The Eccentric Universe

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References

"The Eccentric Universe"
A. Berera, R. V. Buniy and T. W. K.,
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"The Eccentric Universe: Exact Solutions"
A. Berera, R. V. Buniy and T. W. K.,
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"The Eccentric Universe: Density Perturbations"
A. Berera, R. V. Buniy and T. W. K., to appear

Outline

FRW Cosmology

- Homogeneous Isotropic Cosmologies
- Energy-momentum and Einstein's Equations
- Data–Departure from Homogeneous Isotropic Cosmology
- Solutions–FRW Universe

Outline

Planar Symmetric Cosmologies

- Metric and Curvature
- Energy-momentum and Einstein's Equations
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- Planar Symmetric Solutions
- Graphics and Asymptotics

Outline

Density Perturbations

- Density Perturbations in an FRW Universe
- Solutions to Density Perturbation Equations in an FRW Universe
- Sachs-Wolfe Effect in FRW Cosmology
- Density Perturbations in a Planar Symmetric Universe
- Solutions to Density Perturbation Equations in a Planar Symmetric Universe
- Sachs-Wolfe Effect in a Planar Symmetric Cosmology

Homogeneous isotropic cosmologies

Metric: $g_{\mu\nu} = \mathrm{diag}\,(1, -e^{2a}, -e^{2a}, -e^{2a})$

Christoffel symbols:

$$\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = \dot{a}\,\epsilon^{2a},$$
$$\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \dot{a}.$$

Ricci tensor:

$$R^0{}_0 = -(3\ddot{a} + 3\dot{a}^2),$$
$$R^1{}_1 = R^2{}_2 = R^3{}_3 = -(\ddot{a} + 3\dot{a}^2).$$

Energy-momentum

Energy-momentum: $T^{\mu}{}_{\nu} = (8\pi G)^{-1}\mathrm{diag}\,(\xi, \eta, \eta, \eta).$

Einstein equations: $G^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}Rg^{\mu}{}_{\nu} = T^{\mu}{}_{\nu}$

gives

$$3\dot{a}^2 = \xi,$$

$$2\ddot{a} + 3\dot{a}^2 = \eta.$$

Solution: FRW Universe

Conservation of Stress-Energy

Covariant conservation of the stress-energy

$$\dot{\xi} + 3\dot{a}(\xi - \eta) = 0$$

is a direct consequence of Einstein's Equations.

Data and the Evidence for a Departure from Homogeneous Isotropic Cosmology

WMAP: The first year WMAP results contain some unusual large-scale features. (astro-ph/0302207, 0302209, and 0302217).

Power Spectrum:

–*suppression*

suppression of power at large angular scales in quadrupole C_2 and octupole C_3 (also seen in the COBE)

–*alignment*

quadrupole and octupole are aligned;
$\ell = 2$ and 3 power concentrated in a plane P;
P inclined 30° to Galactic plane;
powers mostly in $m = \pm\ell$ modes,
i.e., power along axis suppressed relative to orthogonal plane.

Planar symmetry

Metric: $$g_{\mu\nu} = \mathrm{diag}\,(1, -e^{2a}, -e^{2a}, -e^{2b})$$

Christoffel symbols:

$$\Gamma^0_{11} = \Gamma^0_{22} = \dot{a}\,\epsilon^{2a}, \quad \Gamma^0_{33} = \dot{b}\,\epsilon^{2b},$$
$$\Gamma^1_{01} = \Gamma^2_{02} = \dot{a}, \quad \Gamma^3_{03} = \dot{b},$$

Ricci tensor:

$$R^0{}_0 = -(2\ddot{a} + \ddot{b} + 2\dot{a}^2 + \dot{b}^2),$$
$$R^1{}_1 = R^2{}_2 = -(\ddot{a} + 2\dot{a}^2 + \dot{a}\dot{b}),$$
$$R^3{}_3 = -(\ddot{b} + \dot{b}^2 + 2\dot{a}\dot{b}).$$

Energy-momentum

Energy-momentum: $T^{\mu}{}_{\nu} = (8\pi G)^{-1}\mathrm{diag}\,(\xi, \eta, \eta, \zeta).$

Einstein equations:

$$
\begin{aligned}
&\dot{a}^2 + 2\dot{a}\dot{b} = \xi, \\
&\ddot{a} + \ddot{b} + \dot{a}^2 + \dot{a}\dot{b} + \dot{b}^2 = \eta, \\
&2\ddot{a} + 3\dot{a}^2 = \zeta.
\end{aligned}
$$

Conservation of the Stress-Energy

Covariant conservation of the stress-energy is a direct consequence of Einstein's Equations:

$$\dot{\xi} + 2\dot{a}(\xi - \eta) + \dot{b}(\xi - \zeta) = 0.$$

Thermodynamics

Isotropic part of $T^{\mu}{}_{\nu}$: λ, ρ, and p.

Ansotropic part of $T^{\mu}{}_{\nu}$: Stresses and tensions from B-fields, strings and walls.

Split components via:

$$\xi = \lambda + \rho + \tilde{\xi},$$

$$\eta = \lambda - p + \tilde{\eta},$$

$$\zeta = \lambda - p + \tilde{\zeta},$$

where tildes are for anisotropic parts.

Thermodynamics

As in the isotropic case

$$T\frac{\mathrm{d}p}{\mathrm{d}T} = \rho + p$$

Entropy in a volume V is

$$S = (\rho + p)V/T.$$

Taking $V = V_i\, e^{2a+b}$ we find

$$\dot{S}/S = 2\dot{a} + \dot{b} + \dot{\rho}/(\rho + p).$$

Entropy in a comoving volume is conserved, so

$$\dot{\rho} + (2\dot{a} + \dot{b})(\rho + p) = 0.$$

Thermodynamics

Integrate for equation of state $p = w\rho$ to find

$$\rho = \rho_i \, e^{-(1+w)(2a+b)}.$$

Isotropic part of $T^{\mu}{}_{\nu}$ is conserved locally.
Since total energy-momentum is conserved locally,
the anisotropic part is also conserved,

$$\dot{\tilde{\xi}} + 2\dot{a}(\tilde{\xi} - \tilde{\eta}) + \dot{b}(\tilde{\xi} - \tilde{\zeta}) = 0.$$

Key to finding our exact solutions.

Universes with planar symmetry

Table: The components of the energy momentum for various contributions to the matter. Note $T^{\mu}{}_{\nu}$ is traceless for B-fields and radiation.

	ξ	η	η	ζ
vacuum energy	λ	λ	λ	λ
radiation	ρ	$-\frac{1}{3}\rho$	$-\frac{1}{3}\rho$	$-\frac{1}{3}\rho$
matter (dust)	ρ	0	0	0
magnetic field	ϵ	$-\epsilon$	$-\epsilon$	ϵ
strings	ϵ	0	0	ϵ
walls	ϵ	ϵ	ϵ	0

Solutions: Cosmological Constant plus B-Fields (ΛB)

Conservation of stress-energy gives:

$$\dot{\epsilon} + 4\dot{a}\epsilon = 0.$$

Solve for $\dot{a}$ and plug into Einstein's eqs. to find:

$$\epsilon\ddot{\epsilon} - \frac{11}{8}\dot{\epsilon}^2 + 2\epsilon^2(\lambda + \epsilon) = 0.$$

This has a general solution:

$$t - t_i = \frac{1}{4}\int_{\epsilon}^{\epsilon_i} d\epsilon \left(\frac{1}{3}\lambda\epsilon^2 + \frac{4}{3}\epsilon_i^{\frac{1}{4}}\epsilon^{\frac{11}{4}} - \epsilon^3\right)^{-\frac{1}{2}}$$

Invert to get $\epsilon(t)$, then solve for $a(t)$ and $b(t)$.

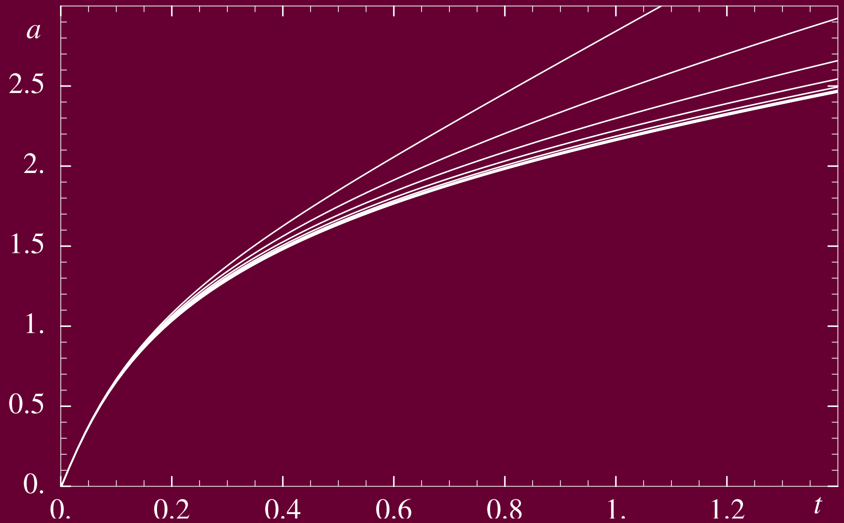
Eccentricity

Eccentricity: The eccentricity of an ellipse, with semi-major axis $A = e^a$, and semi-minor axis $B = e^b$, is $e_s = \frac{\sqrt{A^2 - B^2}}{B} = \sqrt{e^{2(a-b)} - 1}$. We are interested in prolate and oblate spheroids. If a cross section that is tangent to the symmetry axis of the spheroid is an ellipse with axes A along the symmetry axis and B normal to that axis, then either one can be larger. An appropriate measure for our purposes is the ratio $e_p = \frac{A}{B} = e^{a-b}$ which we will call the pseudo-eccentricity.

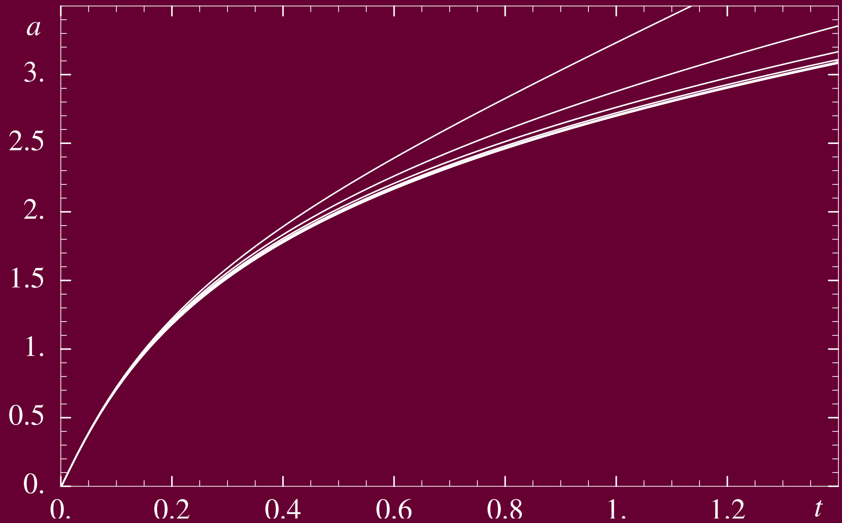
Expansion parameter $a(t)$

$a(t)$ for (M, S and W) + Λ + w with $\lambda = 1$, $\rho_\mathrm{i} = 10$, $\epsilon_\mathrm{i} = 200$. Curves are for w from -1 to 1 with step 0.2 from top to bottom.

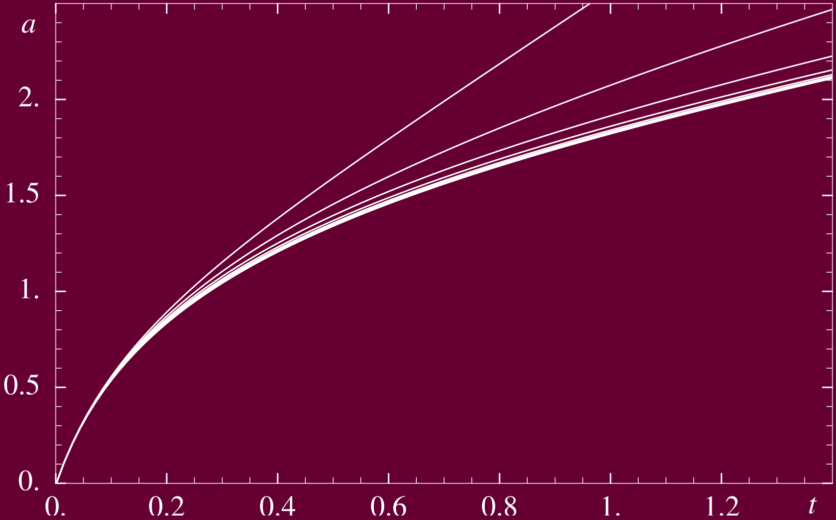
a(t) in a Universe with Λ + B-fields+ matter



a(t) in a Universe with Λ + strings + matter



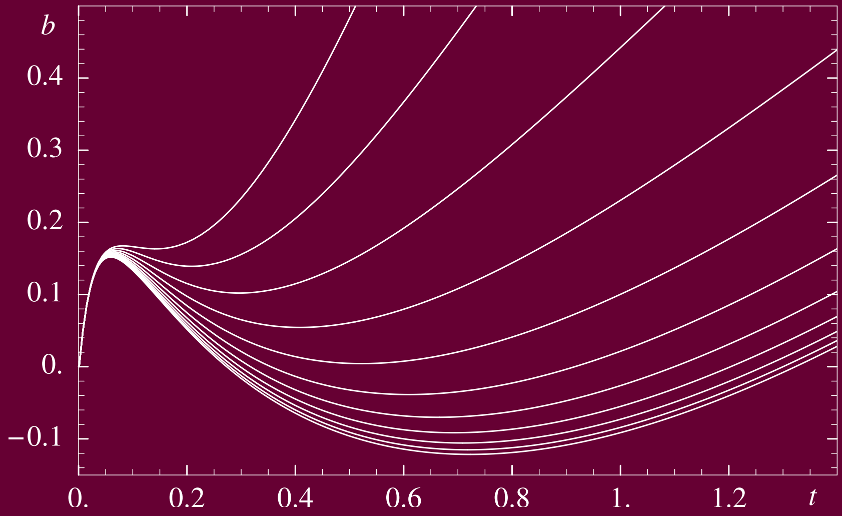
a(t) in a Universe with Λ + walls + matter



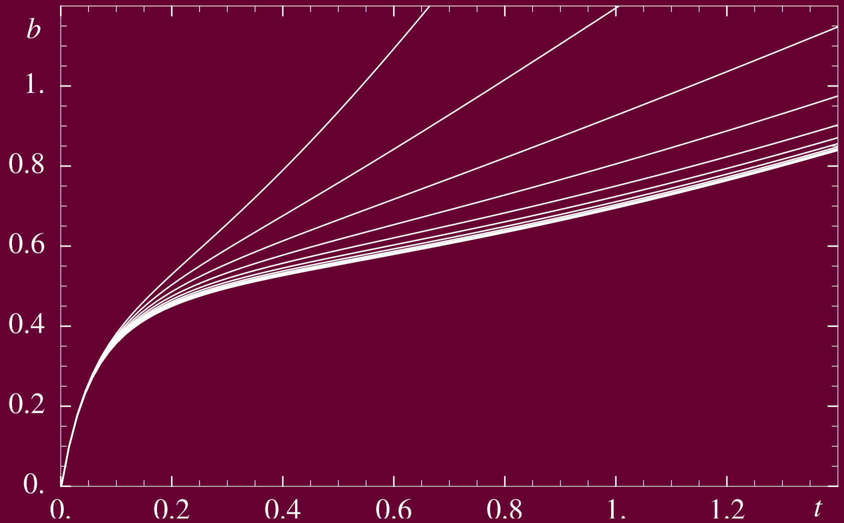
Expansion parameter $b(t)$

$b(t)$ for (M, S and W) + Λ + w with $\lambda = 1$, $\rho_\mathrm{i} = 10$, $\epsilon_\mathrm{i} = 200$. Curves are for w from -1 to 1 with step 0.2 from top to bottom.

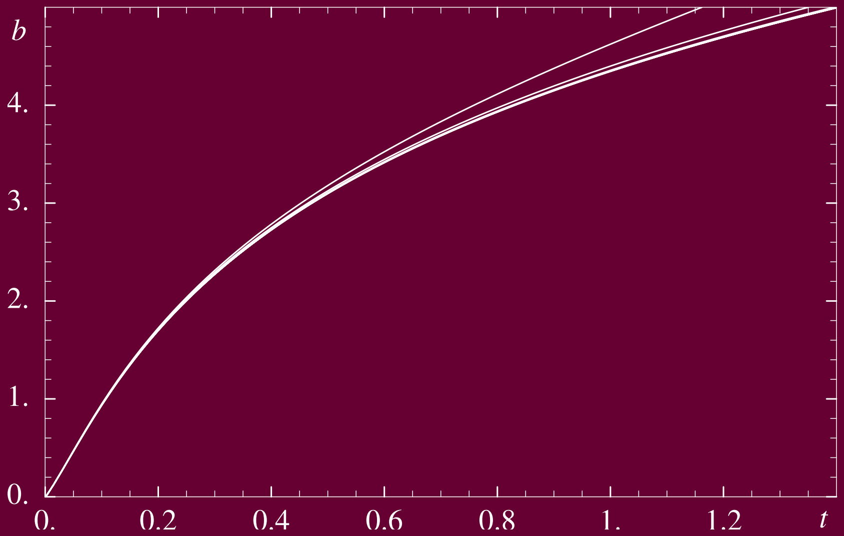
b(t) in a Universe with Λ + B-fields+ matter



b(t) in a Universe with Λ + strings + matter



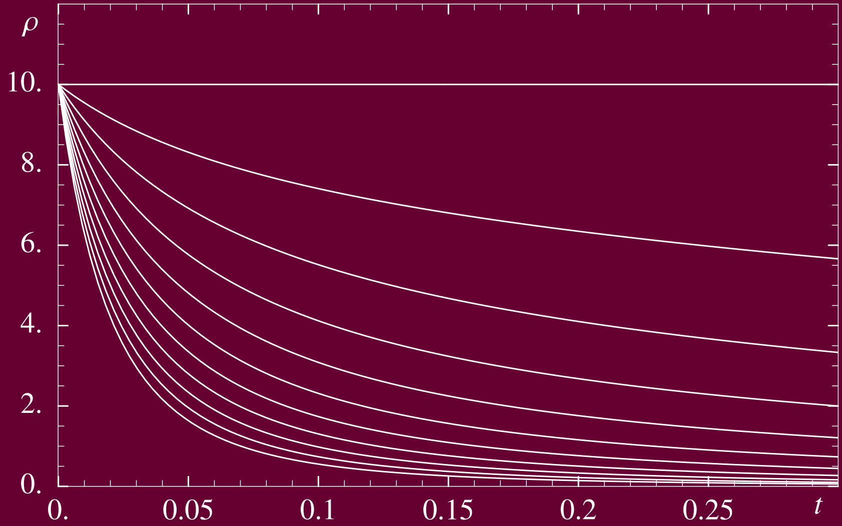
b(t) in a Universe with Λ + walls + matter



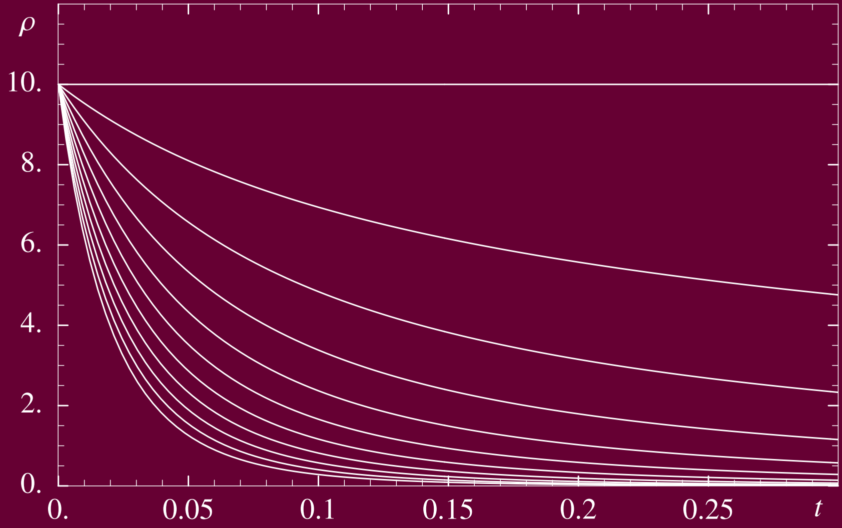
Matter density $\rho(t)$

$\rho(t)$ for (M, S and W) + Λ + w with $\lambda = 1$, $\rho_\mathrm{i} = 10$, $\epsilon_\mathrm{i} = 200$. Curves are for w from -1 to 1 with step 0.2 from top to bottom.

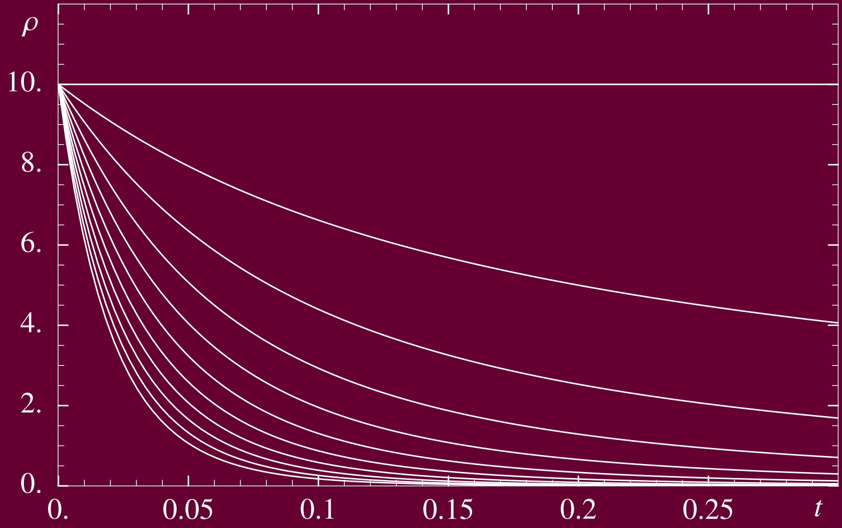
$\rho(t)$ in a Universe with Λ + B-fields + matter



$\rho(t)$ in a Universe with Λ + strings + matter



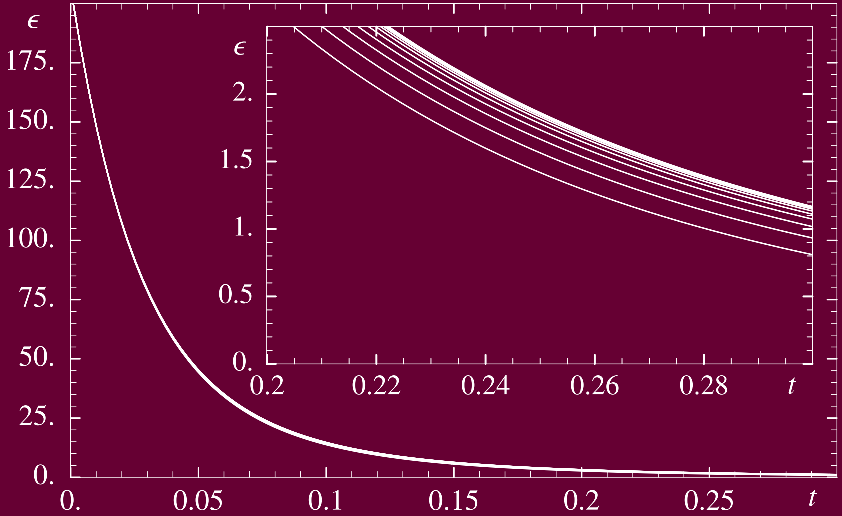
$\rho(t)$ in a Universe with Λ + walls + matter



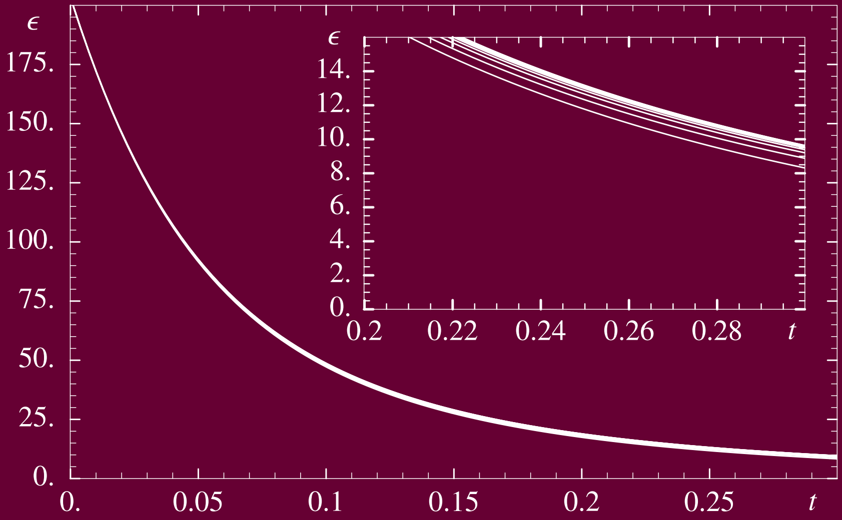
Energy density $\epsilon(t)$

$\epsilon(t)$ for (M, S and W) + Λ + w with $\lambda = 1$, $\rho_{\mathrm{i}} = 10$, $\epsilon_{\mathrm{i}} = 200$. Curves are for w from -1 to 1 with step 0.2 from top to bottom.

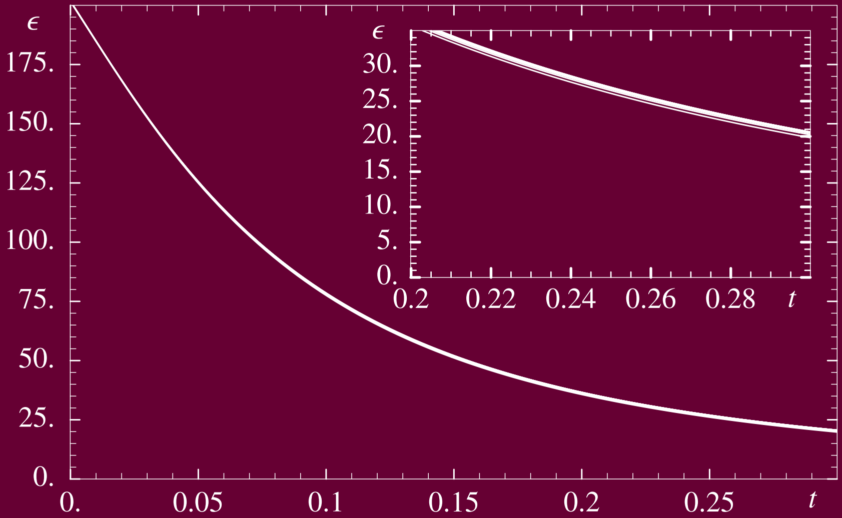
$\epsilon(t)$ in a Universe with Λ + B-fields + matter



$\epsilon(t)$ in a Universe with Λ + strings + matter



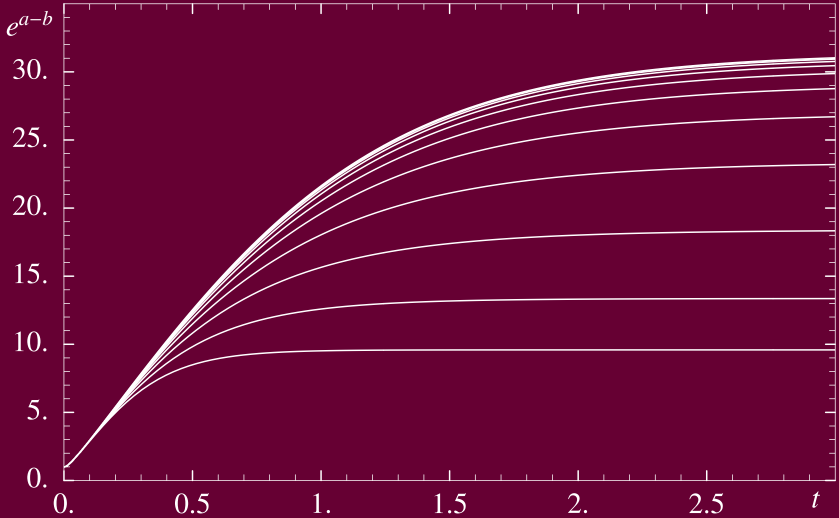
$\epsilon(t)$ in a Universe with Λ + walls + matter



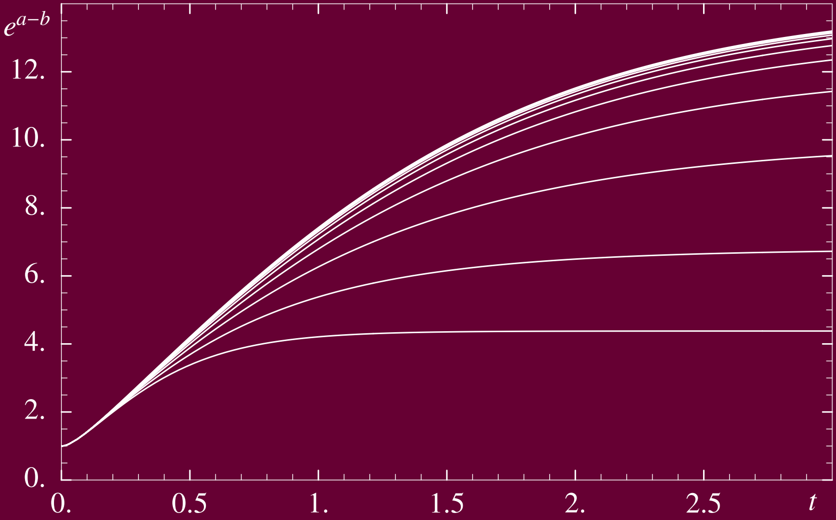
Pseudo-eccentricity e^{a-b}

e^{a-b} for (M, S and W) + Λ + w with $\lambda = 1$, $\rho_{\mathrm{i}} = 10$, $\epsilon_{\mathrm{i}} = 200$. Curves are for w from -1 to 1 with step 0.2 from top to bottom for M and S and from bottom to top for W.

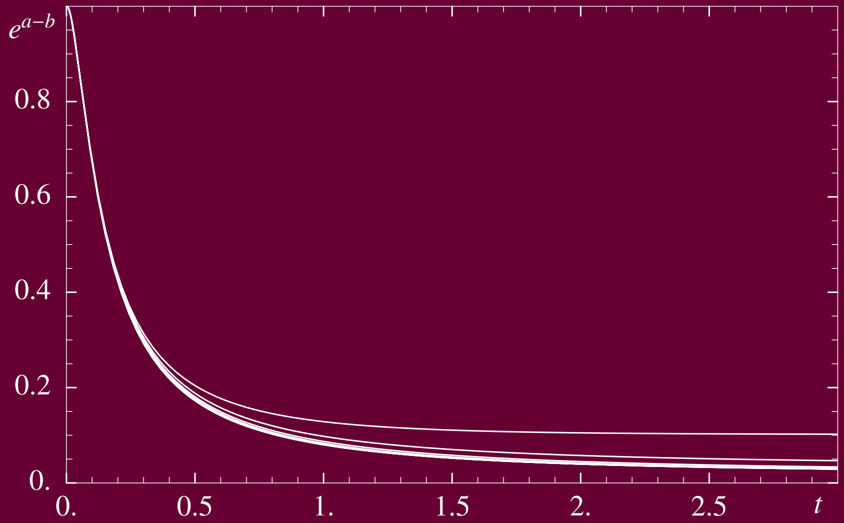
Pseudo-eccentricity in a Universe with Λ + B-fields + matter



Pseudo-eccentricity in a Universe with Λ + strings + matter



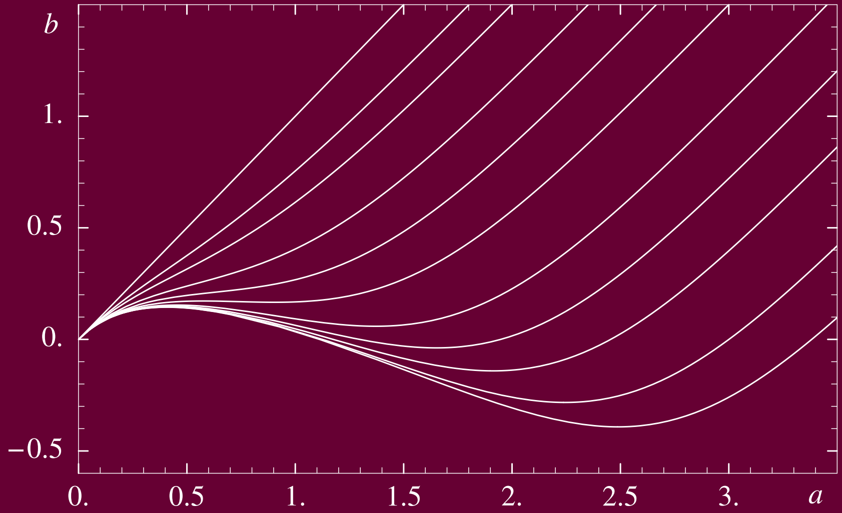
Pseudo-eccentricity in a Universe with Λ + walls + matter



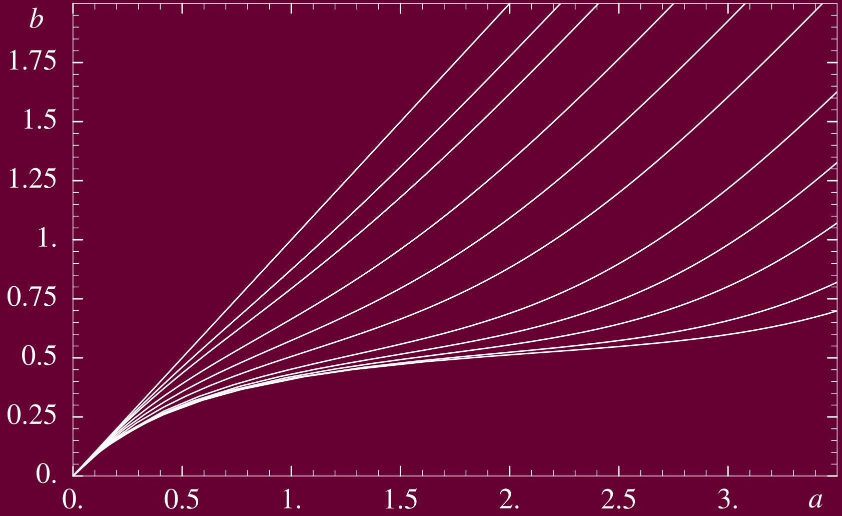
a vs. b

Expansion parameters a vs b for the case (M, S and W) + Λ with $\lambda = 1$, $\rho_{\mathrm{i}} = 0$. Curves are for $\epsilon_{\mathrm{i}} = 0, 1, 2, 5, 10, 20, 50, 100, 200, 500, 1000$ from top to bottom for M and S and from top to bottom for W.

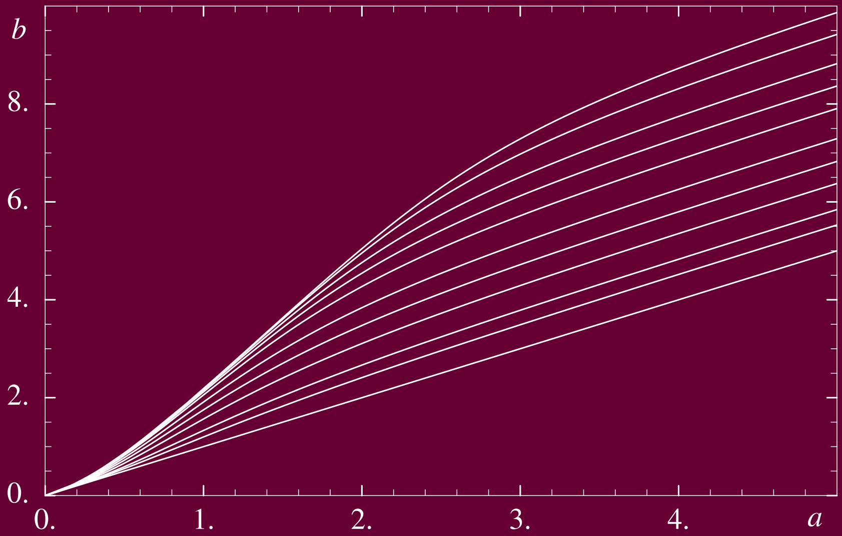
a(t) vs b(t) in a Universe with Λ + B-fields+ matter



a(t) vs b(t) in a Universe with Λ + strings + matter



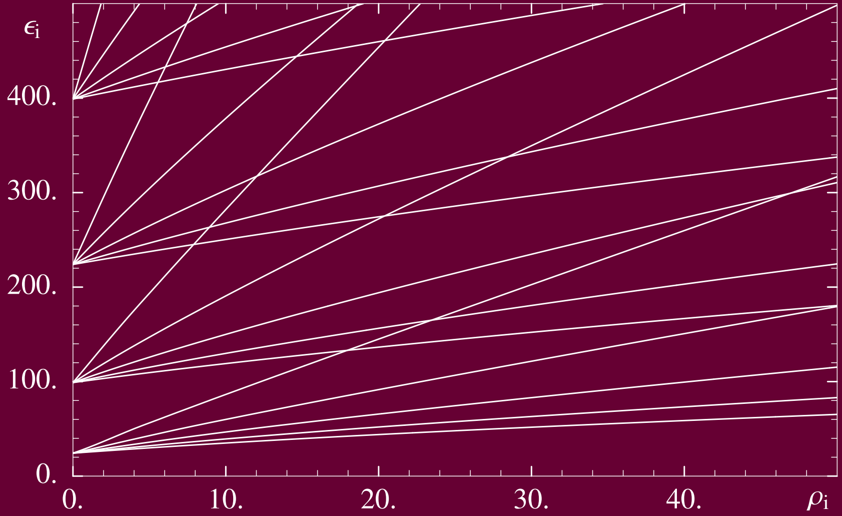
a(t) vs b(t) in a Universe with Λ + walls + matter



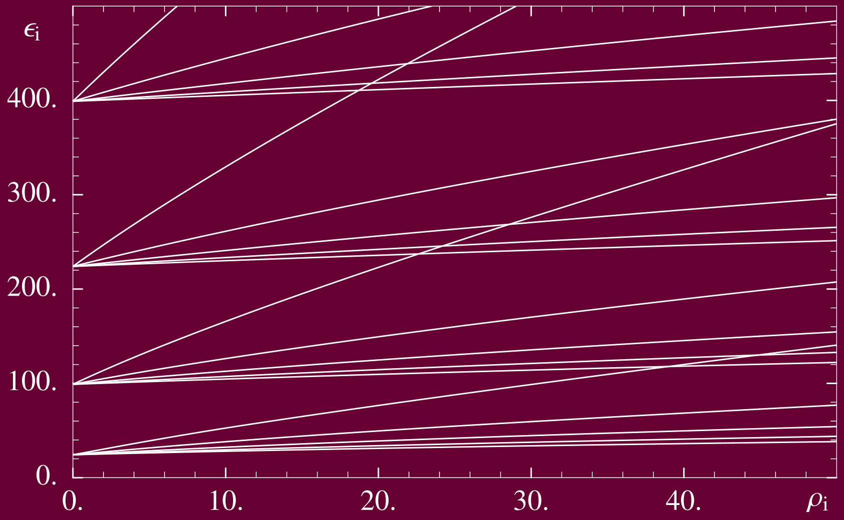
Asymptotic value of pseudo-eccentricity

Asymptotic value of the pseudo-eccentricity for the case (M, S and W) $+ \Lambda + w$ with $\lambda = 1$ as a function of ρ_{i} and ϵ_{i}. Sets of curves are for e^{a-b} equal to $20, 15, 10, 5$ from top to bottom; the abscissa corresponds to $\epsilon^{a-b} = 1$. Curves in each set are for w equal to $-0.5, -0.25, 0, 0.25, 0.5$ from top to bottom.

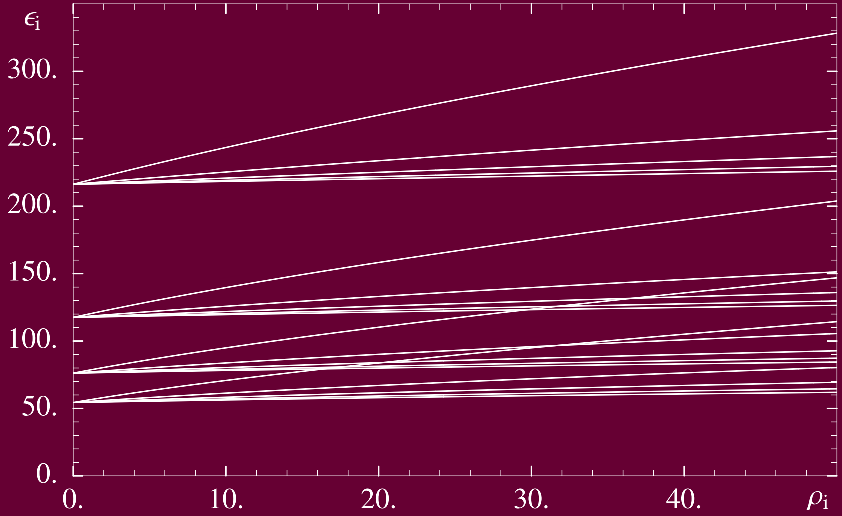
e^{a-b} for $t \to \infty$ in a Universe with Λ + B-fields + matter as a function of ρ_i and ϵ_i



e^{a-b} for $t \to \infty$ in a Universe with Λ + strings + matter as a function of ρ_i and ϵ_i



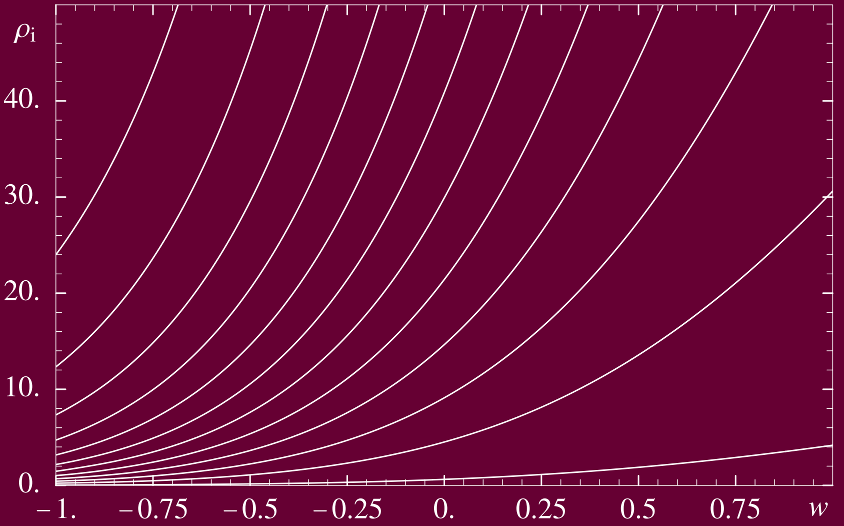
e^{a-b} for $t \to \infty$ in a Universe with Λ + walls+ matter as a function of ρ_i and ϵ_i



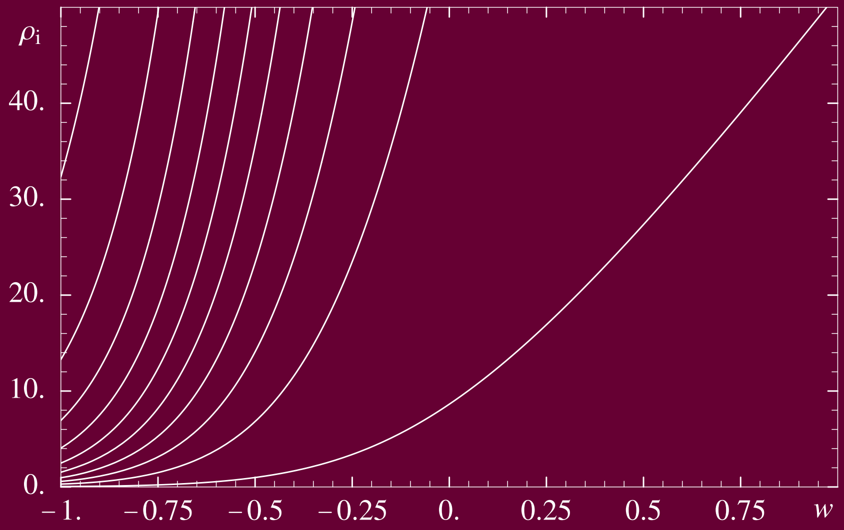
Asymptotic value of pseudo-eccentricity

Asymptotic value of the pseudo-eccentricity for the case (M, S and W) + Λ + w with $\lambda = 1$ as a function of w and $\rho_{\rm i}$ for $\epsilon_{\rm i} = 200$. Curves are for e^{a-b} from 4 to 22 with step 2 from top to bottom for M and S and from top to bottom for W.

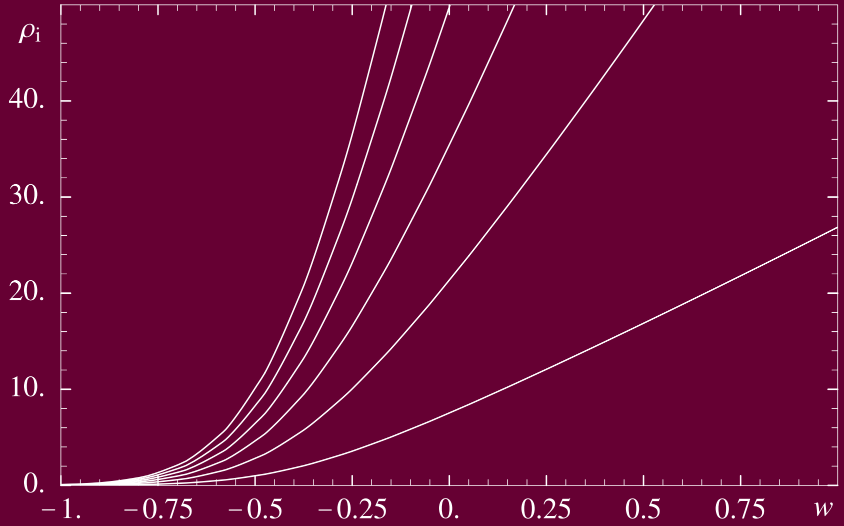
e^{a-b} for $t \to \infty$ in a Universe with Λ + B-fields + matter as a function of w and ϵ_i



e^{a-b} for $t \to \infty$ in a Universe with Λ + strings + matter as a function of w and ϵ_i



e^{a-b} for $t \to \infty$ in a Universe with Λ + walls + matter as a function of w and ϵ_i



Asymptotics: Planar Symmetry

For each choice of an anisotropic component, magnetic fields (M), strings (S) or walls (W), matter with $w = 0$ or with $0 < w < 1$ is included and cosmological constant is either present (Λ) or absent. Only the leading terms in asymptotics are given and $\tilde{t} = (\lambda/3)^{\frac{1}{2}} t$.

Asymptotics: Planar Symmetry

Table: Summary of large-time behavior for a Universe with uniform B-Fields.

	ϵ	ρ	e^a	e^b	e^{a-b}
$M\Lambda w$	$e^{-4\tilde{t}}$	$e^{-3(1+w)\tilde{t}}$	$e^{\tilde{t}}$	$e^{\tilde{t}}$	≥ 1
$M\Lambda 0$	$e^{-4\tilde{t}}$	$e^{-3\tilde{t}}$	$e^{\tilde{t}}$	$e^{\tilde{t}}$	≥ 1
Mw	$t^{-\frac{8}{3}}$	t^{-2}	$t^{\frac{2}{3}}$	$t^{\frac{2(1-2w)}{3(1+w)}}$	$t^{\frac{2w}{1+w}}$
$M0$	$t^{-\frac{8}{3}}$	t^{-2}	$t^{\frac{2}{3}}$	$t^{\frac{2}{3}}$	≥ 1

Asymptotics: Planar Symmetry

Table: Summary of large-time behavior with Strings and Walls.

	ϵ	ρ	e^a	e^b	e^{a-b}
$S\Lambda w$	$e^{-2\tilde{t}}$	$e^{-3(1+w)\tilde{t}}$	$e^{\tilde{t}}$	$e^{\tilde{t}}$	≥ 1
$S\Lambda 0$	$e^{-2\tilde{t}}$	$e^{-3\tilde{t}}$	$e^{\tilde{t}}$	$e^{\tilde{t}}$	≥ 1
Sw	t^{-2}	t^{-2}	t	$t^{-\frac{2w}{1+w}}$	$t^{\frac{1+3w}{1+w}}$
$S0$	t^{-2}	t^{-2}	t	t^{-2}	≥ 1
$W\Lambda 0$	$e^{-\tilde{t}}$	$e^{-3\tilde{t}}$	$e^{\tilde{t}}$	$e^{\tilde{t}}$	≤ 1
$W0$	t^{-2}	$t^{-\frac{10}{3}}$	$t^{\frac{2}{3}}$	t^{2}	$t^{-\frac{4}{3}}$

Density Perturbations: Planar Symmetry

Synchronous Gauge:

$$\delta g_{00} = 0,\ \delta g_{i0} = 0,\ \delta g_{ij} = e^{a_i + a_j} h_{ij}.$$

Variations of the Christoffel symbols:

$$\begin{aligned}
&\delta\Gamma^{\mu}_{00} = 0, \quad \delta\Gamma^{0}_{i0} = 0, \\
&\delta\Gamma^{0}_{ij} = -\tfrac{1}{2} e^{a_i + a_j} \left[(\dot{a}_i + \dot{a}_j) h_{ij} + \dot{h}_{ij} \right], \\
&\delta\Gamma^{i}_{j0} = \tfrac{1}{2} e^{a_j - a_i} \left[(\dot{a}_j - \dot{a}_i) h_{ij} - \dot{h}_{ij} \right],
\end{aligned}$$

$$\delta\Gamma^{k}_{ij} = \tfrac{1}{2} \left(e^{a_i + a_j - 2a_k} h_{ij,k} - e^{a_j - a_k} h_{kj,i} - e^{a_i - a_k} h_{ki,j} \right).$$

Density Perturbations: Planar Symmetry

Variations of the Ricci tensor:

$$\delta R_{00} = \sum_k \left(\tfrac{1}{2}\ddot{h}_{kk} + \dot{a}_k \dot{h}_{kk} \right),$$

$$\begin{aligned} \delta R_{i0} \quad &= \quad \sum_k \Big\{ \tfrac{1}{2}\dot{h}_{kk,i} - \tfrac{1}{2}\dot{a}_i \, h_{kk,i} + \tfrac{1}{2}\dot{a}_k h_{kk,i} \\ &\quad + \tfrac{1}{2} e^{a_i - a_k} \left[(\dot{a}_i - \dot{a}_k) h_{ik,k} - \dot{h}_{ik,k} \right] \Big\}, \end{aligned}$$

Density Perturbations: Planar Symmetry

Variations of the Ricci tensor δR_{ij}:

$$
\begin{aligned}
\delta R_{ij} \quad = \quad & \tfrac{1}{2} e^{a_i + a_j} \Bigg\{ -\ddot{h}_{ij} - \delta_{ij} \dot{a}_i \dot{h} - \dot{h}_{ij} \sum_k \dot{a}_k \\
& -h_{ij} \Bigg[\ddot{a}_i + \ddot{a}_j - (\dot{a}_i - \dot{a}_j)^2 + (\dot{a}_i + \dot{a}_j) \sum_k \dot{a}_k \Bigg] \Bigg\} \\
& + \tfrac{1}{2} \sum_k \Big(h_{kk,ij} + e^{a_i + a_k - 2a_k} h_{ij,kk} - e^{a_j - a_k} h_{jk,ik} - e^{a_i - a_k} h_{ik,jk} \Big) .
\end{aligned}
$$

Density Perturbations: Planar Symmetry

Newtonian Approximation: Peculiar velocities are small–drop time derivatives compared to space derivatives ($\partial_t = v\partial_x \ll \partial_x$ since $v \ll 1$).

Variation of Einstein Equation in Newtonian Approximation:

$$\sum_k \left(\ddot{h}_{kk} + 2\dot{a}_k \dot{h}_{kk} \right) = \delta\rho, \tag{1}$$

$$\sum_k \left\{ \dot{h}_{kk,i} - \dot{a}_i h_{kk,i} + \dot{a}_k h_{kk,i} + e^{a_i - a_k} \left[(\dot{a}_i - \dot{a}_k) h_{ik,k} - \dot{h}_{ik,k} \right] \right\} = 0, \tag{2}$$

$$\sum_k \left(h_{kk,ij} + e^{a_i + a_k - 2a_k} h_{ij,kk} - e^{a_j - a_k} h_{jk,ik} - e^{a_i - a_k} h_{ik,jk} \right) = 0. \tag{3}$$

Density Perturbations: Isotropic Case

Newtonian Approximation: Again $\partial_t = v\partial_x \ll \partial_x$ since $v \ll 1$.

Variations in Newtonian Approximation: Isotropic implies $a_1 = a_2 = a_3 = \ln R$ so that $\frac{\dot{R}}{R} = \dot{a}_i$. With the definition $h = h_{kk}$ we find

$$\ddot{h} + 2\frac{\dot{R}}{R}\dot{h} = \delta\rho,$$

$$\dot{h}_{,i} - \dot{h}_{ik,k} = 0,$$

$$h_{,ij} + h_{ij,kk} - h_{jk,ik} - h_{ik,jk} = 0.$$

Compare Padmanabhan, "Structure Formation in the Universe" p-224 (recall we absorb a factor of $8\pi G$ into $\delta\rho$.)

Density Perturbations: Isotropic Solutions

Solving for the Metric Variation h_{ij}: spherical symmetry ansatz

$$h_{ij} = -\frac{1}{4\pi}\partial_i\partial_j \int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\dot{h}, \quad i,j = 1,2,3$$

Newtonian potential due to density contrast:

$$\phi = -\frac{1}{4\pi}\rho R^2 \int \frac{\delta(\mathbf{x}',t)d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

and since ρR^3 is constant:

$$\frac{\partial(\phi R)}{\partial t} = -\frac{1}{4\pi}\rho R^3 \int \frac{\dot{\delta}(\mathbf{x}',t)d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

Density Perturbations: Isotropic Case

Sachs-Wolfe Effect: Further analysis needed to relate $\delta\rho$ to temperature variations. Find

$$\frac{\delta T}{T_0} = \mathbf{n} \cdot (\mathbf{v}_{ob} - \mathbf{v}_{em}) - \frac{1}{3}(\phi_0 - \phi_{em})$$

where $\mathbf{v}_{ob}$ and $\mathbf{v}_{em}$ are the peculiar velocities of the observer and emitting surface (last scattering surface) and the last term is due to the variations of the potential at the observation point and at the source.

Density Perturbations: Planar Symmetry Solutions

Solving for the Metric Variation h_{ij}: planar symmetry ansatz

$$\begin{aligned} h_{ij} &= \partial_i \partial_j \alpha, \quad i, j = 1, 2, \\ h_{33} &= \partial_3 \partial_3 \beta, \\ h_{i3} &= \partial_i \partial_3 \gamma, \quad i = 1, 2, \end{aligned}$$

where α, β and γ are functions of space-time. Eq. (3) is satisfied by

$$\gamma = \frac{1}{2}(\alpha + \beta).$$

With this relation, Eq. (2) leads to

$$F \equiv \dot{\alpha} - 2\dot{a}\alpha = \dot{\beta} - 2\dot{b}\beta.$$

Density Perturbations: Planar Symmetry Solutions

From Eq. (1) we find

$$\left[e^{-2a}(\partial_1\partial_1 + \partial_2\partial_2) + e^{-2b}\partial_3\partial_3\right]\dot{F} = \delta\rho.$$

Assuming $\delta\rho$ has an associated potential ϕ yields

$$4\pi\left[e^{-2a}(\partial_1\partial_1 + \partial_2\partial_2) + e^{-2b}\partial_3\partial_3\right]\phi = \delta\rho.$$

Comparing gives

$$\dot{F} = 4\pi\phi.$$

Density Perturbations: Planar Symmetry Solutions

Sachs-Wolfe Effect: Further numerical analysis needed to relate $\delta\rho$ to temperature variations and find

$$\left(\frac{\delta T}{T_0}\right)_\theta = f_\theta(\mathbf{n} \cdot (\mathbf{v}_{ob} - \mathbf{v}_{em})) + g_\theta((\phi_0 - \phi_{em})),$$

where $\mathbf{v}_{ob}$ and $\mathbf{v}_{em}$ are the peculiar velocities of the observer and emitting surface (last scattering surface), and the last term is due to the variations of the potential at the observation point and at the source.