Nonlocal Condensate Model for QCD Sum Rules

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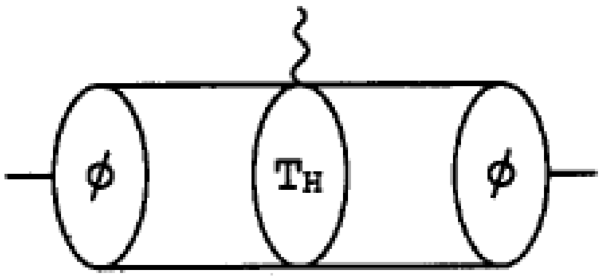
Outline

- Concepts
- Local and non-local condensates
- Summary

Pion form factor



$$\langle \pi(P_2)|J_\mu^{em}(0)|\pi(P_1)\rangle = F_\pi(Q^2)(P_1+P_2)_\mu$$
$$= \frac{1}{(2\pi)^4}\int \mathrm{d}^4z\mathrm{d}^4q\,\exp(iq\cdot z)\langle \pi(P_2)|J_\mu^{em}(z)|\pi(P_1)\rangle$$

The pion form factor can be written as the convolution of a hard-scattering amplitude T_H and wave function $\phi(x)$

$$F_\pi(Q^2) = \int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,\phi(x_2,\mu^2)T_\mathrm{H}(x_1,x_2,Q^2/\mu^2,\alpha_s(\mu^2))\phi(x_1,\mu^2)$$

Concepts

- Basic idea : Describing the nonperturbative contribution by a set of phenomenologically effective Feynman rules ------- “quark-hadron duality”.
- How to do it ?
  - Dispersion relation : a phenomenological procedure which connect perturbative and non-perturbative corrections with the *lowest-lying resonances* in the corresponding channels by using of the *Borel* improved dispersion relations
  - Borel transformation : a) An improved expansion series b) Give a selection rule of s_0

Dispersion relation

Firstly, consider a polarization operator $\Pi_{\mu\nu}(Q^2=-q^2)$ which was defined as the vacuum average of the current product:

$$\begin{aligned}\Pi_{\mu\nu}(Q^2) &= i\int dx e^{iqx}\langle\Omega|\mathrm{T}\left[j_\mu(x)j_\nu(0)\right]|\Omega\rangle \\ &\equiv (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(Q^2),\end{aligned}$$

Where the state $|\Omega\rangle$ is the exact vacuum which contain non-perturbative information.

Now, we can insert a complete set of states $\sum_\Gamma |\Gamma\rangle\langle\Gamma|$ and the identity

$$\frac{1}{(2\pi)^3}\int d^4p\,\theta(p^0)\,\delta(p^2-m_\Gamma^2)\equiv 1,$$

Between two currents.

$$\begin{aligned}\Pi_{\mu\nu}(Q^2) &= \frac{i}{(2\pi)^3}\sum_\Gamma\int dx\int d^4p\,\theta(p^0)\,\delta(p^2-m_\Gamma^2)e^{iqx}\langle\Omega|j_\mu(x)|\Gamma\rangle\langle\Gamma|j_\nu(0)|\Omega\rangle \\ &\equiv \rho_{\mu\nu}(q^2)\theta(s_0-q^2)+\Pi_{\mu\nu}(q^2)\theta(q^2-s_0),\end{aligned}$$

Here assuming that there exists a threshold value s_0 which can separate the matrix element to lowest resonance state and other higher states.

Because the general structure of $\Pi(q^2)$ can be inferred from OPE and can be given by

$$\Pi(q^2) = \Pi^{pert.}(q^2) + \left[a\frac{\langle GG\rangle}{(q^2)^3} + b\frac{\alpha_s\langle\bar{\psi}\psi\rangle^2}{(q^2)^4} + \dots\right]$$

And via the dispersion relation, the function $\Pi(q^2)$ can be written as

$$\Pi(q^2) = \int_0^\infty ds\,\frac{\rho(s)}{s - q^2 - i\epsilon},$$

with the spectral function $\rho(s)$ is

$$\rho(s) \equiv \frac{1}{\pi}\mathrm{Im}\Pi(\mathrm{s}).$$

Thus, we can obtain a duality relation between the hadronic resonance and quark contributions

$$\rho^{res}(q^2) = \theta(s_0 - q^2)\left[\Pi^{pert.}(q^2) + a\frac{\langle GG\rangle}{(q^2)^3} + b\frac{\alpha_s\langle\bar{\psi}\psi\rangle^2}{(q^2)^4} + \ldots\right]$$

Such that

$$\begin{aligned}\int_0^{s_0} ds\,\frac{\rho^{res}(s)}{s-q^2-i\epsilon} &= \frac{1}{\pi}\int_0^{s_0} ds\,\frac{1}{s-q^2-i\epsilon} \\ &\times \mathrm{Im}\left[\Pi^{\mathrm{pert.}}(\mathrm{s}) + \mathrm{a}\frac{\langle\mathrm{GG}\rangle}{\mathrm{s}^3} + \mathrm{b}\frac{\alpha_\mathrm{s}\langle\bar{\psi}\psi\rangle^2}{\mathrm{s}^4} + \ldots\right].\end{aligned}$$

The Borel transformation

$$\hat{L}_M \Pi(Q^2) = \lim_{\substack{Q^2, n\to\infty \\ Q^2/n = M^2}} \frac{1}{(n-1)!}(Q^2)^n \left[-\frac{d}{dQ^2}\right]^n \Pi(Q^2).$$

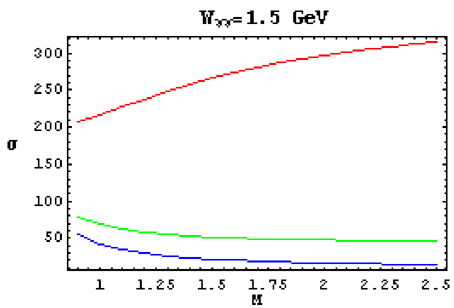
The meaning of the operator $\hat{L}_M$ becomes clear if we act on a particular term in the power expansion:

$$\hat{L}_M \left(\frac{1}{Q^2}\right)^k = \frac{1}{(k-1)!}\left(\frac{1}{M^2}\right)^k.$$

By apply the Borel transformation $\hat{L}_M$ we have

$$\begin{aligned} \hat{L}_M \int_0^{s_0} ds \frac{\rho^{res}(s)}{s - q^2 - i\epsilon} &= \frac{1}{\pi M^2} \sum_{m_\Gamma^2 < s_0} e^{-m_\Gamma^2/M^2} \langle 0|J_\mu(0)|\Gamma\rangle\langle\Gamma|J_\nu(0)^\dagger|0\rangle \\ &= \int_0^{s_0} ds \frac{e^{-s/M^2}}{\pi M^2} \mathrm{Im}\left[\Pi^{pert.}(s) + a\frac{\langle GG\rangle}{s^3} + b\frac{\alpha_s\langle\bar{\psi}\psi\rangle^2}{s^4} + \ldots\right]. \end{aligned}$$

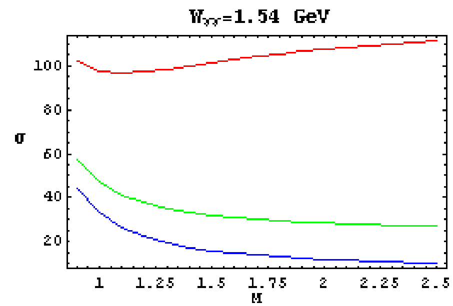
The choice of s_0 in two-photon process



(a) $s_0 = 0.555$

(b) $s_0 = 0.545$

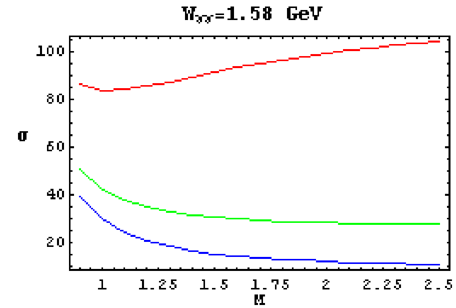
(c) $s_0 = 0.535$



(a) $s_0 = 0.58$

(b) $s_0 = 0.57$

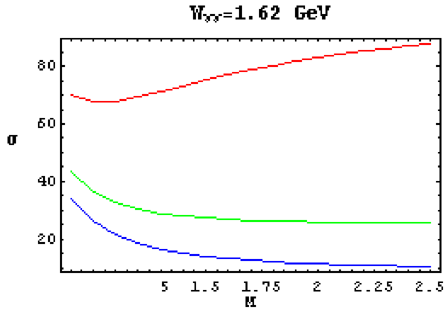
(c) $s_0 = 0.56$



(a) $s_0 = 0.61$

(b) $s_0 = 0.60$

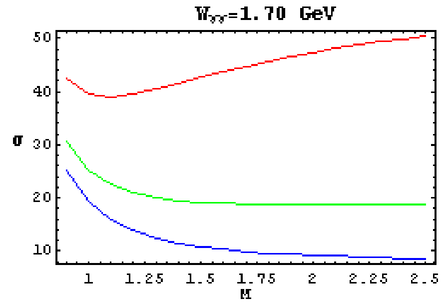
(c) $s_0 = 0.59$



(a) $s_0 = 0.64$

(b) $s_0 = 0.63$

(c) $s_0 = 0.62$



(a) $s_0 = 0.70$

(b) $s_0 = 0.69$

(c) $s_0 = 0.68$

Pion form factor in QSR

$$\langle\pi(P_2)|J_\mu^{em}(0)|\pi(P_1)\rangle = F_\pi(Q^2)(P_1+P_2)_\mu$$
$$= \frac{1}{(2\pi)^4}\int \mathrm{d}^4z\mathrm{d}^4q\,\exp(iq\cdot z)\langle\pi(P_2)|J_\mu^{em}(z)|\pi(P_1)\rangle$$

Consider the three-point function $\Gamma_{\sigma\mu\lambda}$:

$$\Gamma_{\sigma\mu\lambda}(p_1^2,p_2^2,q^2) = i\int \mathrm{d}^4y\mathrm{d}^4z\,\exp(ip_2\cdot y - iq\cdot z) \times\langle 0|T\left(\eta_\sigma(y)J_\mu(z)\eta_\lambda^\dagger(0)\right)|0\rangle\ ,$$

where

$$J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d, \qquad \eta_\sigma = \bar{u}\gamma_\sigma\gamma_5 d$$

are the electromagnetic and axial currents, respectively, of *up* and *down* quarks.

Again, insert the complete set of states and identity:

$$\begin{aligned}\Gamma_{\sigma\mu\lambda}(p_1^2,p_2^2,q^2) &= i\sum_{\Gamma\Gamma'}\int \mathrm{d}^4y\mathrm{d}^4z\mathrm{d}^4p_i\mathrm{d}^4p_f\cdot\delta(p_i^2-m_\Gamma^2)\delta(p_f^2-m_{\Gamma'}^2) \\ &\quad\times e^{(ip_f\cdot y-iq\cdot z)}\langle 0|\eta_\sigma(y)|\Gamma'\rangle\langle\Gamma'|J_\mu(z)|\Gamma\rangle\langle\Gamma|\eta_\lambda^\dagger(0)|0\rangle \\ &= f_\pi^2 p_{1\lambda}p_{2\sigma}(2\pi)^2\delta(p_1^2-m_\pi^2)\delta(p_2^2-m_\pi^2)(p_1+p_2)_\mu F_\pi(q^2) \\ &\quad+\Gamma_{\sigma\mu\lambda}\left[1-\theta(s_0-p_1^2)\theta(s_0-p_2^2)\right],\end{aligned}$$

With the matrix element $\langle 0|\eta_\sigma(y)|\pi(p)\rangle$ is given by PCAC:

$$\langle 0|\eta_\sigma(y)|\pi(p)\rangle = if_\pi p_\sigma e^{-ip\cdot y}.$$

$$f_\pi^2 e^{-2m_\pi^2/M^2} p_{1\lambda} p_{2\sigma}(p_1+p_2)_\mu F_\pi(q^2)$$

$$= \frac{1}{\pi}\int_0^{s_0}\int_0^{s_0} \mathrm{ds_1 ds_2 e}^{-(\mathrm{s_1+s_2})/\mathrm{M}^2} \times \mathrm{Im}\Big[\Gamma^{\mathrm{pert.}}_{\sigma\mu\lambda}(\mathrm{s_1,s_2}) + \mathrm{A}_{\sigma\mu\lambda}(\mathrm{s_1,s_2})\langle\mathrm{GG}\rangle$$

$$+ B_{\sigma\mu\lambda}(s_1,s_2)\alpha_s\langle\bar{\psi}\psi\rangle^2 + \ldots\Big].$$

Local and non-local condensate

An exact propagator : $\langle\Omega|\mathrm{T}(q(x)\overline{q}(y))|\Omega\rangle = \langle\Omega|\overbrace{q(x)\overline{q}(y)}|\Omega\rangle + \langle\Omega| : q(x)\overline{q}(y) : |\Omega\rangle.$

The Wick theorem : $\mathrm{T}(q(x)\overline{q}(y)) = \overbrace{q(x)\overline{q}(y)} + : q(x)\overline{q}(y) : .$

The normal ordering :

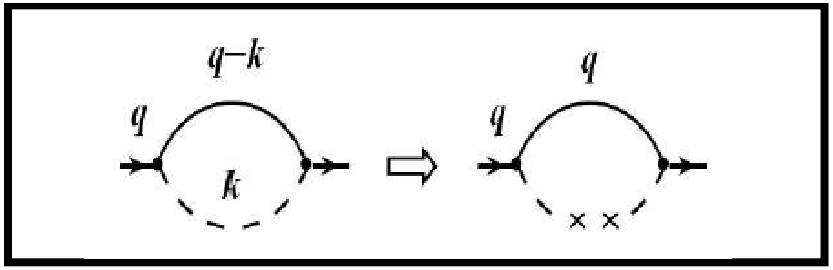
$$\begin{cases} \langle 0| : q(x)\overline{q}(y) : |0\rangle = 0. \\ \langle\Omega| : q(x)\overline{q}(y) : |\Omega\rangle \neq 0. \end{cases}$$

Operator product expansion

In the QSR approach it is assumed that the confinement effects are sufficiently soft for the Taylor expansion:

$$\langle \overline{q}(0)q(z)\rangle = \sum_{n=0}^{\infty} \frac{z^{\mu_1}\dots z^{\mu_2}}{n!}\langle \overline{q}(0)\partial_{\mu_1}\dots\partial_{\mu_n} q(0)\rangle$$

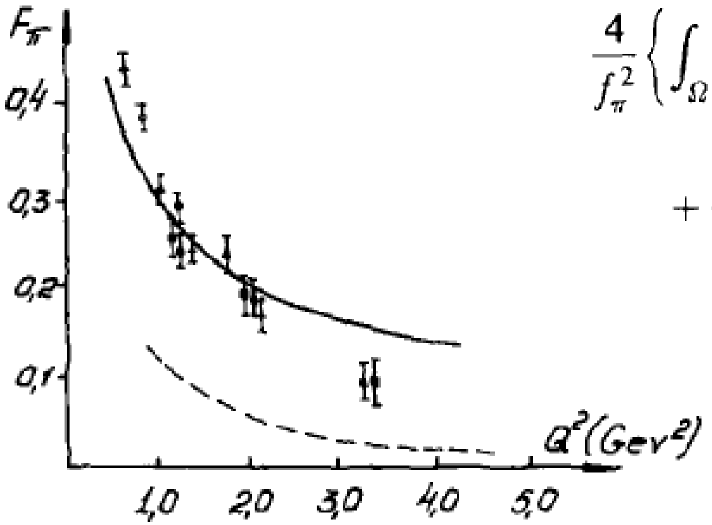
$$= \langle \overline{q}q\rangle + z^{\mu}\langle \overline{q}\partial_{\mu} q\rangle + \frac{z^{\mu_1}z^{\mu_2}}{2}\langle \overline{q}\partial_{\mu_1}\partial_{\mu_2} q\rangle + \dots$$



$$I(q^2) = \int d^4k \cdot \frac{D(k^2)}{(q-k)^2} \Rightarrow \frac{1}{q^2}\cdot\langle \overline{q}q\rangle.$$

Local condensate result of pion form factor

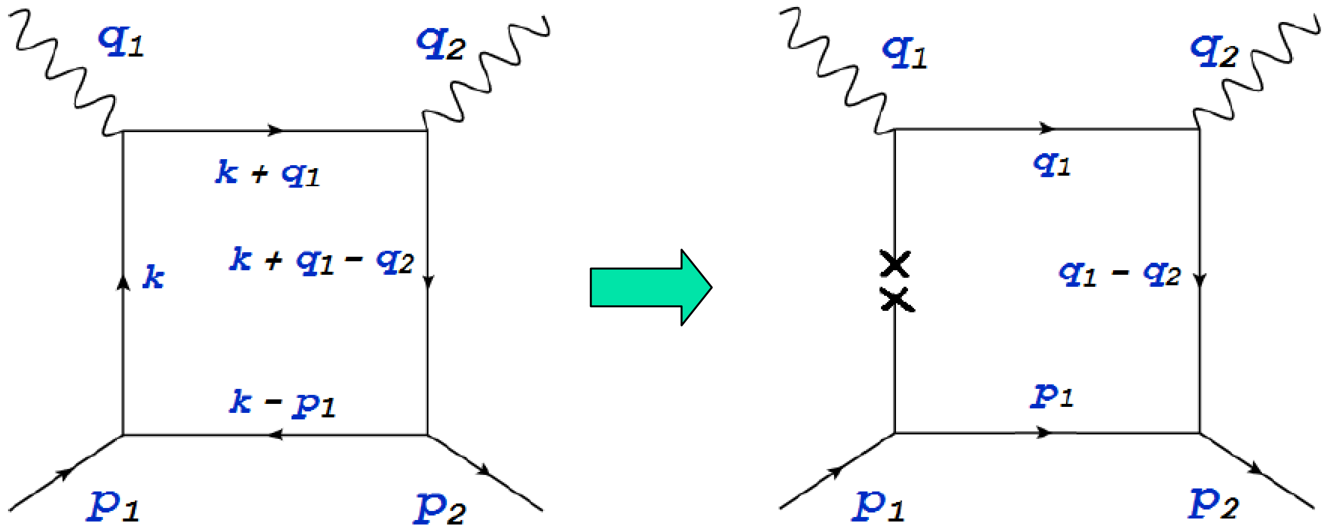
B.L. Ioffe and A.V. Smilga, NPB216(1983)373-407



$$\frac{4}{f_\pi^2}\left\{\int_\Omega \mathrm{d}s\,\mathrm{d}s'\,\mathrm{e}^{-(s+s')/M^2}\rho^{(0)}(s,s',Q^2)+\frac{\alpha_s}{48\pi M^2}\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle\right.$$

$$\left.+\frac{52\pi}{81M^4}\alpha_s\langle 0|\bar{\psi}\psi|0\rangle^2\left(1+\frac{2Q^2}{13M^2}\right)\right\}=F_\pi(Q^2),$$

The infrared divergence problem



$$\langle \bar{q} q \rangle \cdot \frac{1}{q_1^2} \frac{1}{p_1^2} \frac{1}{(q_1 - q_2)^2}$$

Non-local condensate models

In 1986, S. V. Mikhailov and A. V. Radyushkin proposed:

$$\begin{aligned}\langle \overline{q}(0)q(z)\rangle &= \langle \overline{q}q\rangle \int_0^\infty e^{sz^2/4} f_s(s)ds, \\ \langle \overline{q}(0)\gamma_\mu q(z)\rangle &= -iz_\mu \langle \overline{q}q\rangle \int_0^\infty e^{sz^2/4} f_v(s)ds\end{aligned}$$

with

$$\begin{aligned}f_s(s) &= \delta(s) - (\lambda^2/2)\delta'(s) + \cdots \\ f_v(s) &= A[\delta'(s) - \frac{57}{80}\lambda^2\delta''(s)] + \cdots\end{aligned}$$

Other models

$$\begin{aligned}f_s(s) &= N_1 \cdot \exp(-\Lambda^2/s - s^2\sigma_1^2) \quad \text{—Gaussian decay} \\ f_v(s) &= N_2 \cdot \exp(-\Lambda^2/s - s\sigma_2) \quad \text{—exponential decay} \\ &\vdots\end{aligned}$$

$$\langle \overline{q(0)}q(z)\rangle \sim \langle \overline{q}q\rangle \exp(-\lambda_q^2|z^2|/8)$$

A.P. Bakulev, S.V. Mikhailov and N.G. Stefanis, hep-ph/0103119

A.P. Bakulev, A.V. Pimikov and N.G. Stefanis, 0904.2304

Compare with the simplest gauge invariant non-local condensate :

$$\langle \overline{q(0)} q(z) \rangle = \langle \overline{q} q \rangle + \frac{z^2}{8} \langle \overline{q} q \rangle (\lambda_q^2 - m_q^2) + \cdots$$

$$\langle \overline{q(0)} \gamma^\mu q(z) \rangle = -i \frac{m_q z^\mu}{4} \langle \overline{q} q \rangle + \cdots$$

Must obey following constrain condition

$$\int_0^\infty \mathrm{ds} \cdot \mathrm{f_s(s)} = 1$$

$$\int_0^\infty \mathrm{ds} \cdot \mathrm{s} \cdot \mathrm{f_s(s)} = \frac{(\lambda_q^2 - m_q^2)}{2}$$

$$\int_0^\infty \mathrm{ds} \cdot \mathrm{f_v(s)} = m_q$$

$$\langle \overline{q} i g \hat{G}^{\mu\nu} \sigma_{\mu\nu} q \rangle \equiv m_0^2 \langle \overline{q} q \rangle$$

$$\lambda_q^2 = \frac{m_0^2}{2} - m_q^2$$

Local condensate:

$$\varphi_\pi(x) + \varphi_{\pi'}(x) e^{-m_{\pi'}^2/M^2} + \ldots = \frac{\delta(x) + \delta(1-x)}{2} + a\langle \bar{q} D^2 q \rangle \left\{ \delta'(x) + \delta'(1-x) \right\} + \ldots \, .$$

Nonlocal condensate:

$$\begin{aligned}\varphi_\pi(x) \quad &+ \quad \varphi_{\pi'}(x) e^{-m_{\pi'}^2/M^2} + \varphi_{\pi''}(x) e^{-m_{\pi''}^2/M^2} + \ldots \equiv \Phi(x, M^2) \\ &= \quad \frac{M^2}{2} \left(1 - x + \frac{\lambda_q^2}{2M^2} \right) f(xM^2), + (x \to 1-x)\end{aligned}$$

The Källén-Lehmann representation

The exact fermion's propagator :

$$\langle\Omega|\mathrm{T}\psi(\mathrm{x})\overline{\psi}(\mathrm{y})|\Omega\rangle = \mathrm{i}\int\frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}\mathrm{e}^{-\mathrm{ik}\cdot(\mathrm{x}-\mathrm{y})}\int_0^\infty \mathrm{d}\mu^2\frac{\not{k}\rho_1(\mu^2)+\rho_2(\mu^2)}{\mathrm{k}^2-\mu^2+\mathrm{i}\epsilon}$$

$$= iZ_2S^r(x-y;m_\gamma) + i\int\frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}e^{-ik\cdot(x-y)}\int_{m_\gamma}^\infty \mathrm{d}\mu^2\frac{\not{k}\rho_1(\mu^2)+\rho_2(\mu^2)}{\mathrm{k}^2-\mu^2+\mathrm{i}\epsilon}$$

Non-perturbative part (normal ordering)

Renormalized perturbative part

Recast the equation into:

$$\langle \Omega | \mathrm{T}(q(z)\overline{q}(0)) | \Omega \rangle = \frac{1}{16\pi^2} \int_0^\infty ds \exp\left(\frac{z^2}{4}s\right) \int_0^\infty d\mu^2 \exp\left(-\frac{\mu^2}{s}\right) \left[\frac{i\not{z}}{2} s \rho_1^q(\mu^2) + \rho_2^q(\mu^2)\right].$$

And set the nonperturbative piece as:

$$\langle \Omega | : q(z)\overline{q}(0) : | \Omega \rangle \;=\; \frac{1}{16\pi^2} \int_0^\infty ds \exp\left(\frac{z^2}{4}s\right) \int_{[s,m_\gamma^2]}^\infty d\mu^2 \exp\left(-\frac{\mu^2}{s}\right) \left[\frac{i\not{z}}{2} s \rho_1^q(\mu^2) + \rho_2^q(\mu^2)\right]$$

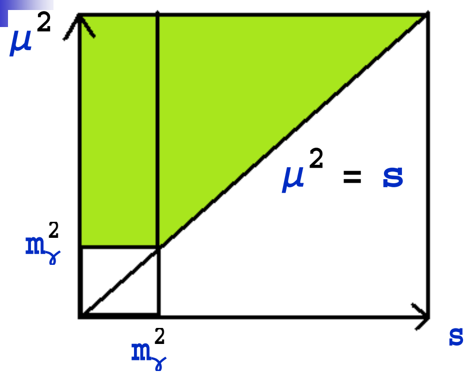
Here $[s, m_\gamma^2]$ means that for s larger than m_γ^2 then the lower bound is s otherwise is m_γ^2

The quark condensate contribution can be obtained by the normal ordering

$$\begin{aligned}\langle \overline{q}(0)q(z)\rangle &\equiv -\mathrm{Tr}\left[\langle\Omega| : q(z)\overline{q}(0) : |\Omega\rangle\right] \\ &= \langle\overline{q}q\rangle\left[1+\frac{z^2}{4}\left(\frac{\lambda_q^2}{2}-\frac{m_q^2}{2}\right)+\cdots\right], \\ \langle \overline{q}(0)\gamma_\mu q(z)\rangle &\equiv -\mathrm{Tr}\left[\gamma_\mu\langle\Omega| : q(z)\overline{q}(0) : |\Omega\rangle\right] \\ &= -i\frac{z_\mu}{4}\langle\overline{q}q\rangle\left(m_q+\cdots\right),\end{aligned}$$

The weight functions are parameterized as

$$\rho_1^q(\mu^2) = N_1\exp(-a\mu^2), \quad \rho_2^q(\mu^2) = N_2\mu\exp(-a\mu^2),$$



$$\int_{[s,m_\gamma^2]}^{\infty} d\mu^2 = \int_0^{m_\gamma^2} ds \int_{m_\gamma^2}^{\infty} d\mu^2 + \int_{m_\gamma^2}^{\infty} ds \int_s^{\infty} d\mu^2$$

small s region

$$f_s(s) \propto \frac{e^{-\frac{m\gamma^2 (1+a\,s)}{s}}\, s}{(1 + a\, s)}$$

$$f_v(s) \propto \frac{s * \mathrm{Erfc}\left[m\gamma \sqrt{a + \frac{1}{s}}\right]}{\sqrt{a + \frac{1}{s}}}$$

large s region

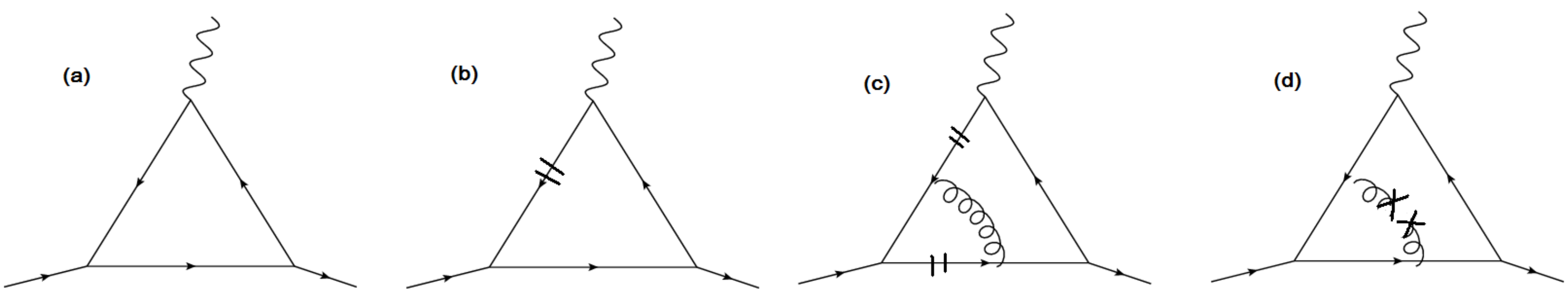
$$f_s(s) \propto \frac{e^{-1-a\,s}\, s}{(1 + a\, s)}$$

$$f_v(s) \propto \frac{s * \mathrm{Erfc}\left[\sqrt{1 + a\, s}\right]}{\sqrt{a + \frac{1}{s}}}$$

The dressed propagator for the quark is then given by

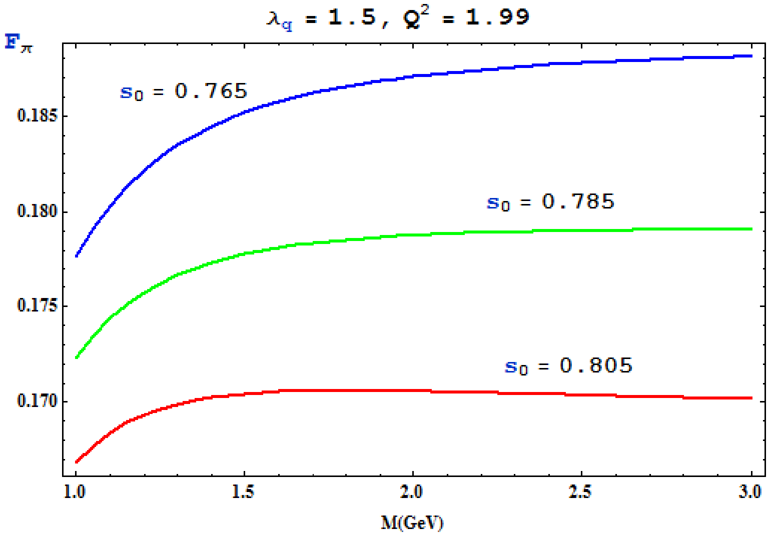
$$S^q(p) = \frac{\not p + m_q}{p^2 - m_q^2} + \frac{1}{2} i \frac{(\gamma^\alpha \not p \gamma^\beta G_{\alpha\beta} - m_q \gamma_\alpha G^{\alpha\beta} \gamma_\beta)}{(p^2 - m_q^2)^2}$$
$$+ \frac{\pi^2 \langle G^2 \rangle m_q \not p (m_q + \not p)}{(p^2 - m_q^2)^4} + \left[\not p \hat{I}_1^q(\mu) + \hat{I}_2^q(\mu) \right] \frac{\exp[-(p^2 - \mu^2)/\mu^2]}{p^2 - \mu^2}$$

With the definitions $\hat{I}^q_{1,2}(\mu) f(\mu) \equiv \int_0^\infty d\mu^2 \rho^q_{1,2}(\mu^2) f(\mu).$



Expected results

(a)



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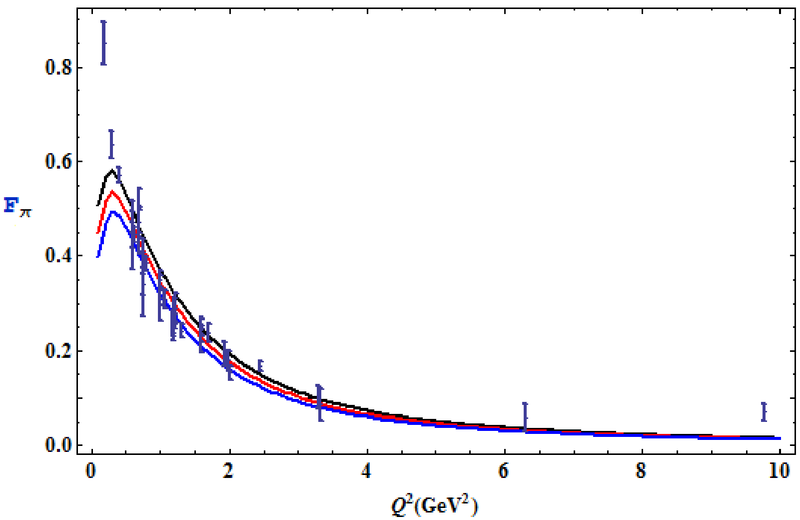

Summary

- The infrared divergence problem can be solved by our nonlocal condensate model.
- The applicable energy region can be extended to 10 GeV^2 in the calculation of pion form factor.
- Can we use the Källén-Lehmann representation to improve the QSR approach?