An Algebraic Approach to Color-Kinematic Duality and Formulations of Yang-Mills Amplitudes

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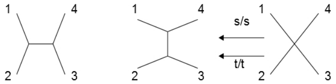
Based on work in collaboration with Yi-Jian Du and Bo Feng
arXiv[hep-th]:1212.6168, 1105.3503

Outline

- BCJ duality – A brief introduction
  - possible mechanism: gauge DOF? algebra? or...?
  - cubic prescription for YM amplitudes
- Duality and formulations of YM and gravity amplitudes
- Remaining thoughts

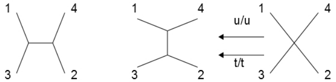
BCJ relation

- Definition of kinematic numerators through absorbing 4-pt contributions into cubic graphs



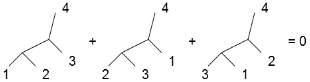
$$A(1234) = \frac{n_s}{s} - \frac{n_t}{t}$$

$$A(1324) = -\frac{n_u}{u} + \frac{n_t}{t}$$



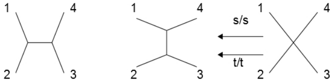
- Jacobi-like identity [Bern, Carrasco, Johansson(08)]

$$n_s + n_t + n_u = 0$$



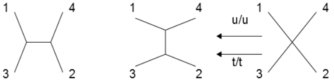
BCJ relation

- Definition of kinematic numerators through absorbing 4-pt contributions into cubic graphs



$$A(1234) = \frac{n_s + s\Delta}{s} - \frac{n_t + t\Delta}{t}$$

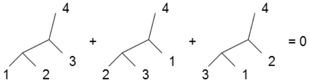
$$A(1324) = -\frac{n_u + u\Delta}{u} + \frac{n_t + t\Delta}{t}$$



- Jacobi-like identity [Bern, Carrasco, Johansson(08)]

$$n_s + n_t + n_u \quad + (s+t+u)\Delta = 0$$

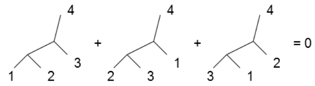
Generalized gauge invariance



BCJ relation

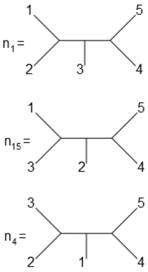
An algebraic-like identity is satisfied between kinematic dependent numerators

$$
\begin{aligned}
f^{12e} f^{e34} + f^{23e} f^{e14} + f^{31e} f^{e24} &= 0 \\
\updownarrow \quad & \\
n_s + n_t + n_u &= 0
\end{aligned}
$$

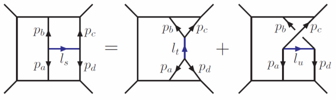


- BCJ numerators and Jacobi identities at 5-points (tree level):

$$
\begin{aligned}
A(12345) &= \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}, \\
A(14325) &= \frac{n_6}{s_{14}s_{25}} + \frac{n_5}{s_{43}s_{51}} + \frac{n_7}{s_{32}s_{14}} + \frac{n_8}{s_{25}s_{43}} + \frac{n_2}{s_{51}s_{32}}, \\
A(13425) &= \frac{n_9}{s_{13}s_{45}} + \frac{n_5}{s_{34}s_{51}} + \frac{n_{10}}{s_{42}s_{13}} - \frac{n_8}{s_{25}s_{34}} + \frac{n_{11}}{s_{51}s_{42}}, \\
A(12435) &= \frac{n_{12}}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{n_3}{s_{43}s_{12}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{43}}, \\
A(14235) &= \frac{n_{14}}{s_{14}s_{35}} - \frac{n_{11}}{s_{42}s_{51}} - \frac{n_7}{s_{23}s_{14}} - \frac{n_{13}}{s_{35}s_{42}} - \frac{n_2}{s_{51}s_{23}}, \\
A(13245) &= \frac{n_{15}}{s_{13}s_{45}} - \frac{n_2}{s_{32}s_{51}} - \frac{n_{10}}{s_{24}s_{13}} - \frac{n_4}{s_{45}s_{32}} - \frac{n_{11}}{s_{51}s_{24}},
\end{aligned}
$$



BCJ relation



Validity check at loop-level:

- $\mathcal{N} = 4$ SYM
  - At 4-pts, verified up to four loops

    [Bern, Carrasco, Johansson(10)]

    [Bern, Dixon,Dunbar,Perelstein, Rozowsky(98)]

    [Bern, Carrasco, Dixon, Johansson, Roiban(12)]
  - At 5-pts, up to three loops

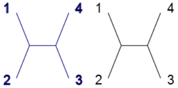
    [Carrasco, Johansson(12)]

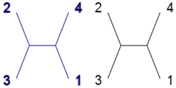
    [Yuan(12)]
- pure YM. Two-loop checked

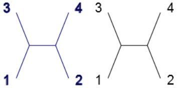
  [Bern, Carrasco, Johansson(10)]

A new version of KLT relations

Double-copy expression

$$M(1^\alpha, 2^\beta, 3^\gamma, 4^\delta) = \frac{1}{s}$$ 

$$+ \frac{1}{t}$$ 

$$+ \frac{1}{u}$$ 

$$= \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

- A simpler tree level fromula

$$M = \prod_{\text{cubic graphs } i} \frac{c_i n_i}{D_i}$$

- seems to generalize to loop levels!

$$M = \int \frac{d^D l_k}{(2\pi)^D} \prod_{\text{cubic graphs } i} \frac{c_i n_i(l_k)}{D_i(l_k)}$$

$\mathcal{N} = 8$ supergravity, 4-pts up to four loops

[Bern, Carrasco, Johansson(10)]

[Bern, Dixon,Dunbar,Perelstein, Rozowsky(98)]

[Bern, Carrasco, Dixon, Johansson, Roiban(12)]

5-pts, checked up to two loops

[Carrasco, Johansson(12)]

A new version of KLT relations

- Double-copy vs string low-energy limit KLT

  Heterotic string theory → A "color" KLT relation for example

$$M_{full\ YM} = \sum_{\alpha,\beta} \frac{\tilde{A}_{scalar}(n,\alpha,1)\mathcal{S}[\alpha|\beta]A_{YM}(1,\beta,n)}{s_{123\ldots n-1}}$$

[Kawai, Lewellen, Tye(86)]

[Bern, Freitas, Wong(00)]

[Bjerrum-Bohr, Feng, Damgaard, Sondargaard(10)]

[Bjerrum-Bohr, Damgaard, Sondargaard, Vanhove(10)]

$$\begin{aligned} M(1^{\alpha},2^{\beta},3^{\gamma},4^{\delta}) \quad = \quad & \frac{\tilde{A}(4321)s_{21}(s_{31}+s_{32})A(1234)}{s_{123}} + \frac{\tilde{A}(4321)s_{21}s_{31}A(1234)}{s_{123}} \\ & + \frac{\tilde{A}(4231)s_{21}s_{31}A(1234)}{s_{123}} + \frac{\tilde{A}(4321)(s_{21}+s_{23})s_{31}A(1324)}{s_{123}} \end{aligned}$$

Analytic construction of numerators

Algebra of generators of area-preserving diffeomorphism

- Self-dual YM
- Light-cone gauge YM

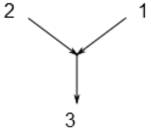
$$\begin{aligned} \mathrm{MHV}_{3-pt} &\rightarrow L_k = e^{-ik\cdot x}(-k_\perp \partial_+ + k_+ \partial_\perp), \\ \overline{\mathrm{MHV}}_{3-pt} &\rightarrow \bar{L}_k = e^{-ik\cdot x}(-\bar{k}_\perp \partial_+ + k_+ \bar{\partial}_\perp). \end{aligned}$$

[Bjerrum-Bohr, Damgaard, Monteiro, O'Connell(11)(12)]

Algerba of generators of diffeomorphism (in Fourier basis)

$$\begin{aligned} x^\mu &\rightarrow g^a(x) = x^a + \int d^D k\, \epsilon^a(k) e^{ik\cdot x} \\ f(x) &\rightarrow f(g^a(x)) = f(x) + \int d^D k\, \epsilon^a e^{ik\cdot x} \partial_a f(x) \end{aligned}$$

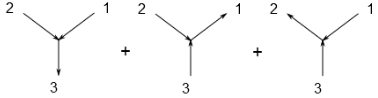
$$\begin{aligned} T^{k,a} &= e^{ik\cdot x}\partial_a, \\ \left[T^{k_1,a}, T^{k_2,b}\right] &= (-i)(\delta_a{}^c k_{1b} - \delta_b{}^c k_{2a})\, e^{i(k_1+k_2)\cdot x}\partial_c \\ &= f^{(k_1,a),(k_2,b)}{}_{(k_1+k_2,c)} T^{k_1+k_2,c}. \end{aligned}$$



[Du, Feng, CF(12)]

Analytic construction of numerators

- structure constants are *NOT* totallly anti-symmetric

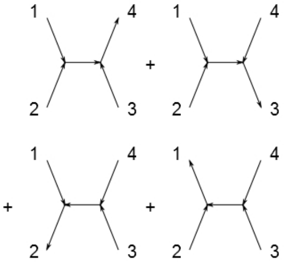


$$\eta_{ab}(k_1 - k_2)_c + \eta_{bc}(k_2 - k_3)_a + \eta_{ca}(k_3 - k_1)_b$$
$$= f^{1,2}{}_3 + f^{2,3}{}_1 + f^{3,1}{}_2$$

- Observed numerator relations are the collective work of four sets of Jacobi identities

$$n_s^* + n_t^* + n_u^* = 0$$
$$\updownarrow$$
$$f^{3,4}{}_e f^{2,e}{}_1 + f^{2,3}{}_e f^{4,e}{}_1 + f^{4,2}{}_e f^{3,e}{}_1 = 0$$
$$f^{3,4}{}_e f^{e,1}{}_2 + f^{4,1}{}_e f^{e,3}{}_2 + f^{1,3}{}_e f^{e,4}{}_2 = 0$$
$$f^{1,2}{}_e f^{4,e}{}_3 + f^{4,1}{}_e f^{2,e}{}_3 + f^{2,4}{}_e f^{1,e}{}_3 = 0$$
$$f^{1,2}{}_e f^{e,3}{}_4 + f^{2,3}{}_e f^{e,1}{}_4 + f^{3,1}{}_e f^{e,2}{}_4 = 0$$

$n_s^* =$



Analytic construction of numerators

What about 4-pt vertex?

$$
\begin{aligned}
A(1234) &= \frac{n_s^*}{s} - \frac{n_t^*}{t} + X_1^* \\
A(1324) &= -\frac{n_u^*}{u} + \frac{n_t^*}{t} + X_2^*
\end{aligned}
$$

Analytic construction of numerators

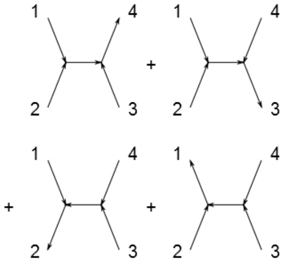
What about 4-pt vertex?

$$
\begin{aligned}
A(1234) &= \sum_i c_i \epsilon_i \cdot \left(\frac{n_s^*}{s} - \frac{n_t^*}{t} + X_1^* \right) \\
A(1324) &= \sum_i c_i \epsilon_i \cdot \left(-\frac{n_u^*}{u} + \frac{n_t^*}{t} + X_2^* \right)
\end{aligned}
$$

Average over reference momenta, subject to the constraints

$$
\begin{aligned}
c_1 + c_2 &= 1, \\
(c_1 \epsilon_1 + c_2 \epsilon_2) \cdot X_1^* &= 0, \\
(c_1 \epsilon_1 + c_2 \epsilon_2) \cdot X_2^* &= 0.
\end{aligned}
$$

$$\sum_i c_i \epsilon_i \cdot n_s^* = \sum_i c_i \epsilon_i \cdot$$



Formulations of YM amplitudes: A sketchy review

- Color-ordered formulation

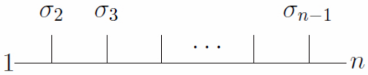
$$M(1^\alpha, 2^\beta, 3^\gamma, \ldots, n^\delta) = \sum_{\sigma \in S_{n-1}} tr(T^\alpha T^{\sigma_2} T^{\sigma_3} \ldots T^{\sigma_n}) \times A(p_1, p_{\sigma_2}, \ldots, p_{\sigma_n})$$

properties of $SU(N)$ color algebra

$$\begin{aligned} tr(T^\alpha T^\beta) &= \delta^{\alpha\beta} \\ f^{\alpha\beta\gamma} &= tr([T^\alpha, T^\beta] T^\gamma) \\ (T^\alpha)_{ij} (T^\alpha)_{kl} &= \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \end{aligned}$$

- Del Duca-Dixon-Maltoni chain

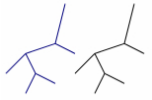
$$M(1^\alpha, 2^\beta, 3^\gamma, \ldots, n^\delta) = \sum_{\sigma \in S_{n-2}} f^{\alpha\sigma_2\rho_2} f^{\rho_2\sigma_3\rho_3} \ldots f^{\rho_{n-2}\sigma_{n-1}\sigma_n} \times A(p_1, p_{\sigma_2}, \ldots, p_{\sigma_{n-1}}, p_n)$$



[Del Duca, Dixon, Maltoni(00)]

Formulations of YM amplitudes/Conclusion

$tr(T\, T \ldots T) \times A(1\sigma)$
color-ordered formulation

$\sum \frac{1}{p_i^2}$ 
double-copy formulation

$\tilde{A}(1\sigma) \times \tau_{1\sigma}$
dual color-ordered formulation
[Bern, Dennen(11)]

?

KK — relation

Feynman rules

BCFW
[Bjerrum-Bohr, Feng, Damgaard, Sondergaard, Vanhove(10)]

$\times A(1\sigma n)$
Del Duca-Dixon-Maltoni chain(00)

Jacobi identity

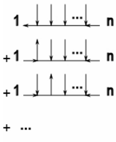
[Du, Feng, CF(11)]

$$(-)^n \sum_{\alpha,\beta} \frac{\tilde{A}(n,\alpha,1)\mathcal{S}[\alpha|\beta]A(1,\beta,n)}{s_{12\ldots n-1}}$$

KLT

Jacobi identity

[Du, Feng, CF(12)]

$\tilde{A}(1\sigma n)\times$


Remaining Thoughts

- BCJ-dual to traces

$$
\begin{aligned}
tr(T^\alpha T^\beta) &= \delta^{\alpha\beta} \\
f^{\alpha\beta\gamma} &= tr([T^\alpha, T^\beta]T^\gamma) \\
&\downarrow \\
M &= tr(T^1 T^2 \ldots T^n)A(12\ldots n) + \ldots
\end{aligned}
$$

- Color-kinematic duality in 3-dimensions

$$
A(1234) = \sum_i c_i \epsilon_i \cdot \left(\frac{n_s}{s} - \frac{n_t}{t} \right)
$$

[Du, Feng, CF, work in progress]

- dual Del Duca-Dixon-Maltoni chain at loop level
- A proof of double-copy expression at loop-level

  Improved large-z behavior [Boels, Isermann(12)]

  $(+ + + \cdots +)$, LC construction at 1-loop [Boels, Isermann, Monteiro, O'Connell(13)]

- Algebra and double-copy from string perspective

  [Bjerrum-Bohr, Damgaard, Johansson, Sondergaard(11)]