Quantum Entanglement & its implications in Theoretical Physics

1. bi-partite quantum entanglement
2. Quantum entanglement in Many-body systems
3. evaluation of entanglement entropy

Topics:
- Basics of EE
- Topological order
- Firewall
- Replica trick
- Holographic EE

I. bi-partite entanglement

product states: $|00\rangle$, $|11\rangle$

$$|00\rangle + |01\rangle + |10\rangle + |11\rangle$$
$$= (|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$$

$$|00\rangle + |01\rangle = |0\rangle(|0\rangle + |1\rangle)$$

entangled states: $\left.\begin{matrix} |00\rangle + |11\rangle \\ |01\rangle + |10\rangle \end{matrix}\right\}$ Bell state

Quantify bi-partite entanglement:

(i) Schmidt decomposition

$$|\psi\rangle = \sum_i c_i |i\rangle_A |\tilde{i}\rangle_B$$

# of non-zero c_i = Schmidt rank

Schmidt rank of $\begin{cases} \text{product state} = 1 \\ \text{Bell state} = 2 \end{cases}$

(ii) Von Neumann entropy of the reduced density matrix

$$\rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi|$$

$$S_e = -\rho_A \log \rho_A$$

$$\rho_A^{(\text{product})} = |0\rangle\langle 0| \rightarrow S_e^{(\text{product})} = 0$$

$$\rho_A^{(\text{Bell})} = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$\rightarrow S_e^{(\text{Bell})} = 2$$

Maximal

Bell state = maximally entangled state!

(iii) CHSH function

Alice has two observables A_0, A_1

Bob " " " B_0, B_1

$$CHSH = |\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle|$$

- Local hidden variable $\Rightarrow$ $CHSH \le 2$
- Quantum Mechanics $\Rightarrow$ $CHSH \le 2\sqrt{2}$

  " = " when the quantum state is the Bell state.

- Causality (no-signaling) $\Rightarrow$ $CHSH \le 4$

Why Q.M. is special among no-signaling theories?

A proposopal: Information causality

— Information gain over a non-local correlation cannot succeed the amount of information send by Alice to Bob

(No predictive power of future by non-local correlation)

(iv) local unitary operations

product state $\xleftrightarrow[\quad]{\text{1-party unitary}}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\times\quad\quad\quad\;$ Bell state

Product state $\xleftrightarrow{\text{bi-partite unitary}}$ Bell state

$\Rightarrow$ One can classify quantum entanglement by classifying LUs

(related to topological order)

(V) Mutual information

$$I_2(A:B) = S(A) + S(B) - S(AB)$$

e.g. product state $|00\rangle$.

$$\Rightarrow I_2(A:B) = 0$$

e.g. Bell's state $|00\rangle + |11\rangle$

$$I_2(A:B) = 1 + 1 - 0 = 2$$

II. Quantum entanglement in Many-body systems

1. Not much understanding about the multi-partite quantum entanglement e.g. how to quantify?

Except: Quantum entanglement has monogomy property — If A & B are maximally entangled, then neither A nor B can be entangled with a 3rd party C.

Monogamy of Entanglement:

For tri-partite system: A, B, C

the entanglement measure $\mathcal{E}$ should satisfy

$$\mathcal{E}_{A:B} + \mathcal{E}_{A:C} \leq \mathcal{E}_{A:BC}$$

e.g. $\mathcal{E} = I_2$ (for pure state only)

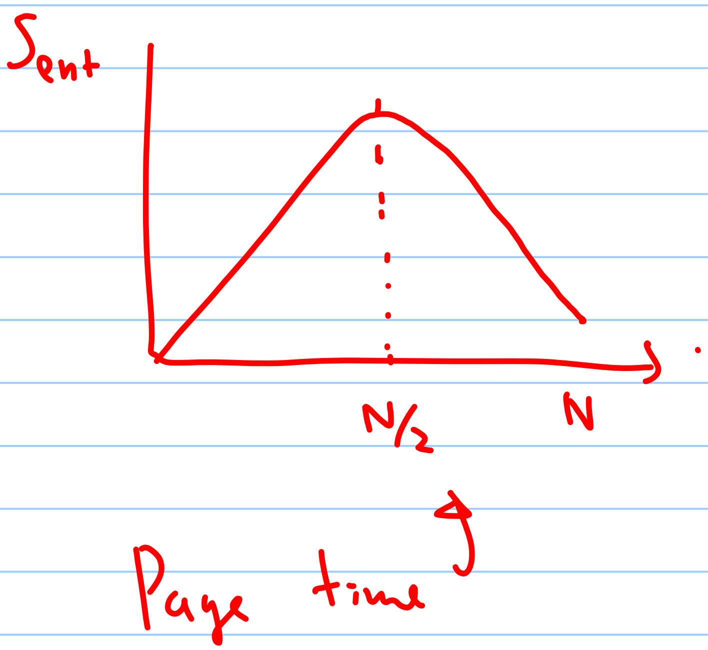
for $|GHZ\rangle = |000\rangle + |111\rangle$.

$\Rightarrow I_2(A:B) = I_2(A:C) = 1$, $I(A:BC) = 2$

for $(\underbrace{|00\rangle}_{AB} + \underbrace{|11\rangle}_{AB}) \otimes \underbrace{|0\rangle}_{C}$.

$\Rightarrow \quad I_2(A:B) = I_2(A:BC) = 1$, $I_2(A:C) = 0$

This property raise the issue of firewall recently by the observation: After Page's time, the black hole is maximally entangled with early time Hawking radiation, so that it cannot be entangled with late time Hawking radiation.



AMPS argument :

free falling observer

$$(|00\rangle + |11\rangle) \otimes |R\rangle$$

Hawking pair — earlier Hawking radiation

Asymptotic observer

$$((|00\rangle + |11\rangle) \otimes |0\rangle \otimes |R/_{\gamma}\rangle$$

inside BH — earlier HR — late HR — rest of earlier HR

$\times$ by monogamy of entanglement !!

$\Rightarrow$ Give up the smoothness of the near horizon regime seen by free falling observer (Minkowski vaccum) as implicity assumed in formulating the complementarity principle of black hole.

2. Folklore: EE of ground state of many-body system obey area law:

i.e., $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

in $(d+1)$-dim.

$$S_A \sim \frac{\text{Area of entangling hypersurface}}{\epsilon^{d-1}} + \dots$$

ϵ = UV cutoff

For pure state $S_A = S_B$

For $(1+1)$-dim. CFT

$$S_L \sim \frac{c}{3} \log L/\epsilon + \dots$$

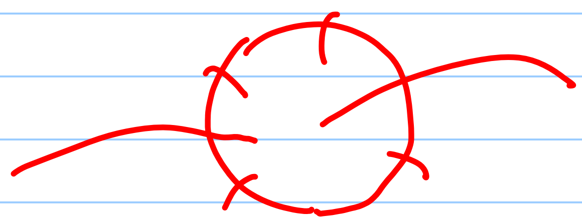
In fact, the UV-independent refinement UV also contains useful information such as RG flow of c functions. (Hertzberg & Wilczek)

※ For (1+1)-dim. deformed CFT,

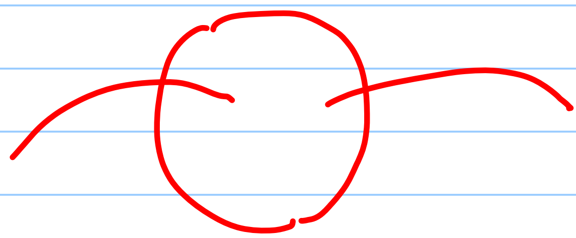
c function is the UV-independent refinement of EE

$\Rightarrow$ RG flow of EE is monotonically decreasing according to c theorem.

This indicates the local nature of EE for ground state:



$\downarrow$ RG (coarse-graining)



* For a thermal system, the UV-divergent piece obeys area law, but the refinement obey area law at UV regime and volume law at IR regime.

$\downarrow$ thermal entropy

* If the refinement contains the scale-independent piece which cannot be removed by coarse graining.

This is called long range entanglement

— signature of intrinsic topological order

e.g. (2+1)-dim. topological order such FQHE or toric-code

$$S_e \sim \alpha A - \gamma$$

$\uparrow$ topological EE (Kitaev-Preskill)

For string net picture of toric code GS: (simplest Z_2 TO state)

$$|\Psi\rangle = \sum_{\{c\}} |c\rangle$$

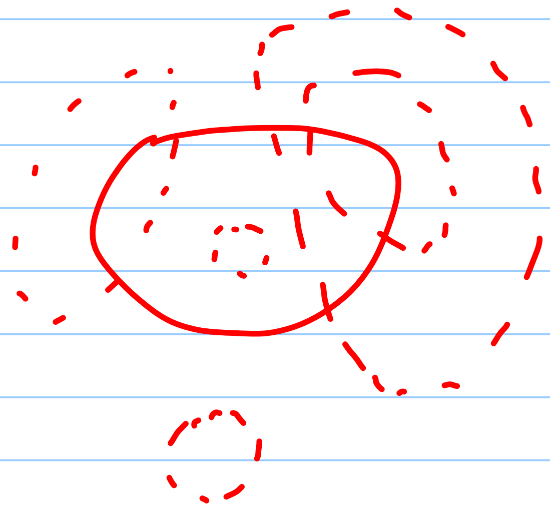
= equal weight supperposition of all closed string configurations

$\uparrow$

a highly entangled state

$\Rightarrow \gamma \sim$ even # of crossings of the entangling surface

$$\gamma = \ln_2 2 = 1$$



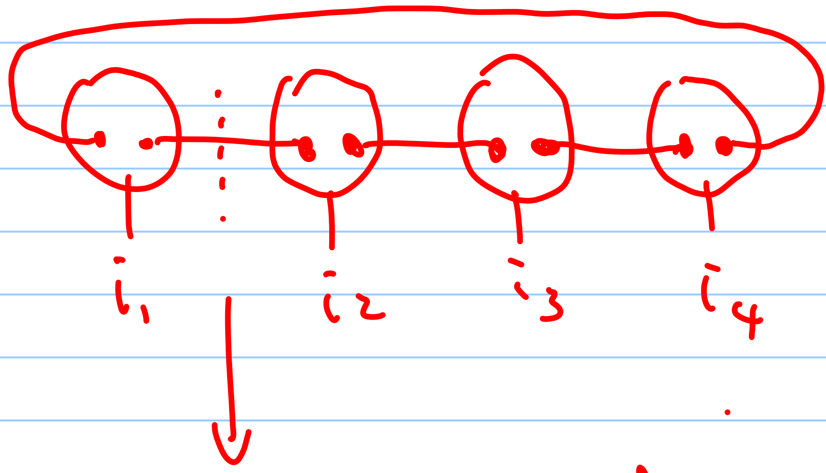
robust against small perturbations

3. Matrix product state (MPS)

a way to expose the short range entanglement (SRE) of

GS is the MPS ansatz

$$|\psi\rangle = \sum_{i_1,\dots,i_N} \mathrm{Tr}\left(A^{i_1} A^{i_2} \cdots A^{i_N}\right) |i_1, i_2, \dots, i_N\rangle$$

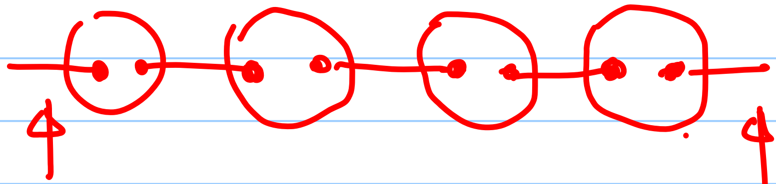


Neighboring virtual particles are entangled

$\Rightarrow$ this ensure area law of 1D system

Moreover, the virtual particle could be in projective representation

$\Rightarrow$ For open chain, one may have "fractional" edge states.

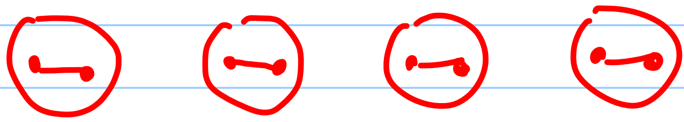


dangling edge mode

in projective repres. of on-site symmetry

This is called symmetry-protected topological order (SPT).

In contrast, a trivial product state



Also, no dangling edge modes

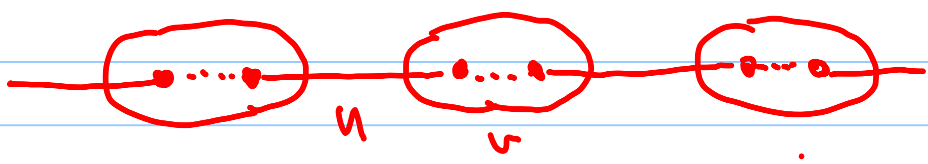
$\Rightarrow$ classify SPT by group cohomology

On the other hand, SPT phase has interesting non-perturbative TQFT phenomenon such as Witten effect.

1D topological superconductor is the physical realization of the above picture.

= Dirac fermion $(a, a^\dagger)$

Majorana fermions.
$$\begin{cases} \gamma_1 = (a + a^\dagger)/2 \\ \gamma_2 = (a - a^\dagger)/2i \end{cases}$$



$$\gamma_1^\dagger = \gamma_1$$
$$\gamma_2^\dagger = \gamma_2$$

tuning coupling u, v:

dimer state → SPT

w/o robust edge mode — w/ Majorana edge mode

Hilbert space of Majorana fermion

$= [\text{Hilbert space of Dirac fermion}]^{1/2}$

⇒ nontrivial statistics

⇒ Majorana fermion = Ising non-Abelian anyon

$$\boxed{\begin{aligned} \gamma_1 &\to \gamma_2 \\ \gamma_2 &\to -\gamma_1 \end{aligned}}$$

entities for topological quantum computation

How to understand entanglement in the language of QFT?

4. MERA (or QSRG)

One can remove SRE by LUs (disentangler)

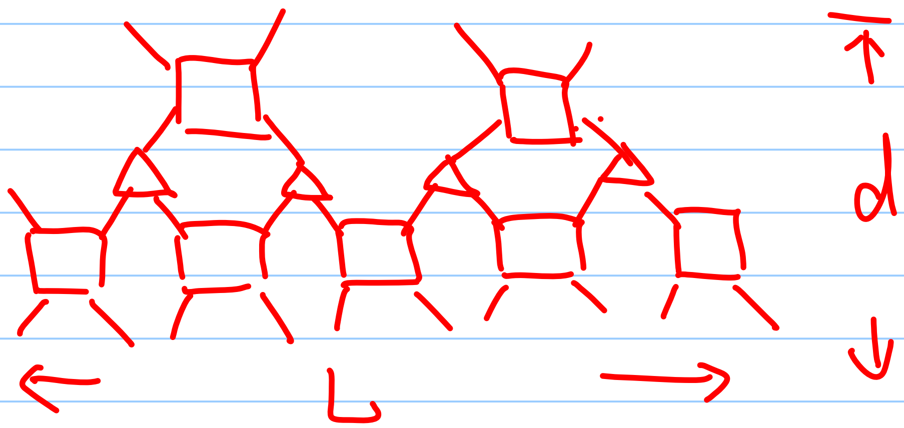
$\Rightarrow$ pattern of $\begin{cases} \text{SPT} \\ \text{TO} \end{cases}$

= pattern of $\begin{cases} \text{Symmetric SRE} \\ \text{LRE} \end{cases}$

= pattern of $\begin{cases} \text{Symmetric LUs} \\ \text{LUs} \end{cases}$

Furthermore, one can do coarse graining by merging neighboring site (isometry)

$\Rightarrow$ This forms a RG procedure for quantum state w/ SRE removed.



$\square$ = disentangler

$\triangle$ = isometry

For 1D CFT, one has $d \sim \ln L$

which resemble "AdS" geometry.

It is called "multi-scale entanglement renormalization ansatz" (MERA)

This is the starting point of MERA/AdS

More evidence like power law of correlation functions is inherited from geometric picture.

More details see Evenbly + Vidal

A deep question:

Could entanglement renormalization yield an elegant way of deriving bulk EOM of AdS space from RGE of dual CFT?

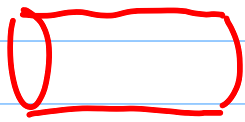
Key issue: No Canonical RG "frame" once away from fixed point.

Maybe need entanglement to fix the "frame".

Simple examples are discussed in the formulation of continuous MERA.

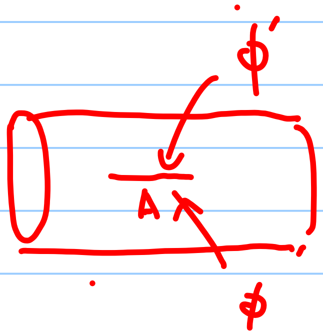
III. Evaluation of entanglement entropy

1. Replica method for free QFT

$Z \sim$ 

path integral w/ periodic condition imposed along euclidean time direction

reduced density matrix

$\langle \phi | \rho_A | \phi' \rangle \sim$ 

The entanglement entropy

$$S_A = - \mathrm{Tr}\, \rho_A \log \rho_A$$

$$= - \lim_{n \to 1} \frac{\partial}{\partial n} \mathrm{Tr}\, \rho_A^n$$

$$= \lim_{n \to 1} S_A^{(n)}$$

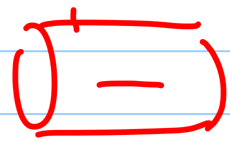
Rényi entropy

$$S_A^{(n)} = \frac{1}{1-n} \ln \mathrm{Tr}\, \rho_A^n$$

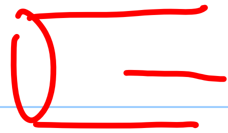
Sometimes, the mutual information

$$I_{A:B}^{(n)} = S_A^{(n)} + S_B^{(n)} - S_{A \cup B}^{(n)}$$

Is an interesting derived quantity.

To evaluate $\mathrm{Tr}\, \rho_A^n$, we need to glue n copies of 

With b.c. $\phi_{i+1} = \phi_i' \pmod n$

For bi-partition case, i.e. 

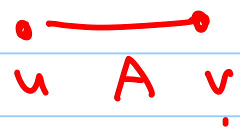
$\mathrm{Tr}\,\rho_A^n$ can be thought as a path integral over the fields on a cone ($\otimes M_\perp$) with conic deficit $\delta = 2\pi\left(1-\frac{1}{n}\right)$.

Then, it is to extract the saddle point contribution from the tip. One should carefully regularize the cone.

In the end of days, we will get the area law

$$S_A \sim \frac{A}{\epsilon^{d-1}}$$

On the other hand, one can introduce the twist operator T_n & $\tilde{T}_n$ around the branch points



to take care the b.c. $\phi_{i+1} = \phi_i'$

define $\tilde{\phi}_k = \sum_j e^{2\pi i \frac{k}{n} j} \phi_j$

$$\Rightarrow \left\{ \begin{array}{l} T_n \tilde{\phi}_k = e^{2\pi i \frac{k}{n}} \tilde{\phi}_k \\ \tilde{T}_n \tilde{\phi}_k = e^{-2\pi i \frac{k}{n}} \tilde{\phi}_k \end{array} \right.$$

$$\Rightarrow \quad \mathrm{Tr}\, \rho_A^{(n)} = \langle T_n(u)\, \tilde{T}_n(v) \rangle_{\mathbb{C}}$$

complex plane

For (H1)-CFT, we assume the twist op to be chiral primaries

Then, use the conformal mapping

from cone to complex plane $\mathbb{C}$, and the conformal Ward identity, one find

$$\langle T_n(u) \tilde{T}_n(v) \rangle_{\mathbb{C}} \sim |u-v|^{-2d_n}$$

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

$$\Rightarrow \quad S_A^{(n)} = \frac{c}{6}\left(1+\frac{1}{n}\right) \log \frac{|u-v|}{\epsilon}$$

$$\xrightarrow{n\to 1} \frac{c}{3} \log \frac{|u-v|}{\epsilon}$$

In fact, the EE has only been evaluated for few models.

e.g. Model with Fermi surface is conjectured to have log div correction besides area law.

e.g. EE for interacting theory is hard.

A more detailed way of characterizing the quantum entanglement is the entanglement spectrum, i.e. the eigen-spectrum of H_A with

$$\rho_A = e^{-H_A}.$$

H_A is related to the Hamiltonian for the edge modes. This fact is exploited in "deriving" the bulk/edge correspondence for TO or SPT states.

Moreover, using the fact

$$C_{ij}^{(A)} = \mathrm{Tr}\left(\rho_A \hat{C}_{ij}^{(A)}\right)$$

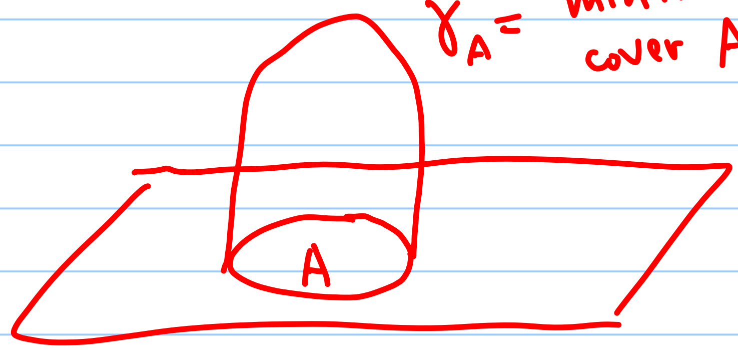
One can derive the S_A for free fermions & free bosons.

2. Holographic EE

Ryu-Takayanagi formula

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$

$\gamma_A =$ AdS minimal surface cover A

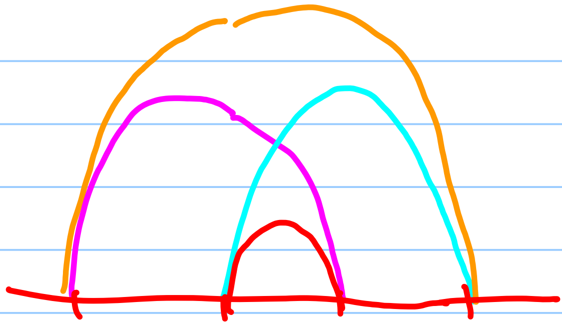


This formula has passed many tests:

(i) for (1+1)-CFT, its dual is AdS_3 or BTZ black hole.

It agrees with CFT results exactly.

(ii) Strong additivity of EE has simple geometric interpretation



$$S(AB) + S(BC) \geq S(ABC) + S(B)$$
$$\geq S(A) + S(C)$$

(iii) Refinements agree with c functions or c-theorem

(iv) Work for time-dependent case such de Sitter space.

Some open issues

(a) Not work well for entangling surface w/ extrinsic curvature (due to inconsistent Graham - Witten anomaly)

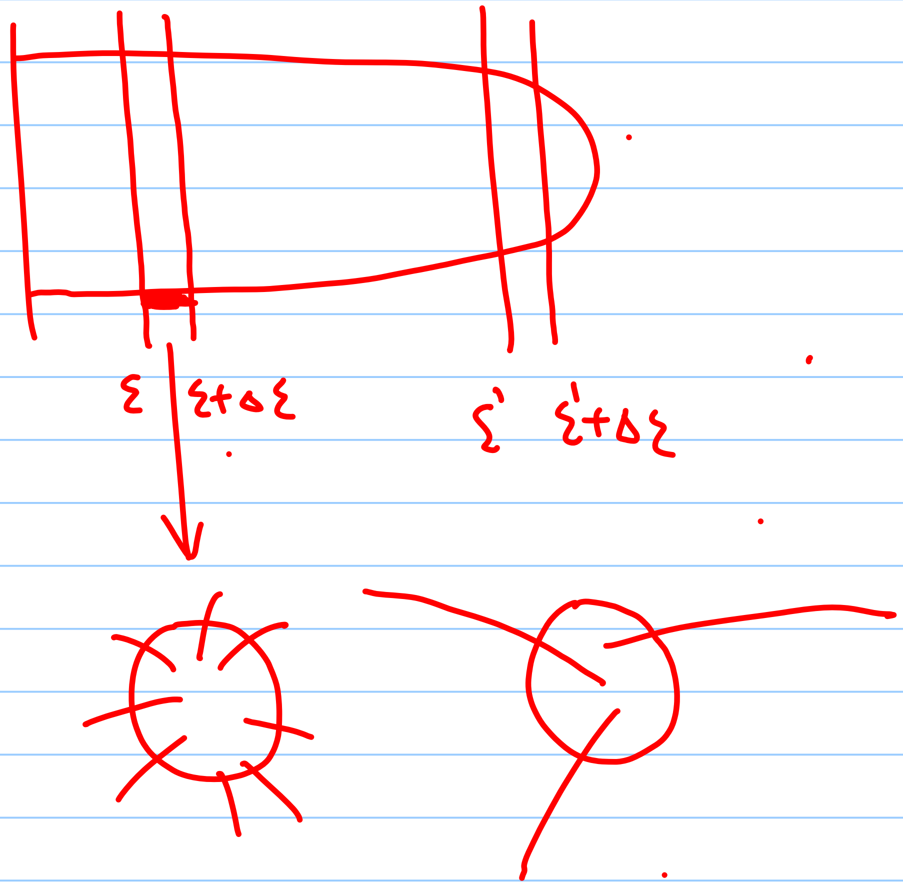
(b) Lack a 1st principle derivation of this formula

(c) Only work for dual gapless systems, while many interesting CMT systems are gapped.

(d) Cannot incorporate the deconfined d.o.f.s (changes outside AdS Black Hole)

— Scale-dependent entanglement

The holographic EE



Suggests a picture of scale-dependent entanglement

$\downarrow$

motivate a scale-dependent Schmidt decomposition

$$|\psi\rangle = \int ds \sum_i c_i(s)\, |i\rangle_s$$

$$\downarrow$$

or $$\rho_A = e^{-\int ds\, \mathcal{H}_A(s)}$$

Similar to cMERA

Many open & important questions remains.

In the past few years, I have learned the topics by working on some related projects.

Here is the list I have worked on:

a) Refine EE & its RG flow for holographic gapped systems

b) Classify SPT for holographic CFTs (With Shih-Hao)

c) SPT & Majorana zero modes in Jackiw-Rebbi model (With SH)

d) Inflation & non-cloning thm

e) Information Causality & noisy computation

f) Quantum state RG for SPT

g) Quantum Decoherence in holographic CFTs
(on-going)

h) Topological order of J-J' model

i) Geometric EE in 1D & 2D lattice spin models