{K}_a^i - N^b H_b(E_j^a, K_a^j) - N H(E_j^a, K_a^j) - N^i G_i(E_j^a, K_a^j)]$$

The sympletic structure now becomes

$$\{E_j^a(x), K_b^i(y)\} = 8\pi\delta_b^a\delta_j^i\delta(x,y),$$
$$\{E_j^a(x), E_i^b(y)\} = \{K_a^j(x), K_b^i(y)\} = 0$$

The Ashtekar-Barbero connection variables

- There is a natural $so(3)$-connection (spin-connection Γ^i_a) that defines the notion of covariant derivative compatible with the dreibein

$$\partial_{[a}e^i_{b]} + \epsilon^i{}_{jk}\Gamma^j_{[a}e^k_{b]} = 0$$

  - Ashtekar-Barbero variable: $A^i_a = \Gamma^i_a + \gamma K^i_a$
  - γ : Immirzi parameter, $\gamma \in \mathrm{R} - \{0\}$.

- With the connection variables, the action becomes

$$I[E^a_j, A^j_a, N_a, N, N^j] = \frac{1}{8\pi}\int dt \int_\Sigma d^3x [E^a_i \dot{A}^i_a$$
$$-N^b H_b(E^a_j, A^j_a) - N H(E^a_j, A^j_a) - N^i G_i(E^a_j, A^j_a)]$$

  - $H_b(E^a_j, A^j_a) = E^a_j F^j_{ab} - (1+\gamma^2) K^i_b G_i = 0$
  - $H(E^a_j, A^j_a) = \frac{E^a_i E^b_j}{\sqrt{\det(E)}}(\epsilon^{ij}{}_k F^k_{ab} - 2(1+\gamma^2)K^i_{[a}K^j_{b]}) = 0$
  - $G_i(E^a_j, A^j_a) = D_a E^a_i = 0$

The Ashtekar-Barbero connection variables

- where $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^j_a A^k_b$ and $D_a E^a_i = \partial_a E^a_i + \epsilon_{ij}{}^k A^j_a E^a_k$

- The Poisson bracket of the new variables are

$$\{E^a_j(x), A^i_b(y)\} = 8\pi\gamma\delta^a_b\delta^i_j\delta(x,y),$$
$$\{E^a_j(x), E^b_i(y)\} = \{A^j_a(x), A^i_b(y)\} = 0$$

- Phase space variables: (A^i_a, E^b_j)

- Series of (Canonical) transformations:

  Metric variables: (q_{ab}, π^{ab})

  $\longrightarrow (e_{ai}, \pi^{ai})+$ 3 gauge constraints (Gauss' Law)

  $\longrightarrow (E^a_i, K^i_a)+$ Gauss' Law

  $\longrightarrow (E^a_i, A^{i\,-}_a = \Gamma^i_a - iK^i_a)+$ Gauss' Law (Ashtekar Variable)

  $\longrightarrow (E^a_i, A^i_a = \Gamma^i_a + \gamma K^i_a)+$ Gauss' Law (Ashtekar-Barbero Variable)

  (related discussion: C.H.C, R.H. Tung, H. L. Yu, PRD **72**, 064016 (2005))

Conceptual Breakthroughs

- Distinction between geometrodynamics and gauge dynamics is bridged. Identify E^a_j as the momentum conjugate to the gauge potential A^i_a;
  $\Rightarrow (E^a_j, A^i_a)$ phase space identical to Yang-Mills Theory.
- Quantum States can be wavefunctions in A-representation $\Psi[A]$, with $E^a_i = (\frac{8\pi G\hbar}{c^3})\frac{\delta}{\delta A^i_a}$. All manipulations done on gauge variables.

Technical Breakthroughs

- Constraints much simpler:
- Exact solution found (e.g. Chern-Simons state, in field theory variables)
- Loop variables: Wilson loops: holonomy elements.
  - Gauss's constraint solved by Ψ[Wilson loops in A] ;
  - $H_a = 0$ solved by Ψ[knot classes of Wilson loops in A].
- Super-Hamiltonian constraint still difficult, but can be made well-defined:
  - Volume V and area $\mathcal{A}$ operators : well-defined operators acting on loop and spin network states and have discrete spectra.
- Derivation of horizon entropy, both for black hole and cosmological horizons.
  - Black hole evaporation via transition from higher $\mathcal{A}$ states to lower $\mathcal{A}$ states.

Technical Breakthroughs

- Matching Bekenstein-Hawking entropy formula for large black holes

$$k_B \ln N = S_{BH} \approx k_B(\frac{\mathcal{A}}{4l_p^2}); \quad \mathcal{A} \gg l_p^2,$$

including quantum logarithmic correction when $\mathcal{A}$ is small,

$$S_{BH} = k_B(\frac{\mathcal{A}}{4l_p^2}) - \frac{1}{2}k_B \ln \frac{\mathcal{A}}{4l_p^2} + K_0.$$

(related discussion: C.H.C, Y. Ling, C. Soo, H. L. Yu, PLB **637**, 12 (2006))

- Resolution of big-bang singularity, curvature bounded and not divergent. (Bojowald)

- Addressing black hole singularity (Ashtekar and Bojowald): Minisuperspace (spherical symmetric) investigation.

(related discussion: C.H.C, C. Soo, H. L. Yu, PRD **76**, 084004 (2007))

The construction of LQG

- Holonomy:

$$U[A,\gamma](s) = \mathcal{P}\exp\int_\gamma A = \mathcal{P}\exp\int_\gamma ds\dot{\gamma}^a A_a^i(\gamma(s))\tau_i$$

- The key idea of LQG is to choose the loop states as the basis states for quantum gravity

$$\Psi_\alpha(A) = TrU[A,\gamma](s)$$

- The spin network state $\Psi_S(A)$: a cylindrical function f_S associated to spin network S whose graph is Γ

$$\Psi_S(A) = \Psi_{\Gamma,f_S}(A) = f_S(U[A,\gamma_1],...,U[A,\gamma_n])$$

Spin networks and quantum geometry

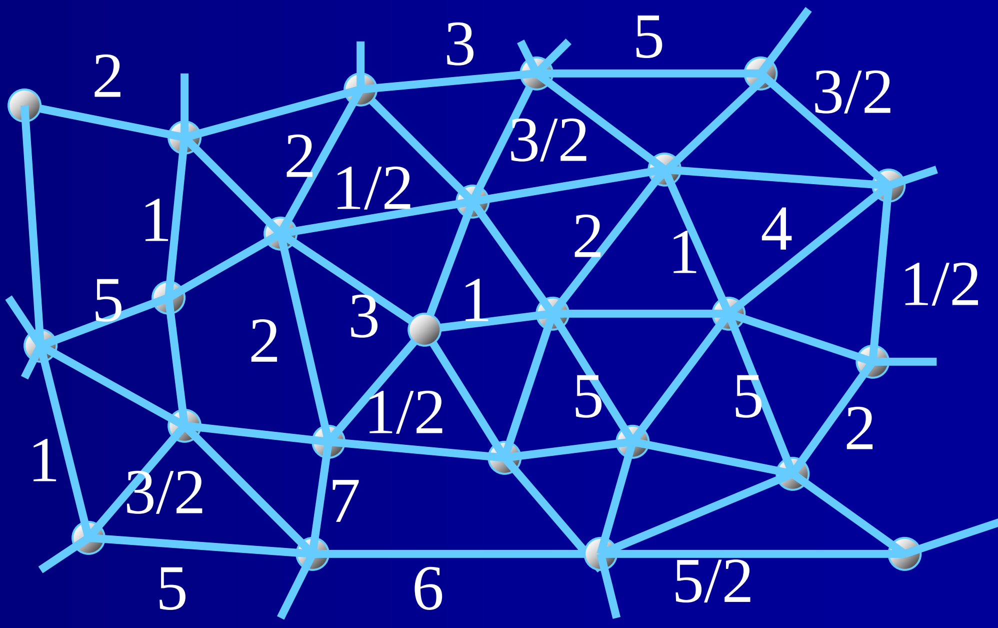
- Spin networks
- Quantum states of gravitational field：

Spin network states

$$\left| \Gamma ,\ j_m ,\ v_n \right\rangle$$

$$j = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$$

$$\left| n ,\ l ,\ m \right\rangle$$



Γ

Quantization of Area

- Rovelli and Smolin (1994); Ashtekar, Lewandowski et al (1995): given a surface

$$A(\mathcal{S}) = \int_{\mathcal{S}} \sqrt{n_a E_i^a n_b E_i^b} d^2\sigma$$

- The quantum area spectrum is

$$A(\mathcal{S})|S\rangle = 8\pi\gamma \sum_P \sqrt{j_P(j_P+1)}|S\rangle$$

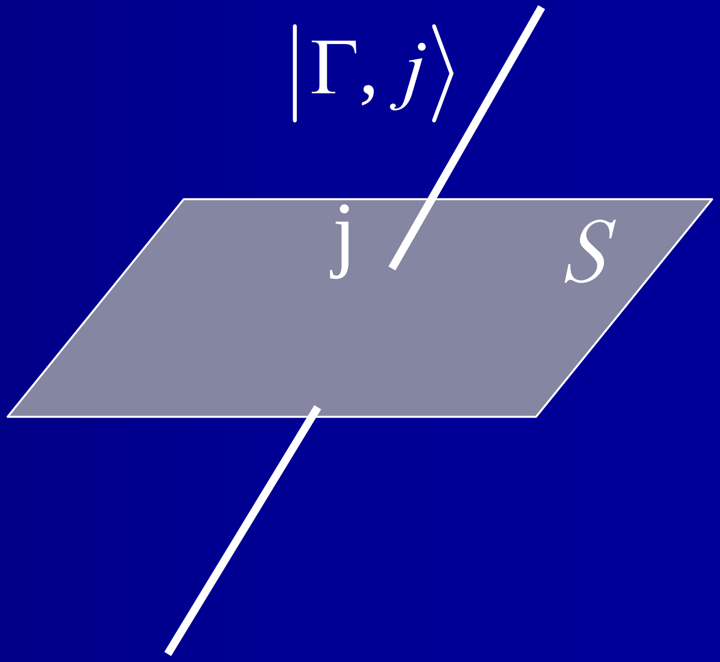
  - The result is topological and background independent.
  - The spectrum of the operator is discrete.
  - The spin of the lines of a spin network can be viewed as "quanta of area".

Discreteness of quantum geometry

- Area spectrum

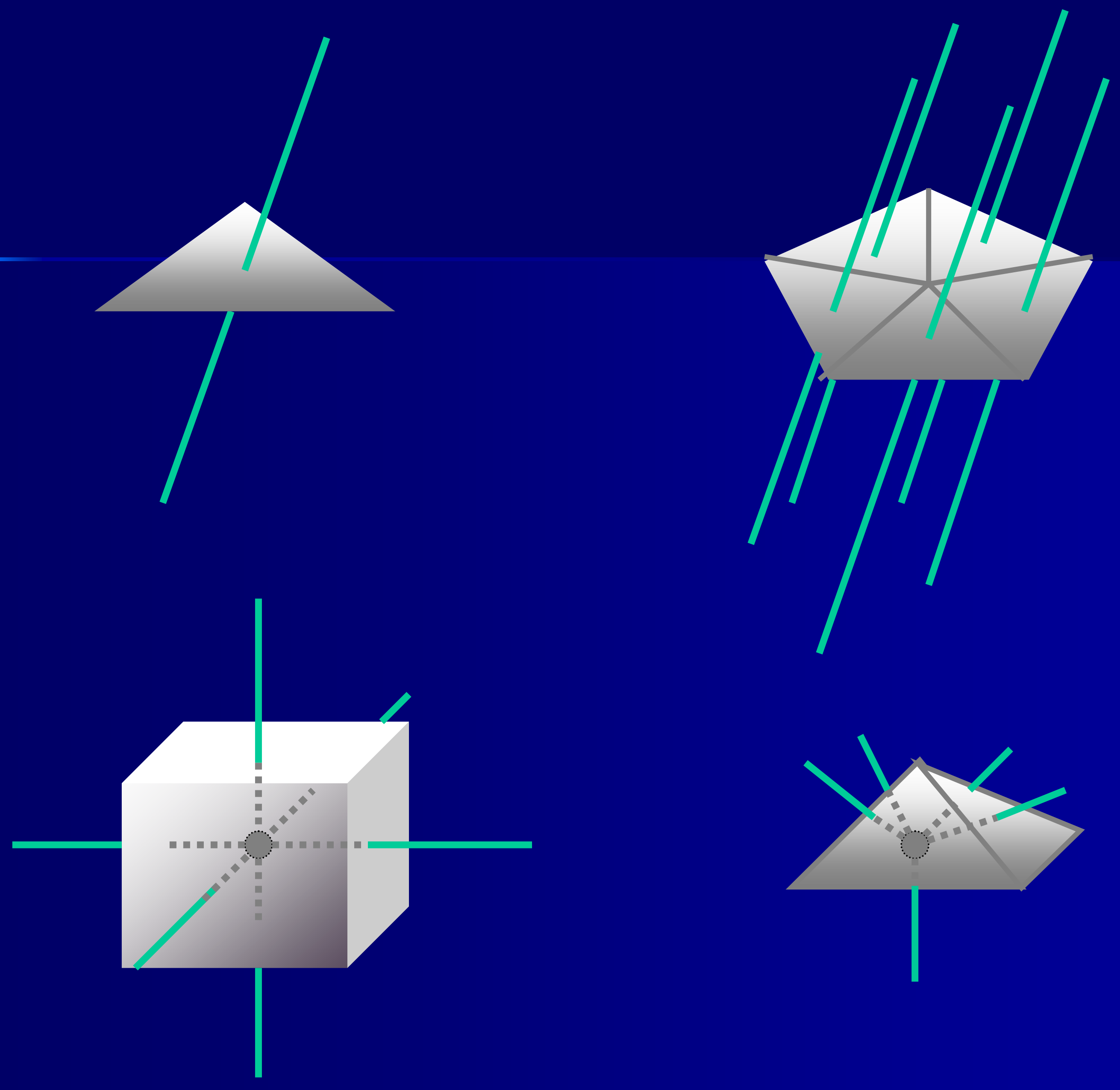
$$A(S) = \int_S d\sigma^2 \sqrt{\det({}^2h)}$$

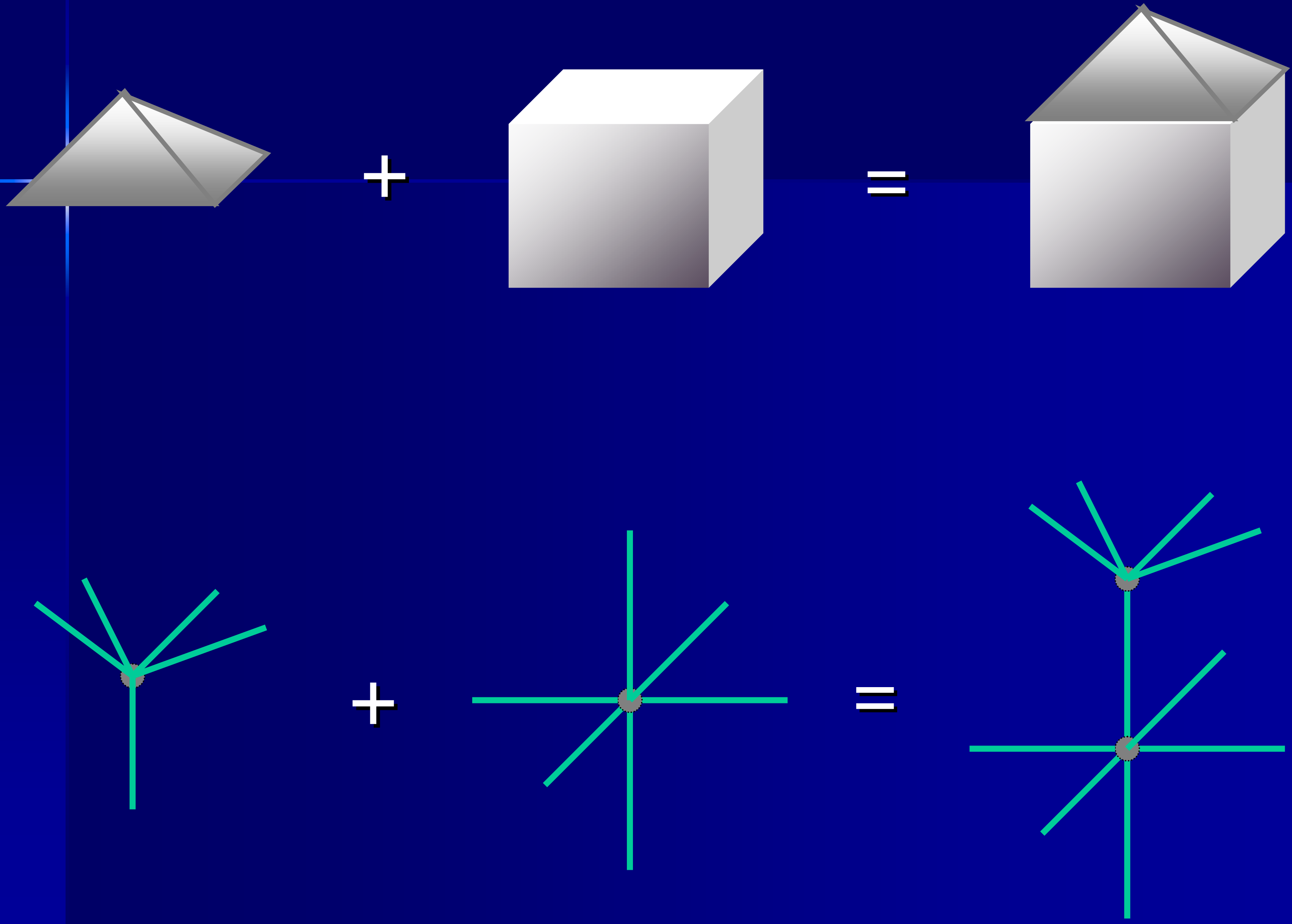
$$= \int_S d\sigma^2 \sqrt{\hat{E}_a^i \hat{E}^{aj} n_i(\sigma) n_j(\sigma)}$$



$$\hat{A}(S)\left|\Gamma, j\right\rangle = 8\pi\gamma l_p{}^2 \sqrt{j(j+1)}\left|\Gamma, j\right\rangle$$

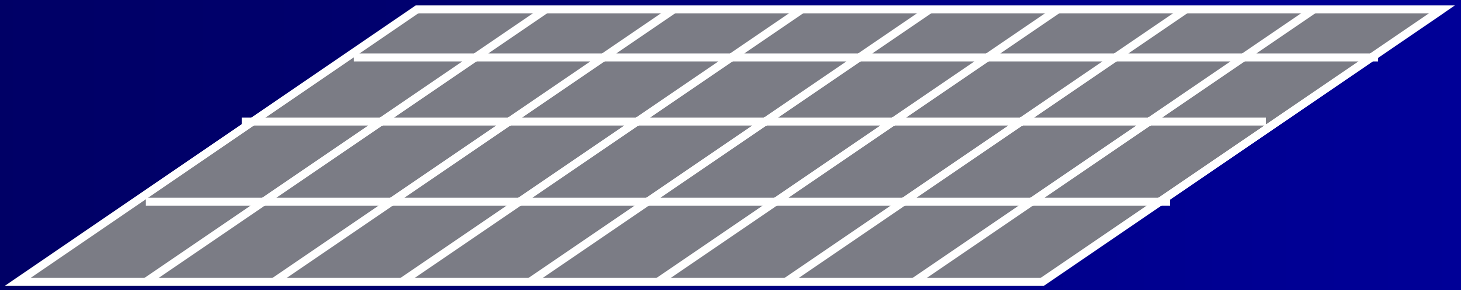
$$Dim(j) = 2j + 1$$





Discreteness of quantum geometry

- Microscopic version of space



$$\langle j | \hat{A} | j \rangle = 8\pi l_p^{\ 2} \sqrt{j(j+1)} \qquad j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

$$\langle l \rangle \geq l_p$$

Discrete horizons from quantum geometry

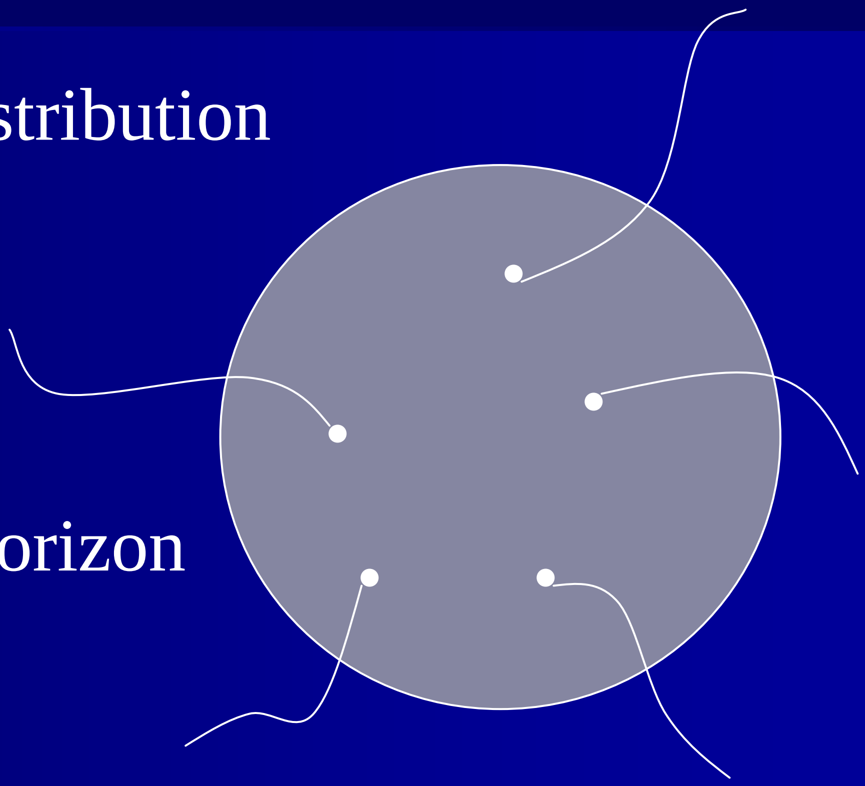
- The most probable distribution

$$\{ j_{\min}, j_{\min}, \ldots, j_{\min} \}$$

- The area of discrete horizon

$$A(j) =$$

$$N 8\pi\gamma l_p{}^2 \sqrt{j_{\min}(j_{\min}+1)}$$



Statistical entropy of black holes

- Bekenstein-Hawking entropy

$$S = \frac{A}{4G}$$

$$S = \ln(\#\ \ of\ \ \ microstates)$$

$$A = \langle \hat{A} \rangle$$

Quantum Black Holes

- In LQG the area of a surface in quantized:

$$A(j) = 8\pi\gamma\sqrt{j(j+1)}$$

where γ is the Immirzi parameter.

- The dimension of the boundary Hilbert space is $\prod_{i=1}^{N}(2j_i+1)$
- With the lowest possible spin j_{min} dominate assumption, the entropy $S = N\ln(2j_{min}+1)$,

$$N = \frac{A}{A(j_{min})} = \frac{A}{8\pi\gamma\sqrt{j_{min}(j_{min}+1)}}$$

$$S = \frac{A\ln(2j_{min}+1)}{8\pi\gamma\sqrt{j_{min}(j_{min}+1)}}$$

Quantum Black Holes

- If we require this to be consistent with Berkenstein-Hawking formula $S = \frac{A}{4}$, we get

$$\gamma = \frac{\ln(2j_{min}+1)}{2\pi\sqrt{j_{min}(j_{min}+1)}}$$

- if $j_{min} = \frac{1}{2}$, $\gamma = \frac{\ln 2}{\pi\sqrt{3}}$
- if $j_{min} = 1$, $\gamma = \frac{\ln 3}{2\pi\sqrt{2}}$

Correspondence Principle and Area spectrum of black hole

- Bohr's Correspondence Principle (1923): an oscillatory frequency of a classical system should be equal to a transition frequency of the corresponding quantum system

$$\Delta M = \hbar \omega_{QNM} = \frac{\hbar \ln 3}{8\pi M}$$

- Since $A = 16\pi M^2$, we have $\Delta A = 32\pi M \Delta M = 4\hbar \ln 3$,

$$4\hbar \ln 3 = \Delta A = A(j_{min}) = 8\pi\gamma\sqrt{j_{min}(j_{min}+1)}$$

$$\gamma = \frac{\ln 3}{2\pi\sqrt{j_{min}(j_{min}+1)}}$$

  - if $j_{min} = \frac{1}{2}$, $\gamma = \frac{\ln 3}{\pi\sqrt{3}}$
  - if $j_{min} = 1$, $\gamma = \frac{\ln 3}{2\pi\sqrt{2}}$

Correspondence Principle and Area spectrum of black hole

- Hence $j_{min} = 1$ is the only value which is consistent with
  - Area Spectrum
  - Black Hole entropy
  - Quasi-normal mode of black hole
- This implies the gauge group of LQG is $SO(3)$!

Estimate of the Immirzi parameter

- Domagala and Lewandowski (2004): Taking the higher spin into account, the configurations should be governed by sequences labelled by

$$\sum_i \sqrt{|m_i|(|m_i|+1)} \leq a \equiv \frac{A}{8\pi\gamma}\ , \quad \sum_i m_i = 0;$$

with $m_i \in -j_i, -j_i+1, ..., j_i$ and $j_i \in N/2$.

- Meissner (2004): the number of states for a given area is given by

$$N(a) = \frac{C_M}{\sqrt{4\pi\beta_M a}} e^{2\pi\gamma_M a}$$

Estimate of the Immirzi parameter

- The black hole entropy is consequently

$$S = \ln N(a) = \left(\frac{\gamma_M}{\gamma}\right)\frac{A}{4} - \frac{1}{2}\ln A + \ln \frac{C_M}{\sqrt{2\beta_M \gamma}}$$

- By matching this to the Bekenstein-Hawking entropy formula for large black holes, $\gamma_M = \gamma$.
- The coefficient of logarithmic correction to the Bekenstein-Hawking entropy formula is $-\frac{1}{2}$.

Values of the Immirzi parameter in *SU(2)*

- Meissner (2004): $\gamma_M = 0.2375\ldots$

$$1 = \sum_{j \in \frac{N}{2}} 2 \exp(-2\pi\gamma_M \sqrt{j(j+1)})$$

- Ghosh and Mitra (2005): $\gamma_M = 0.2739\ldots$

$$1 = \sum_{j \in \frac{N}{2}} (2j+1) \exp(-2\pi\gamma_M \sqrt{j(j+1)})$$

- Tamaki and Nomura (2005): $\gamma_M = 0.2619\ldots$

$$1 = \sum_{j \in \frac{N}{2}} 2\left[\frac{2j+1}{2}\right] \exp(-2\pi\gamma_M \sqrt{j(j+1)})$$

Effective gauge group of LQG

- The dimension of the representation space is dependent upon the global structure of the gauge group. One needs to examine the full physical contents of the theory to determine the actual gauge group.
- A Lie algebra valued connection 1-form always transforms according to the adjoint representation of the gauge group:

$$A' = gAg^{-1} + igdg^{-1} \ ; \quad g \in G.$$

- Since the center of the group, C, commutes with all elements of the Lie algebra, as far as gauge potentials are concerned, the effective gauge group is not G but G/C.
- The gauge group of $SU(N)$ pure Yang-Mills theory in NOT $SU(N)$ but $SU(N)/Z_N$.

Effective gauge group of LQG

- Same conclusion for the quantum theory with loop variables.

$$g(\vec{y})\mathcal{P}\exp(i\int_{\vec{x}\ p}^{\vec{y}} A\,)g^{-1}(\vec{x})$$

where both $g(\vec{y})$ and $g^{-1}(\vec{x})$ are *effectively* elements of G/C i.e. $SO(3)$ (or $SO(3,C)$).

- The configuration space is the space of $SO(3)$ gauge connections modulo the action of $SO(3)$ gauge group, and it is faithfully parametrized by holonomy elements of $SO(3)$ connections with integer spin representations, rather than $SU(2)$ holonomies which include half-integer spin representations.

New estimate of the Immirzi parameter

- Chou, Lin, Soo, Yu: *G/C = SO(3)*

- The expression for the number of states

$$N(a) = \frac{C_M}{\sqrt{4\pi\beta_M a}} e^{2\pi\gamma_M a}$$

remains the same.

- The new Immirzi parameter is $\gamma_M = 0.170\ldots$

$$1 = \sum_{j\in N} 2[\frac{2j+1}{2}] \exp(-2\pi\gamma_M\sqrt{j(j+1)})$$

- The coefficient of logarithmic correction to the Bekenstein-Hawking entropy formula is $-\frac{1}{2}$.

Quantum black hole in SUGRA

- In loop quantization of $N = 1$ supergravity

$$A_{SUGRA}(J) = 8\pi\tilde{\gamma}\sqrt{J(J+\frac{1}{2})}\,, \quad J \in \frac{N}{2}$$

and the degeneracy of state $D(J) = 4J + 1$.

- One finds that

$$\begin{aligned} A_{SUGRA}(\tilde{j}) &= 8\pi\tilde{\gamma}\sqrt{\tilde{j}(\tilde{j}+\frac{1}{2})}\,, \quad \tilde{j} \in \frac{N}{2} \\ &= 8\pi(\frac{\tilde{\gamma}}{2})\sqrt{j(j+1)}\,, \quad j \in N \end{aligned}$$

- The Immirzi parameter now obeys the modified equation

$$1 = \sum_{\tilde{j}\in\frac{N}{2}} 2[\frac{4\tilde{j}+1}{2}] \exp(-2\pi\tilde{\gamma}\sqrt{\tilde{j}(\tilde{j}+\frac{1}{2})})$$

Quantum black hole in SUGRA

$$= \sum_{j \in N} 2[\frac{2j+1}{2}] \exp(-2\pi(\frac{\tilde{\gamma}}{2})\sqrt{j(j+1)})$$

- The result for the SUGRA case will be the same as for the case of $SO(3)$ but with the Immirzi parameter $\tilde{\gamma} = 2\gamma_{SO(3)}$.
- Note that this relation between the Immirzi parameters produces exactly the same area spectrum for *both* pure LQG without supersymmetry and its supersymmetric extension.

Summary

- The effective gauge group for *pure* four-dimensional LQG is $SO(3)$ (or $SO(3,C)$) instead of $SU(2)$ (or SL(2,C)).
- Our observations imply a new value of $\gamma \approx 0.170$ for the Immirzi parameter.
- The results of both pure LQG and the SUSY extension of LQG can be made compatible and $\tilde{\gamma} = 2\gamma_{SO(3)}$.
- The $-\frac{1}{2}$ coefficient of logarithmic correction to the Bekenstein-Hawking entropy formula is robust.

C.H. Chou, Y. Ling, C. Soo and H.L. Yu, *Phys. Lett. B637,12*