Warped AdS/CFT Correspondence

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AdS/CFT Correspondence

$\mathcal{N} = 4$ SYM in 4D $\simeq$ IIB String in $AdS_5 \times S^5$

S^5: $\quad X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = R^2 \quad$ in R^6

AdS_5: $-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -R^2 \quad$ in $R^{2,4}$

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N=4 SYM	Conformal Sym. in 4D: $SO(4,2)$	R-sym.: $SU(4)$
IIB String	Isometry in AdS_5: $SO(4,2)$	Isometry in S^5: $SO(6)$

't Hooft coupling: $\lambda = g_{YM}^2 N = g_s N$, $R = (4\pi g_s N)^{1/4} l_s \sim \lambda^{1/4} l_{pl}$

g_s: string coupling, $2\pi l_s^2$: inverse string tension.

For $\lambda \gg 1$, $R \gg l_{pl}$, SUGRA limit.

- The holographic principle.
- CFT calculation of BH entropy, BH greybody factor.
- PP wave limit.

Let $X_{-1} = R\cosh\rho\cos\tau, \qquad X_0 = R\cosh\rho\sin\tau,$

$$X_1 = R\sinh\rho\sin\theta_1\sin\theta_2\cos\phi, \quad X_2 = R\sinh\rho\sin\theta_1\sin\theta_2\sin\phi, \tag{1}$$

$$X_3 = R\sinh\rho\sin\theta_1\cos\theta_2, \qquad X_4 = R\sinh\rho\cos\theta_1.$$

Global coordinate: $$ds^2 = R^2\left[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2\right] \tag{2}$$

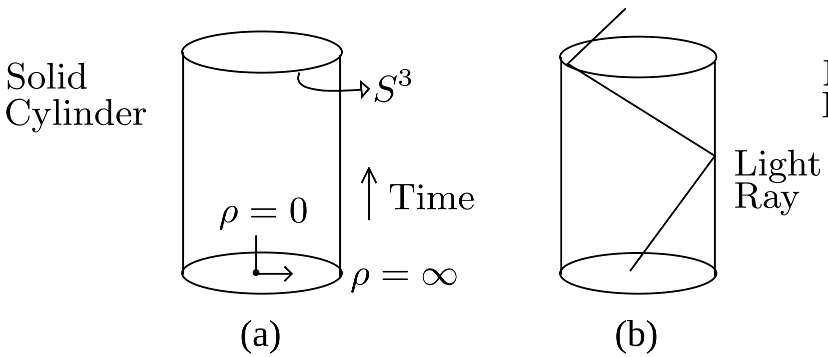


Figure 1. Penrose diagram and the AdS boundary ($S^3 \times R$).

Define $dx = d\rho/\cosh\rho$, $\tan(x/2) = \tanh(\rho/2)$, $\rho \to \infty \sim x = \pi/2$.

$\Rightarrow -\cosh^2\rho d\tau^2 + d\rho^2 = \sec^2 x(-d\tau^2 + dr^2)$.

Let $X_{-1} + X_4 = R/z$, $X_\mu = Rx_\mu/z$ for $\mu = 0, \cdots 3$.

Poincare coordinate: $$ds^2 = R^2 \left(\frac{-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2}{z^2} \right) \qquad (3)$$

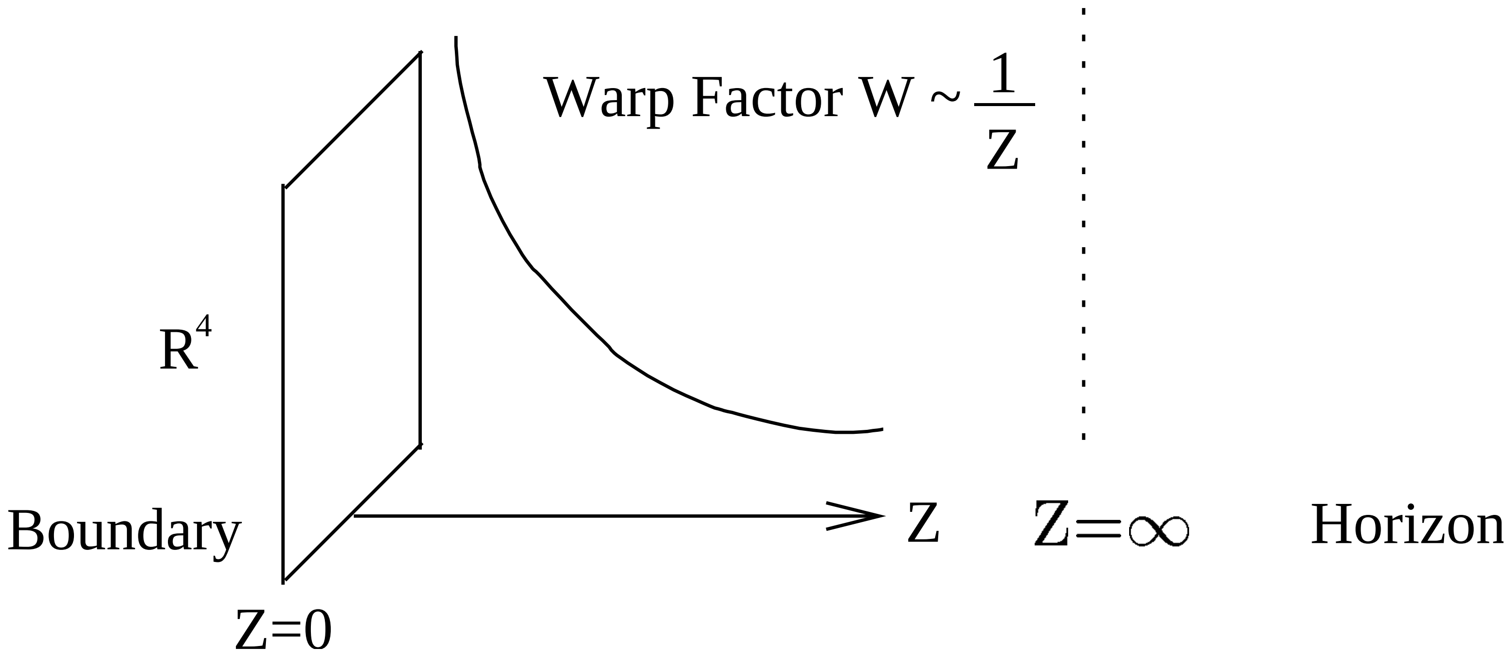
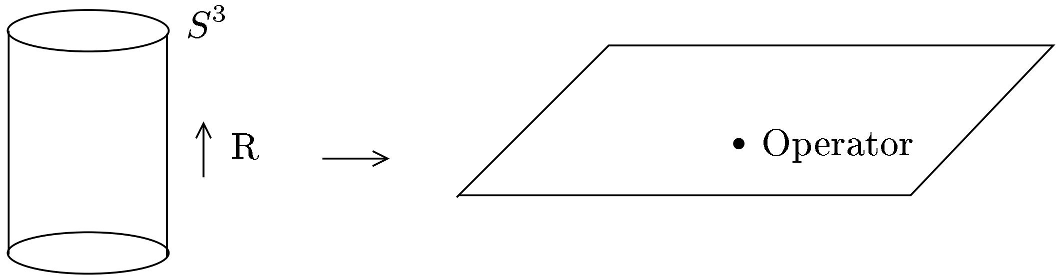


Figure 2. AdS boundary: $S^3 \times R$.



States of CFT on S^3 $\quad = \quad$ Operators of Euclidean CFT on R^4

$$\mathcal{Z}_{bulk}\left[\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})\right] = \langle e^{\int d^4x \phi_0(\vec{x})\mathcal{O}(\vec{x})} \rangle_{\text{Field Theory}} \tag{4}$$

Scalar field of mass m: $S = N^2 \int \frac{dx^4 dz}{z^5}[z^2(\partial\phi)^2 + m^2R^2\phi^2]$ (5)

$$\Rightarrow z^3\partial_z(\frac{1}{z^3}\partial_z\phi) - p^2\phi - \frac{m^2R^2}{z^2}\phi = 0 \tag{6}$$

$$\phi = z^2\left[A_+ I_\nu(pz) + A_- K_\nu(pz)\right], \text{ with } \nu = \sqrt{4 + m^2R^2}. \tag{7}$$

Near $z = 0$, $\phi \sim B_+ z^{2+\nu} + B_- z^{2-\nu}$. (8)

Boundary condition: $\phi(x, z)|_{z=\epsilon} = \epsilon^{2-\nu}\phi_0^r(x)$, $\phi_0^r(x)$ fixed for $\epsilon \to 0$.

The rescaling $x_\mu \to \lambda x_\mu,\ z \to \lambda z$ is a isometry in AdS,

ϕ does not get rescaled, ϕ_0^r has dimension $2-\nu$,

$\Rightarrow$ the correponding operator O has dimension $\Delta \equiv 2+\nu$.

Bulk Green function in AdS_5: $$G_\Delta(z, x^\mu, x'^\mu) = \frac{z^\Delta}{[(x-x')^2 + z^2]^\Delta} \quad (9)$$

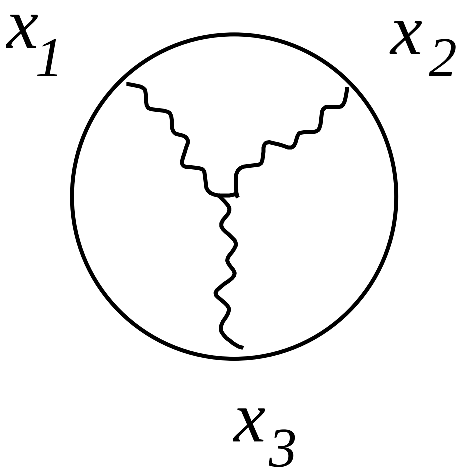


Figure 4. Three point functions.

With a cubic term in the action,

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle \sim \int \frac{d^4x dz}{z^5} G_\Delta(z, x, x_1) G_\Delta(z, x, x_2) G_\Delta(z, x, x_3). \quad (10)$$

$$\begin{cases} 0 > m^2R^2 \geq -4, & 2-\nu < 0, \quad \phi \text{ induces relevant pertubation;} \\ m^2R^2 > 0, & 2-\nu > 0, \quad \phi \text{ induces irrelevant pertubation.} \end{cases}$$

$$\begin{cases} (E_{\text{FT}}) = \frac{1}{z}(E_{\text{proper}}); \\ (\text{size})_{\text{FT}} = z(\text{proper size}). \end{cases} \qquad \begin{cases} z \to 0 \sim \text{spatial infinity}, & \text{UV;} \\ z \to \infty \sim \text{horizon}, & \text{IR.} \end{cases}$$

IR/UV Correspondence: UV in field theory $\sim$ IR in gravity .

Blackhole $\Rightarrow$ Hawking temperature(T_H) $\Rightarrow$ thermal effect.

Simplest black hole in Poicare coordinate (S^5 supressed):

$$ds^2 = \frac{R^2}{z^2}\left[-\left(1-\frac{z^4}{z_0^4}\right)dt^2 + d\vec{x}^2 + \left(1-\frac{z^4}{z_0^4}\right)^{-1}dz^2\right]. \quad (11)$$

For $z \simeq z_0$, $ds^2 \propto \left[-16\left(1-\frac{z}{z_0}\right)^2 dt^2 + dz^2 + \ldots\right]$. (12)

Period in imaginary time: $\beta = \pi z_0/2$.

Bekenstein Hawking entropy: $$S_{BH} = \frac{\text{(Area)}}{4G_N} \Rightarrow \frac{S_{BH}}{V} = \frac{\pi^2}{2} N^2 T^3. \quad (13)$$

Weakly coupled field theory entropy: $$\frac{S_{FT}}{V} = \frac{4}{3}\frac{\pi^2}{2} N^2 T^3. \quad (14)$$

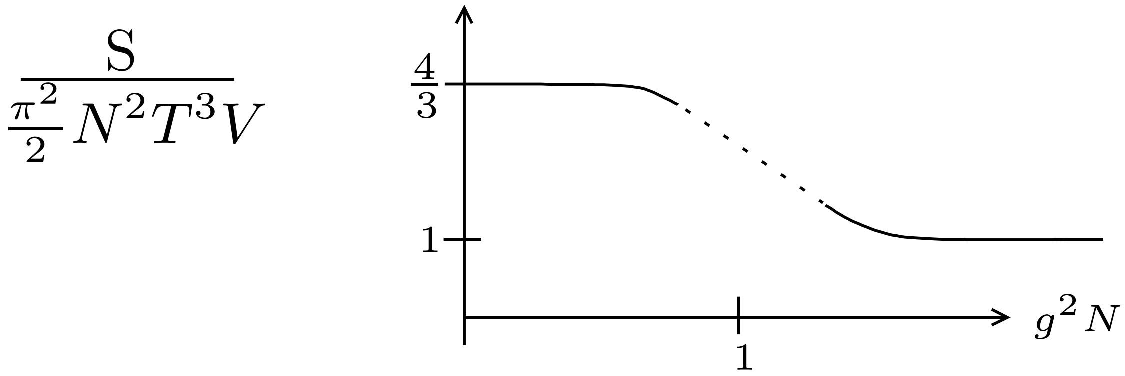


Figure 5. Field theory entropy vs. Bekenstein Hawking entropy.

g^2N correction to S_{FT} and R^4 to S_{BH} go in the right direction.

$\mathrm{AdS}_3/\mathrm{CFT}_2$ Correspondence

CFT interpretation of BH QMN

The BTZ blackhole ($R = 1$), part of SUGRA:

$$ds^2 = -\left[\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}\right] dt^2 + \left[\frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)}\right] dr^2 + r^2 \left[d\theta - \frac{r_+ r_-}{r^2} dt\right]^2 . \quad (15)$$

$M = (r_+^2 + r_-^2),\ J = 2r_+ r_-.$

$T_H = (r_+^2 - r_-^2)/(2\pi r_+),\ \mathcal{A}_H = 2\pi r_+,\ \Omega_H = J/(2r_+^2).$

Locally equivalent to AdS_3:

$$ds^2 = -\sinh^2 \mu (r_+ dt - r_- d\phi)^2 + d\mu^2 + \cosh^2 \mu (r_+ d\phi - r_- dt)^2 . \quad (16)$$

The dual CFT on the boundary is (1+1)-D.

Two independent copies of CFT with

$T_L = (r_+ - r_-)/(2\pi),\ T_R = (r_+ + r_-)/(2\pi),\ 1/T_L + 1/T_R = 2/T_H.$

Operators $\mathcal{O}$ in (1+1)-D CFT are charaterized by

the conformal weights: (h_L, h_R):

$$h_L + h_R = \Delta, \quad h_R - h_L = \pm s. \tag{17}$$

$$\begin{cases} \text{Scalar:} & \Delta = \sqrt{1+m^2}; \\ \text{Spinor and vector:} & \Delta = 1 + |m|. \end{cases}$$

Retarded Green function: $$D_{\text{ret}}(x, x') = i\theta(t-t')\langle[\mathcal{O}(x), \mathcal{O}(x')]\rangle_T, \tag{18}$$
$$= i\theta(t-t')\bar{\mathcal{D}}(x, x').$$

$\bar{\mathcal{D}}(x, x') = \mathcal{D}_+(x, x') - \mathcal{D}_-(x, x')$, $x^\pm = t \pm \sigma$.

$$\mathcal{D}_+(x) = \frac{(\pi T_R)^{2h_R}}{\sinh^{2h_R}(\pi T_R x^- - i\epsilon)} \frac{(\pi T_L)^{2h_L}}{\sinh^{2h_L}(\pi T_L x^+ - i\epsilon)}; \tag{19}$$

$$\mathcal{D}_-(x) = \frac{(\pi T_R)^{2h_R}}{\sinh^{2h_R}(\pi T_R x^- + i\epsilon)} \frac{(\pi T_L)^{2h_L}}{\sinh^{2h_L}(\pi T_L x^+ + i\epsilon)}. \tag{20}$$

With $k_\pm = (\omega \mp k)/2$, Fourier transform of $\bar{\mathcal{D}}(x)$:

$$\bar{\mathcal{D}}(k_+, k_-) = \Gamma\left(h_L + \frac{ik_+}{2\pi T_L}\right)\Gamma\left(h_R + \frac{ik_-}{2\pi T_R}\right)\Gamma\left(h_L - \frac{ik_+}{2\pi T_L}\right)\Gamma\left(h_R - \frac{ik_-}{2\pi T_R}\right). \quad (21)$$

Two sets of poles in the lower complex plane of ω:

$$\omega_L = \quad k - 4\pi i T_L(n + h_L); \quad \omega_R = -k - 4\pi i T_R(n + h_R). \quad (22)$$

$n = 0, 1, 2, \ldots$. They characterize they decay of the perturbation.

These poles coincide precisely with the QMN in the BTZ backgound!

Scalar perturbation satisfies $$\left[\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu) - m^2\right]\Phi = 0. \quad (23)$$

Let $\Phi = e^{-i(k_+x^+ + k_-x^-)}f(\mu)$, $x^\pm = r_\pm t - r_\mp \phi$, $z = \tanh^2\mu$,

$$\left(k_+ + k_-\right)\left(r_+ - r_-\right) = \omega - k; \quad \left(k_+ - k_-\right)\left(r_+ + r_-\right) = \omega + k.$$

$$z(1-z)\tilde{f}''(z) + (1-z)\tilde{f}'(z) + \left[\frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{m^2}{4(1-z)}\right]\tilde{f}(z) = 0. \quad (24)$$

$$\tilde{f}(z) = z^{\alpha}(1-z)^{\beta_s}\ {}_2F_1(a_s, b_s; c_s; z), \tag{25}$$

$$\alpha = -ik_+/2,\ \beta_s = \left(1 - \sqrt{1+m^2}\right)/2$$

$$a_s = -i(k_+ - k_-)/2 + \beta_s,\ b_s = -i(k_+ + k_-)/2 + \beta_s,\ c_s = 1 + 2\alpha.$$

QNM: vanishing Dirichlet condition at infinity.

$$\text{Flux: } \mathcal{F} = \sqrt{g}\frac{1}{2i}\left(f^*\partial_\mu f - f\partial_\mu f^*\right) \propto \left|\frac{\Gamma(c_s)\Gamma(c_s - a_s - b_s)}{\Gamma(c_s - a_s)\Gamma(c_s - b_s)}\right|^2. \tag{26}$$

$\mathcal{F}$ vanishes if

$$\begin{cases} c_s - a_s = -n \\ c_s - b_s = -n \end{cases} \Rightarrow i(k_+ \pm k_-)/2 = n + \left(1 + \sqrt{1+m^2}\right)/2. \tag{27}$$

For spinors,

$$\beta_f = -\left(m + 1/2\right)/2, \;\; c_f = 1/2 + 2\alpha.$$

$$a_f = -i(k_+ - k_-)/2 + \beta_f + 1/2, \; b_f = -i(k_+ + k_-)/2 + \beta_f.$$

For vectors,

$$\beta_v = m/2, \;\; c_v = 1 + 2\alpha.$$

$$a_v = -i(k_+ - k_-)/2 + \beta_v, \; b_v = -i(k_+ + k_-)/2 + \beta_v.$$

CFT interpretation of BH Greybody Factor

$$ds^2 = -\left[\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}\right] dt^2 + \left[\frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)}\right] dr^2 + r^2 \left[d\theta - \frac{r_+ r_-}{r^2} dt\right]^2 . \quad (28)$$

Consider massless scalar Φ. Let $\Phi = e^{-i\omega t + im\theta} R_{\omega,m}(r)$,

$$\partial_r^2 R_{\omega,m}(r) + \left[-\frac{1}{r} + \frac{2r}{r^2 - r_+^2} + \frac{2r}{r^2 - r_-^2}\right] \partial_r R_{\omega,m}(r)$$
$$+ \frac{r^4}{(r^2 - r_+^2)^2 (r^2 - r_+^2)^2} \left[\omega^2 - m^2 + \frac{Mm^2 - J\omega m}{r^2}\right] R_{\omega,m}(r) = 0. \qquad (29)$$

Let $z = (r^2 - r_+^2)/(r^2 - r_-^2)$, $A_1 = \left(\frac{\omega - m\Omega_H}{4\pi T_H}\right)^2$, $B_1 = -\left(\frac{\omega r_-^2 - m\Omega_H r_+^2}{4\pi T_H r_+ r_-}\right)^2$

$$z(1-z)\tilde{R}''_{\omega,m}(z) + (1-z)\tilde{R}'_{\omega,m}(z) + \left[\frac{A_1}{z} + B_1\right] \tilde{R}_{\omega,m}(z) = 0. \qquad (30)$$

$z = 0 \sim$ horizon, $z \to \infty \sim$ spatial infinity.

In-going into BH at horizon:

$\tilde{R}_{\omega,m}(z) = z^{\alpha}\ {}_2F_1(a,b;c;z),\ \alpha = i\sqrt{A_1},$

$a+b = 2\alpha,\ ab = -A_1 - B_1,\ c = 1+2\alpha$. At horizon, $z=0$,

$$\text{Flux: } \mathcal{F} = 2\mathcal{A}_H(\omega - m\Omega_H). \tag{31}$$

BTZ $\sim AdS_3$, hard to tell in-coming from out-going.

$$R^{\text{in}}_{\omega,m} = A_i\left(1 - i\frac{c}{r^2}\right), \quad R^{\text{out}}_{\omega,m} = A_o\left(1 + i\frac{c}{r^2}\right). \tag{32}$$

$$\Rightarrow \mathcal{F}_{\text{in}} = 8\pi c|A_i|^2; \quad \sigma = \frac{\mathcal{F}}{\mathcal{F}_{\text{in}}}. \tag{33}$$

In low energy limit $\omega \to 0$, $\sigma_{\text{abs}} \simeq \dfrac{\sigma^{\text{m=0}}}{\omega} = \dfrac{\mathcal{A}_H}{\pi c}\dfrac{|\Gamma(a+1)\Gamma(b+1)|^2}{|\Gamma(a+b+1)|^2}$. (34)

Choose c so that $\sigma_{\text{abs}}|_{\omega=0} = \mathcal{A}_H$.

Decay rate:

$$\Gamma = \frac{\sigma_{\text{abs}}}{e^{\omega/T_H} - 1}$$
$$= 4\pi^2 \omega^{-1} T_L T_R e^{-\omega/(2T_H)} \left| \Gamma\left(1 + i\frac{\omega}{4\pi T_L}\right) \Gamma\left(1 + i\frac{\omega}{4\pi T_R}\right) \right|^2 . \qquad (35)$$

$$\text{From CFT, } \Gamma = \int d\sigma^- \, e^{-i\omega(\sigma_- - i\epsilon)} \left[\frac{2T_L}{\sinh(2\pi T_L \sigma^-)}\right]^2$$
$$\times \int d\sigma^+ \, e^{-i\omega(\sigma_+ - i\epsilon)} \left[\frac{2T_R}{\sinh(2\pi T_R \sigma^+)}\right]^2 . \qquad (36)$$

The result is identical to that obtained from the gravity side.

Warped $\mathrm{AdS}_3/\mathrm{CFT}_2$ Correspondence

Warped BTZ Black Hole

The warped BTZ black hole is a solution of TMG:

$$I_{TMG} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\left(R + 2/\ell^2\right) + \frac{\ell}{96\pi G\nu}\int_{\mathcal{M}} d^3x\sqrt{-g}\varepsilon^{\lambda\mu\nu}\Gamma^r_{\lambda\sigma}\left(\partial_\mu\Gamma^\sigma_{r\nu} + \frac{2}{3}\Gamma^\sigma_{\mu\tau}\Gamma^\tau_{\nu r}\right). \tag{37}$$

where $\varepsilon^{\tau\sigma u} = +1/\sqrt{-g}$, $\nu = \frac{m_g\ell}{3}$.

For $\nu = 1/3$, critical chiral gravity theory.

$$\text{EOM: } G_{\mu\nu} - \frac{1}{\ell^2}g_{\mu\nu} + \frac{\ell}{3\nu}\varepsilon_\mu{}^{\alpha\beta}\nabla_\alpha\left(R_{\beta\nu} - \frac{1}{4}g_{\beta\nu}R\right) = 0. \tag{38}$$

$$ds^2 = -N^2(r)dt^2 + \ell^2 R^2(r)[d\theta + N_\phi(r)dt]^2 + \frac{\ell^4 dr^2}{4R^2(r)N^2(r)}, \tag{39}$$
$$R^2(r) \equiv \frac{r}{4}\left[3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+ r_-(\nu^2+3)}\right],$$
$$N^2(r) \equiv \frac{\ell^2(\nu^2+3)(r-r_+)(r-r_-)}{4R^2},\ N_\theta(r) \equiv \frac{2\nu r - \sqrt{r_+ r_-(\nu^2+3)}}{2R^2}.$$

$$T_H = \frac{\nu^2+3}{4\pi}\left\{\frac{r_+ - r_-}{2\nu r_+ - \sqrt{(\nu^2+3)r_+ r_-}}\right\},\ \mathcal{A}_H = \pi\left\{2\nu r_+ - \sqrt{r_+ r_-(\nu^2+3)}\right\}.$$

$\nu > 1$, streched AdS_3, $\nu = 1$, BTZ limit,

$\nu < 1$, squashed AdS_3, closed time-like curve.

t, r, θ differnent from the BTZ case.

Warped AdS_3 (locally equvalent to warped BTZ):

$$ds^2 = \frac{\ell^2}{\nu^2+3}\left[-\cosh^2\sigma d\tau^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2+3}(du + \sinh\sigma d\tau)^2\right]. \tag{40}$$

Greybody Factor in Warped BTZ

Consider scalar Φ with mass μ. Let $\Phi = e^{-i\omega t + im\theta}\phi(r)$,

$$\frac{d^2\phi(r)}{dr^2} + \frac{2r - r_+ - r_-}{(r-r_+)(r-r_-)}\frac{d\phi(r)}{dr} - \frac{(\alpha r^2 + \beta r + \gamma)}{(r-r_+)^2(r-r_-)^2}\phi = 0, \tag{41}$$

$$\alpha = -\frac{3\omega^2(\nu^2-1)}{(\nu^2+3)^2} + \frac{\mu^2\ell^2}{\nu^2+3},\ \gamma = -\frac{4m\left[m - \omega\sqrt{r_+r_-(\nu^2+3)}\right]}{(\nu^2+3)^2} + \frac{\mu^2\ell^2 r_+ r_-}{\nu^2+3},$$

$$\beta = -\frac{\omega^2(\nu^2+3)(r_+ + r_-) - 4\nu\left[\omega^2\sqrt{r_+r_-(\nu^2+3)} - 2m\omega\right]}{(\nu^2+3)^2} + \frac{\mu^2\ell^2(r_+ + r_-)}{\nu^2+3}.$$

$z = \frac{r-r_+}{r-r_-}$, $\quad z = 0 \sim$ horizon, $z = 1 \sim$ spatial infinity.

$$z(1-z)\tilde{\phi}''(z) + (1-z)\tilde{\phi}'(z) + \left[\frac{A}{z} + \frac{B}{1-z} + C\right]\tilde{\phi}(z) = 0; \tag{42}$$

$$A = \frac{4(\omega\Omega_+^{-1} + m)^2}{(r_+ - r_-)^2(\nu^2+3)^2},\ B = -\alpha,\ C = -\frac{4(\omega\Omega_-^{-1} + m)^2}{(r_+ - r_-)^2(\nu^2+3)^2}.$$

$$\Omega_+^{-1} = \nu r_+ - \frac{\sqrt{r_+ r_-(\nu^2+3)}}{2}, \qquad \Omega_-^{-1} = \nu r_- - \frac{\sqrt{r_+ r_-(\nu^2+3)}}{2}.$$

$$\phi(z) = z^p (1-z)^q u(z), \text{ with } p = -i\sqrt{A},\ q = \frac{1}{2}\left(1 - \sqrt{1+4\alpha}\right),$$

$$\Rightarrow z(1-z)u''(z) + \{c - (a+b+1)z\}u'(z) - abu(z) = 0, \qquad (43)$$

$$\text{with} \qquad a = p + q + \sqrt{C},\ b = p + q - \sqrt{C},\ c = 2p + 1.$$

In-going at horizon: $u(z) = z^p(1-z)^q {}_2F_1(a,b;c;z)$. (44)

At $r \to \infty$: $\phi \simeq A_{in}\left(r^{-h_-^*} - \frac{i\eta}{\pi} r^{-h_+^*}\right) + A_{out}\left(r^{-h_-^*} + \frac{i\eta}{\pi} r^{-h_+^*}\right)$. (45)

$h_\pm^* = \frac{1}{2}(1 \pm \Delta^*)$, $\quad \Delta^* = \sqrt{1+4\alpha} = \sqrt{1 - \frac{12\omega^2(\nu^2-1)}{(\nu^2+3)^2} + \frac{4\mu^2\ell^2}{\nu^2+3}}$.

η is chosen so that $\sigma_{\text{abs}}|_{\omega=0} = \mathcal{A}_H$.

Retaining ω dependence,

$$\sigma_{abs} \propto \left| \frac{\Gamma\left[\frac{1}{2}(1+\sqrt{1+4\alpha}) - i\frac{\omega}{4\pi\tilde{T}_R}\right]\Gamma\left[\frac{1}{2}(1+\sqrt{1+4\alpha}) - i\frac{\omega}{4\pi\tilde{T}_L}\right]}{\Gamma\left[1 - i\frac{\omega}{2\pi T_H}\right]\Gamma\left[\sqrt{1+4\alpha}\right]} \right|^2, \quad (46)$$

$$\tilde{T}_L = \frac{\nu^2+3}{8\pi\nu}, \; \tilde{T}_R = \frac{(\nu^2+3)(r_+ - r_-)}{8\pi\left[\nu(r_+ + r_-) - \sqrt{r_+ r_-(\nu^2+3)}\right]}. \quad (47)$$

$$2/T_H = 1/\tilde{T}_L + 1/\tilde{T}_R. \quad (48)$$

Waped AdS_3, $\Phi = e^{-i\omega_g \tau + iku}\phi_g(\sigma)$:

$$\frac{d^2\phi_g}{d\sigma^2} + \tanh\sigma \frac{d\phi_g}{d\sigma} + \left[(\omega_g\,\mathrm{sech}\sigma + k\tanh\sigma)^2 - \frac{(\nu^2+3)k^2}{4\nu^2} - \frac{\mu^2\ell^2}{\nu^2+3}\right]\phi_g = 0. \quad (49)$$

Define $\phi_g = z_g^{(\omega_g+ik)/2}(1-z_g)^{(\omega_g-ik)/2}f(z_g),\ z_g \equiv \dfrac{1+i\sinh\sigma}{2}.$

$$z_g(1-z_g)f'' + \Big[c_g - (1+a_g+b_g)z_g\Big]f' - a_g b_g f = 0 \tag{50}$$

$$a_g b_g = \omega_g(\omega_g+1) + \frac{3k^2(\nu^2-1)}{4\nu^2} - \frac{\mu^2\ell^2}{\nu^2+3},\ a_g+b_g = 1+2\omega_g,$$

$c_g = 1+\omega_g+ik.$

$$\text{For } z_g \to \infty,\ \phi_g(z_g) \to C_+ z_g^{h_+} + C_- z_g^{h_-}, \tag{51}$$

$$\text{with} \quad h_\pm = \frac{1}{2}(1\pm\Delta), \qquad \Delta \equiv \sqrt{1 - \frac{3k^2(\nu^2-1)}{\nu^2} + \frac{4\mu^2\ell^2}{\nu^2+3}}.$$

Boundary of warped $AdS_3 : \sigma \to \infty.$

$x = e^{-\sigma}$ and $t = \tau/2$, local patch:

$$ds^2 \to \frac{\ell^2}{(\nu^2+3)x^2}\left[\frac{3(\nu^2-1)}{\nu^2+3}dt^2 + dx^2 + \frac{4\nu^2}{\nu^2+3}x^2du^2 + \frac{8\nu^2}{\nu^2+3}xdtdu\right]. \tag{52}$$

On the boundary,

$$\left(\frac{1}{\sqrt{-g}}\partial_x\sqrt{-g}g^{xx}\partial_x + \partial_u g^{uu}\partial_u - m^2\right)K(x,u) = 0. \tag{53}$$

$$K(x,u) = e^{iku}x^{h_+} \Rightarrow K(x,u,t) = e^{iku}\left(\frac{x}{|x^2-t^2|}\right)^{h_+}. \tag{54}$$

Bulk field: $\phi(x,t,u) \sim \int dt'du'K_b(x,t,u,t',u')\phi_0(t',u').$ (55)

In the limit $x \to 0$, $\frac{\partial}{\partial x}\phi(x,t,u) \sim x^{h_+-1}\int dt'\,du'\frac{e^{ik(u-u')}\phi_0(t',u')}{|t-t'|^{2h_+}}.$ (56)

$$\begin{aligned} S_{eff} &= \lim_{x\to 0}\left\{-\frac{1}{2}\int dt\,du\sqrt{-g}g^{xx}\phi\partial_x\phi\right\} \\ &\sim \frac{1}{2}\int dt\,du\,dt'\,du'\frac{e^{ik(u-u')}}{|t-t'|^{2h_+}}\phi_0(t,u)\phi_0(t',u'). \end{aligned} \tag{57}$$

From the AdS/CFT dictionary: $e^{-S_{eff}(\phi)} = \left\langle e^{\int \phi_0 \mathcal{O}} \right\rangle$. (58)

$$\text{Correlator: } \left\langle \mathcal{O}_*(u_+, u_-)\mathcal{O}_*(0,0) \right\rangle^L \sim \frac{e^{i\omega u_+}}{|u_-|^{2h_+^*}}, \tag{59}$$

$$\left\langle \mathcal{O}_*(u_+, u_-)\mathcal{O}_*(0,0) \right\rangle^R \sim \frac{e^{i\omega u_-}}{|u_+|^{2h_+^*}}, \quad u_\pm \equiv u \pm t/2c. \tag{60}$$

Assigning $\tilde{T}_L$ and $\tilde{T}_R$ to the left and right sectors.

$$\left\langle \mathcal{O}_*(u_+, u_-)\mathcal{O}_*(0,0) \right\rangle_T \sim (2h_+ - 1)e^{i\omega(u_+ + u_-)} \left[\frac{\pi \tilde{T}_R}{\sinh(\pi \tilde{T}_R u_+)}\right]^{2h_+^*} \left[\frac{\pi \tilde{T}_L}{\sinh(\pi \tilde{T}_L u_-)}\right]^{2h_+^*}. \tag{61}$$

$\mathcal{O}_*(x_+, x_-)$ with conformal dimension $h_+^* \equiv h_+\Big|_{k=k_* \equiv \frac{2\nu}{\nu^2+3}\omega}$,

$$\sigma_{\text{abs}} \sim \int dx_+ dx_- e^{-i\omega(x_+ + x_-)} \left\langle \mathcal{O}_*(x_+, x_-)\mathcal{O}_*(0,0) \right\rangle_T. \tag{62}$$

Discussions

- There is indeed a warped AdS_3/CFT_2 correspondence.

•Gravity side:

$$\sigma_{abs} \propto \left| \frac{\Gamma\left[\frac{1}{2}(1+\sqrt{1+4\alpha}) - i\frac{\omega}{4\pi\tilde{T}_R}\right]\Gamma\left[\frac{1}{2}(1+\sqrt{1+4\alpha}) - i\frac{\omega}{4\pi\tilde{T}_L}\right]}{\Gamma\left[1 - i\frac{\omega}{2\pi T_H}\right]\Gamma\left[\sqrt{1+4\alpha}\right]} \right|^2,$$

$$\tilde{T}_L = \frac{\nu^2+3}{8\pi\nu},\ \tilde{T}_R = \frac{(\nu^2+3)(r_+ - r_-)}{8\pi\left[\nu(r_+ + r_-) - \sqrt{r_+ r_-(\nu^2+3)}\right]},\ 2/T_H = 1/\tilde{T}_L + 1/\tilde{T}_R.$$

- CFT side: $$\left\langle \mathcal{O}_*(u_+, u_-)\mathcal{O}_*(0,0) \right\rangle^L \sim \frac{e^{i\omega u_+}}{|u_-|^{2h_+^*}},$$
$$\left\langle \mathcal{O}_*(u_+, u_-)\mathcal{O}_*(0,0) \right\rangle^R \sim \frac{e^{i\omega u_-}}{|u_+|^{2h_+^*}}.$$

Assigning $\tilde{T}_L$ and $\tilde{T}_R$ to the left and right sectors.

$$\left\langle \mathcal{O}_*(u_+, u_-)\mathcal{O}_*(0,0) \right\rangle_T \sim (2h_+ - 1) e^{i\omega(u_+ + u_-)} \left[\frac{\pi \tilde{T}_R}{\sinh(\pi \tilde{T}_R u_+)}\right]^{2h_+^*} \left[\frac{\pi \tilde{T}_L}{\sinh(\pi \tilde{T}_L u_-)}\right]^{2h_+^*}.$$

$$\mathcal{O}_*(x_+, x_-) \sim \mathcal{O}_*(u_+, u_-) \text{ with } h_+^* \equiv h_+\Big|_{k=k_* \equiv \frac{2\nu}{\nu^2+3}\omega},$$

$$\sigma_{\text{abs}} \sim \int dx_+ dx_- e^{-i\omega(x_+ + x_-)} \left\langle \mathcal{O}_*(x_+, x_-)\mathcal{O}_*(0,0) \right\rangle_T,$$
$$\propto \left| \frac{\Gamma\left[\frac{1}{2}(1+\sqrt{1+4\alpha}) - i\frac{\omega}{4\pi \tilde{T}_R}\right] \Gamma\left[\frac{1}{2}(1+\sqrt{1+4\alpha}) - i\frac{\omega}{4\pi \tilde{T}_L}\right]}{\Gamma\left[1 - i\frac{\omega}{2\pi T_H}\right] \Gamma\left[\sqrt{1+4\alpha}\right]} \right|^2.$$

- $\nu > 1$, superradiant modes appear when

$$\omega^2 > \frac{(\nu^2+3)^2}{12(\nu^2-1)} + \frac{\nu^2+3}{3(\nu^2-1)}\mu^2\ell^2.$$

Conformal weight $h^*_\pm$ becomes complex. $\omega \sim$ angular velocity.

The *effective* mass of scalar is below the B-F bound

$$\mu^2_{eff} \equiv \frac{4\mu^2}{\nu^2+3} - \frac{12(\nu^2-1)\omega^2}{(\nu^2+3)^2\ell^2} < -\frac{1}{\ell^2}.$$

- Tortoise coordinate r^*: $\phi^*(r^*) \equiv z(r)\phi(r)$, $r^* = f(r)$.

Choose $f(r), z(r)$ s.t. $\left[-\frac{d^2}{dr^{*2}} - \omega^2 + U^*(r^*)\right]\phi(r^*) = 0.$

In spatial infinity, the effective potential

$$\lim_{r\to\infty} U^*(r) \to U^*_\infty = \frac{(\nu^2+3)(\nu^2+3+4\mu^2\ell^2)}{12(\nu^2-1)}.$$

Superrandiance: $\omega^2 - U^*_\infty > 0$.

- Comparing the poles of bulk QNM and boundary retarded Green function.

- Dual theory of a rotating background.