Mathieu Moonshine

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Based on work with Katrin Wendland (Freiburg)

`A twist in the M24 moonshine story'; arXiv:1303.3221
`Symmetry-surfing the moduli space of Kummer K3s'; arXiv:1303.2931
`The overarching finite symmetry group of Kummer surfaces in M24'; arXiv:1107.3834

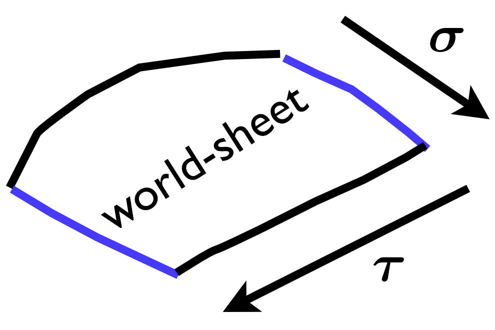
Strings compactified on K3 surfaces

superstring in 10 space-time dimensions

world-sheet:
2-dimensional extended superconformal algebra N= (4,4) at $c = \bar{c} = 6$

target space:
K3 surface X

$X^{\mu}(\tau, \sigma), \mu = 1, ..., 10$



ensures $\mathcal{N} = 1$ **space-time supersymmetry survives compactification**

superstring in 6 space-time dimensions

parameters of non-linear sigma model describing strings propagating on K3 surface :
metric on X and B-field (moduli)

Strings compactified on K3 surfaces

A K3 surface X is a simply connected compact complex Kahler manifold of complex dimension 2 with a nowhere vanishing holomorphic 2-form

Hodge diamond: K3 topology summarized

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

even cohomology

full integral even cohomology lattice of K3 surface X:

$$\underset{(4,20)}{H^*(X,\mathbb{Z})} = \underset{(3,19)}{H^2(X,\mathbb{Z})} \oplus \underbrace{H^0(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})}_{(1,1)}$$

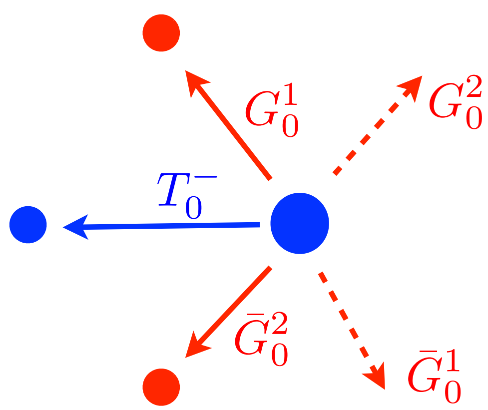
N=4 superconformal algebra at c=6

T. Eguchi, A.T., 1987

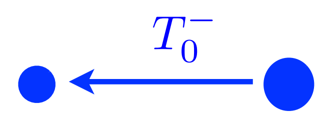
generators: 4 supercharges $G^1_r,\ G^2_r,\ \bar{G}^1_r,\ \bar{G}^2_r, \qquad r\in\mathbb{Z}$

3 SU(2) affine generators at level one $T^{\pm}_r,\ T^3_r$

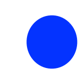
Virasoro generator $L_r \qquad$ (conformal weight h)



massive representation isospin 1/2 (non BPS)



massless isospin 1/2 (BPS)



massless isospin 0 (BPS)

character function (Ramond sector) $\qquad h > 1/4$

$$Ch^R(q,y) = y\, q^h \prod_{n=1}^{\infty} \frac{(1+yq^n)^2(1+y^{-1}q^{n-1})^2}{(1-q^n)}$$

$$= q^h \frac{\vartheta_2^2(q,y)}{\eta(q)^3} = q^h\,(y + y^{-1} + 2 + q(\dots) + \dots)$$

$$Ch^{\widetilde{R}}(q,y) = q^h \frac{\vartheta_1^2(q,y)}{\eta(q)^3} = q^h\,(-y - y^{-1} + 2 + q(\dots) + \dots)$$

$$Ch_0^R(q,y) = y \prod_{n=1}^{\infty} \frac{(1+yq^n)^2(1+y^{-1}q^{n-1})^2}{(1-q^n)} \times\{\text{ corrective series }\}$$

$$= \frac{\vartheta_2^2(q,y)}{\eta(q)^3}\,\mu(q,y) = 1 + q(\dots)$$

$$Ch_0^{\widetilde{R}}(q,y) = \frac{\vartheta_1^2(q,y)}{\eta(q)^3}\,\mu(q,-y) = 1 + q(..)$$

N=(4, 4) partition functions

Superstrings compactified on K3 surface: $10 \rightarrow 6$ dimensions

↓

$N=(4,4)$ SCFT on worldsheet ⟵ 4 bosons and 4 fermions compactified

$c = \bar{c} = 6 = (4 \times 1) + (4 \times \frac{1}{2})$

↓

N=4 partition function:

$$Z_{N=4}(\tau, z; \bar{\tau}, \bar{z}) := \sum_{i=NS,R,\widetilde{NS},\widetilde{R}} \mathrm{tr}_{\mathcal{H}^i \times \mathcal{H}^i}\Big(y^{J_0}\, q^{L_0 - \frac{1}{4}}\, \bar{y}^{\bar{J}_0}\, \bar{q}^{\bar{L}_0 - \frac{1}{4}} \Big)$$

$$q := e^{2\pi i \tau},\ y := e^{2\pi i z},\ \tau, z \in \mathbb{C}, \Im(\tau) > 0.$$

↓

Modular invariant quadratic expression in N=4 characters

$$Z_{N=4}(\tau, z; \bar{\tau}, \bar{z}) := \sum_{i=NS,R,\widetilde{NS},\widetilde{R}} \sum_{a,b} n_{ab}\, Ch^i_a(\tau, z)\, \overline{Ch}^i_b(\bar{\tau}, \bar{z})$$

↓

integer, model dependent

K3 elliptic genus and M24 Moonshine

$$Z_{N=4}(\tau,z;\bar\tau,\bar z) := \sum_{i=NS,R,\widetilde{NS},\widetilde{R}} \sum_{a,b} n_{ab}\, Ch^i_a(\tau,z)\, \overline{Ch}^i_b(\bar\tau,\bar z)$$

modular invariant by itself

$$Z_{N=4,\widetilde{R}}(\tau,z;\bar\tau,\bar z) := \sum_{a,b} n_{ab}\, Ch^{\widetilde{R}}_a(\tau,z)\, \overline{Ch}^{\widetilde{R}}_b(\bar\tau,\bar z)$$

specialisation

$$Z_{N=4,\widetilde{R}}(\tau,z;\bar\tau,\bar z=0) = \sum_{a,b} n_{ab}\, Ch^{\widetilde{R}}_a(\tau,z)\, \underbrace{\overline{Ch}^{\widetilde{R}}_b(\bar\tau,0)}$$

Witten index: $0, 1, -2$

$$:= Z_{K3}(\tau,z) = 24\, Ch^{\widetilde{R}}_0(\tau,z) + 2\{(-1 + 45q + 231q^2 + \ldots\}\, \widetilde{ch}^{\widetilde{R}}(\tau,z)$$

Dimensions of representations of sporadic group Mathieu 24

' Mathieu Moonshine' (T. Eguchi, H. Ooguri and Y. Tashikawa, 2010)

K3 Elliptic genus obtained from the partition function of ANY

$N=(4,4)$ SCFT at $c=\bar c=6$

Character table of Mathieu 24

i	1A	2A	3A	5A	4B	7A	7B	8A	6A	11A	15A	15B	14A	14B	23A	23B	12B	6B	4C	3B	2B	10A	21A	21B	4A	12A
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	23	7	5	3	3	2	2	1	1	1	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
3	252	28	9	2	4	0	0	0	1	−1	−1	−1	0	0	−1	−1	0	0	0	0	12	2	0	0	4	1
4	253	13	10	3	1	1	1	−1	−2	0	0	0	−1	−1	0	0	1	1	1	1	−11	−1	1	1	−3	0
5	1771	−21	16	1	−5	0	0	−1	0	0	1	1	0	0	0	0	−1	−1	−1	7	11	1	0	0	3	0
6	3520	64	10	0	0	−1	−1	0	−2	0	0	0	1	1	1	1	0	0	0	−8	0	0	−1	−1	0	0
7	45	−3	0	0	1	e_7^+	e_7^-	−1	0	1	0	0	$-e_7^+$	$-e_7^-$	−1	−1	1	−1	1	3	5	0	e_7^-	e_7^+	−3	0
8	45	−3	0	0	1	e_7^-	e_7^+	−1	0	1	0	0	$-e_7^-$	$-e_7^+$	−1	−1	1	−1	1	3	5	0	e_7^+	e_7^-	−3	0
9	990	−18	0	0	2	e_7^+	e_7^-	0	0	0	0	0	e_7^+	e_7^-	1	1	1	−1	−2	3	−10	0	e_7^-	e_7^+	6	0
10	990	−18	0	0	2	e_7^-	e_7^+	0	0	0	0	0	e_7^-	e_7^+	1	1	1	−1	−2	3	−10	0	e_7^+	e_7^-	6	0
11	1035	−21	0	0	3	$2e_7^+$	$2e_7^-$	−1	0	1	0	0	0	0	0	0	−1	1	−1	−3	−5	0	$-e_7^-$	$-e_7^+$	3	0
12	1035	−21	0	0	3	$2e_7^-$	$2e_7^+$	−1	0	1	0	0	0	0	0	0	−1	1	−1	−3	−5	0	$-e_7^+$	$-e_7^-$	3	0
13	1035	27	0	0	−1	−1	−1	1	0	1	0	0	−1	−1	0	0	0	2	3	6	35	0	−1	−1	3	0
14	231	7	−3	1	−1	0	0	−1	1	0	e_{15}^+	e_{15}^-	0	0	1	1	0	0	3	0	−9	1	0	0	−1	−1
15	231	7	−3	1	−1	0	0	−1	1	0	e_{15}^-	e_{15}^+	0	0	1	1	0	0	3	0	−9	1	0	0	−1	−1
16	770	−14	5	0	−2	0	0	0	1	0	0	0	0	0	e_{23}^+	e_{23}^-	1	1	−2	−7	10	0	0	0	2	−1
17	770	−14	5	0	−2	0	0	0	1	0	0	0	0	0	e_{23}^-	e_{23}^+	1	1	−2	−7	10	0	0	0	2	−1
18	483	35	6	−2	3	0	0	−1	2	−1	1	1	0	0	0	0	0	0	3	0	3	−2	0	0	3	0
19	1265	49	5	0	1	−2	−2	1	1	0	0	0	0	0	0	0	0	0	−3	8	−15	0	1	1	−7	−1
20	2024	8	−1	−1	0	1	1	0	−1	0	−1	−1	1	1	0	0	0	0	0	8	24	−1	1	1	8	−1
21	2277	21	0	−3	1	2	2	−1	0	0	0	0	0	0	0	0	0	2	−3	6	−19	1	−1	−1	−3	0
22	3312	48	0	−3	0	1	1	0	0	1	0	0	−1	−1	0	0	0	−2	0	−6	16	1	1	1	0	0
23	5313	49	−15	3	−3	0	0	−1	1	0	0	0	0	0	0	0	0	0	−3	0	9	−1	0	0	1	1
24	5796	−28	−9	1	4	0	0	0	−1	−1	1	1	0	0	0	0	0	0	0	0	36	1	0	0	−4	−1
25	5544	−56	9	−1	0	0	0	0	1	0	−1	−1	0	0	1	1	0	0	0	0	24	−1	0	0	−8	1
26	10395	−21	0	0	−1	0	0	1	0	0	0	0	0	0	−1	−1	0	0	3	0	−45	0	0	0	3	0

Sporadic group M24

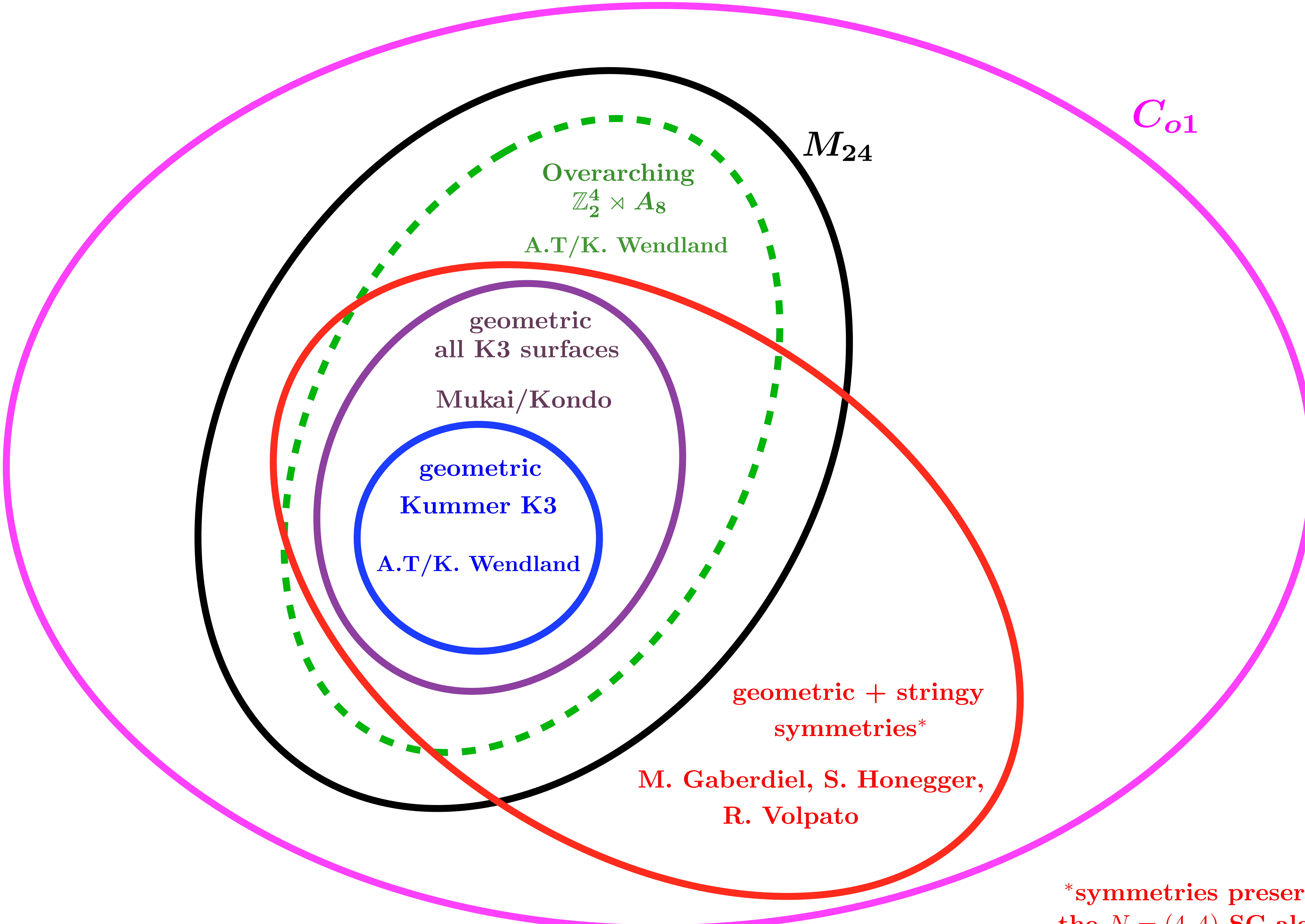
- M_{24} is a finite simple group with $24.23.22.21.20.48 = 244,823,040$ elements
- Steiner system $S(5, 8, 24)$ of 759 octads
- M_{24} is the subgroup of S_{24} that preserves the octads set wise
- Take the extended binary Golay code $\mathcal{G}_{24}$

  Linear code of length 24 and dimension 12 over $\mathbb{F}_2 := \{0, 1\}$ ($\mathcal{G}_{24} \subset \mathbb{F}_2^{24}$)
- M_{24} is the automorphism group of $\mathcal{G}_{24}$

Elusive M24 symmetry

- M_{24} is not a symmetry of the whole partition function
- M_{24} governs the states selected by the elliptic genus, which does not depend on moduli. Hence this hidden M_{24} symmetry is common to all $N = (4, 4)$ SCFTs at $c = \bar{c} = 6$
- techniques from Monster Moonshine study used to provide evidence of M_{24} action (M. Cheng, 2010; M. Gaberdiel, S. Honegger, R. Volpato, 2010; T. Eguchi and Hikami, 2010)
- proof of principle of an M_{24} action (T. Gannon, 2012)
- there are stringy symmetries of $N = (4, 4)$ SCFTs at $c = \bar{c} = 6$ that are not in M_{24} (M. Gaberdiel, S. Honegger, R. Volpato, 2011)
- $\mathbb{Z}_2^4 \rtimes A_8 \subset M_{24}$ overarches the symmetry groups accounting for the geometric symmetries of strings propagating on Kummer K3 surfaces (A. Taormina and K. Wendland, 2011 and 2013)

Symmetry landscape - N=(4,4) SCFT at c=6



*symmetries preserving
the $N = (4,4)$ SC algebra
and the spectral flow generators

From numbers to SCFT states

Strategy (A.T and K. Wendland, 2013)

- the elliptic genus of $K3$,

  $Z_{K3}(\tau,z) = 24\, ch^{\widetilde{R}}(1,0;\tau,z) + 2\{(-1 + 45q + 231q^2 + \ldots\}\, \widetilde{ch}^{\widetilde{R}}(\tau,z)$
  may be obtained from the partition function of any $N=(4,4)$ SCFT at $c=\bar{c}=6$

- choose a $\mathbb{Z}_2$ orbifold theory in 4 dimensions and calculate its partition function in twisted Ramond sector $\widetilde{R}$

- use it to trace back the states contributing to the number 45 in the elliptic genus

- determine the largest group of symmetries acting on them compatible with what we know of the irreducible 45-dimensional representation of M_{24}

Toroidal SCFT in 4d

$$\boldsymbol{T} := \mathbb{R}^{\mathbf{4}}/\boldsymbol{\Lambda}, \quad \boldsymbol{\Lambda} \text{ lattice}$$

holomorphic/antiholomorphic sectors

- **4 real bosonic fields (c=4) $\Longrightarrow$ 4 U(1) currents**

  charges $(p_L, p_R) \in \Gamma(\Lambda, B) \subset \mathbb{R}^{4,4}$

- **4 Majorana fermions (c=2)**

- **N=4 SCA (c=6)**

Partition function ($\widetilde{R}\widetilde{R}$ sector)

$$Z^{\widetilde{R}}(\tau, z; \bar{\tau}, \bar{z}) = \frac{1}{|\eta|^8} \sum_{(p_L, p_R) \in \Gamma(\Lambda, B)} q^{p_L^2/2} \bar{q}^{p_R^2/2} \left| \frac{\vartheta_1(\tau, z)}{\eta} \right|^4$$

G-Orbifold toroidal SCFT

G discrete group, $\mathcal{C} := \mathcal{T}/G$ orbifold CFT

$$Z_{untwisted} = \mathrm{Tr}_{\mathcal{H}}(P q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}) = \frac{1}{|G|} \sum_{g \in G} {}^{g}\Box_{1}$$

↑ projects on G-invariant states

not modular invariant

$$Z_{twisted} = \sum_{h \in G, h \neq 1} \mathrm{Tr}_{\mathcal{H}_h}(P_h q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24})$$

$$= \frac{1}{|G|} \sum_{\substack{g, h \in G \\ h \neq 1, hgh^{-1} = g}} {}^{g}\Box_{h}$$

$$\boxed{Z := Z_{untwisted} + Z_{twisted}}$$

orbifold toroidal partition function

Z2-Orbifold toroidal SCFT on K3- free fields

holomorphic sector

- **2 complex currents and conjugates** $j^1_\pm \sim j^1 \pm i j^2, \; j^2_\pm \sim j^3 \pm i j^4$

  odd under $\mathbb{Z}_2$

- **2 Dirac fermions and conjugates** $\chi^1_\pm, \; \chi^2_\pm$

  odd under $\mathbb{Z}_2$

- **N=4, c=6 SCA:** $T \sim j\,j + \partial\chi\,\chi, \; G^\pm, G'^\pm \sim \chi\,j, \; J^\pm, J^3 \sim \chi\,\chi,$

  ↑ conformal weight h ↑ charge Q

Z2-Orbifold toroidal SCFT on K3 - RR Ground states

- **8 untwisted RR ground states of $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$**

charged ground state	$(h, Q; \bar{h}, \bar{Q})$	uncharged ground state	$(h, Q; \bar{h}, \bar{Q})$
$\sigma_1^{++}\sigma_2^{++}$	$(\frac{1}{4}, 1; \frac{1}{4}, 1)$	$\sigma_1^{++}\sigma_2^{--}$	$(\frac{1}{4}, 0; \frac{1}{4}, 0)$
$\sigma_1^{+-}\sigma_2^{+-}$	$(\frac{1}{4}, 1; \frac{1}{4}, -1)$	$\sigma_1^{--}\sigma_2^{++}$	$(\frac{1}{4}, 0; \frac{1}{4}, 0)$
$\sigma_1^{-+}\sigma_2^{-+}$	$(\frac{1}{4}, -1; \frac{1}{4}, 1)$	$\sigma_1^{-+}\sigma_2^{+-}$	$(\frac{1}{4}, 0; \frac{1}{4}, 0)$
$\sigma_1^{--}\sigma_2^{--} := \boldsymbol{\sigma}$	$(\frac{1}{4}, -1; \frac{1}{4}, -1)$	$\sigma_1^{+-}\sigma_2^{-+}$	$(\frac{1}{4}, 0; \frac{1}{4}, 0)$

RR vacuum **RR massless matter**

- **16 twisted RR ground states $\boldsymbol{T_{\vec{a}}}, \boldsymbol{\vec{a}} \in \mathbb{F}_2^4$,** $\quad \boldsymbol{(h, Q; \bar{h}, \bar{Q}) = (\frac{1}{4}, 0; \frac{1}{4}, 0)}$

recall

point identification:
$$\vec{x} \equiv \vec{x} + \sum_{i=1}^{4} n_i \vec{\lambda}_i, \quad n_i \in \mathbb{Z} \qquad \vec{\lambda}_i \in \Lambda$$
$$\vec{x} \equiv -\vec{x}.$$

16 singularities (fixed points) $\quad \vec{a} := (a_1, a_2, a_3, a_4) \in \mathbb{F}_2^4$

minimal resolution of 16 singularities yields the Kummer K3 surface $\quad X = \widetilde{T(\Lambda)/\mathbb{Z}_2}$

Z2-Orbifold toroidal SCFT on K3 - partition function

- $\mathbb{Z}_2$-orbifold partition function:

$$Z^{\widetilde{R}\widetilde{R}}_{\text{untwisted}}(\tau, z; \bar{\tau}, \bar{z}) = \frac{1}{2|\eta(\tau)|^8} \left(1 + \sum_{\substack{(p_L;p_R)\in\Gamma, \\ (p_L;p_R)\neq(0;0)}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}}\right) \left|\frac{\vartheta_1(\tau,z)}{\eta(\tau)}\right|^4 + 8 \left|\frac{\vartheta_2(\tau,z)}{\vartheta_2(\tau)}\right|^4$$

$$+$$

$$Z^{\widetilde{R}\widetilde{R}}_{\text{twisted}}(\tau, z; \bar{\tau}, \bar{z}) = 8 \left|\frac{\vartheta_3(\tau,z)}{\vartheta_3(\tau)}\right|^4 + 8 \left|\frac{\vartheta_4(\tau,z)}{\vartheta_4(\tau)}\right|^4$$

non-generic contribution

- K3 elliptic genus:

$$Z_{K3}(\tau, z) := Z^{\widetilde{R}\widetilde{R}}_{orb}(\tau, z; \bar{\tau}, \bar{z} = 0)$$

$$q = e^{2i\pi\tau}, \; y := e^{2i\pi z}$$

$$= 24\, ch_0^{\widetilde{R}}(q, y) + 2\{(-1 + 45q + 231q^2 + \ldots\}\, \widetilde{ch}^{\widetilde{R}}(q, y)$$

$$q^{1/4}(-y - y^{-1} + 2 + q(\ldots) + \ldots)$$

Z2-Orbifold toroidal SCFT on K3 - counting states

counting the states with $(h, Q; \bar{h}, \bar{Q}) = (\frac{5}{4}, 1; \frac{1}{4}, \bar{Q})$ from orbifold partition function

- untwisted sector:

$$(4q)\,(y^2)\,(-2\bar{y})\,(y^{-1}\bar{y}^{-1}) + (2qy)\,(2y)\,(1)\,(y^{-1}\bar{y}^{-1}) + (2qy)\,(2y)\,(\bar{y}^2)\,(y^{-1}\bar{y}^{-1})$$

$a_1^K\ \chi_0^1\,\chi_0^2\ \bar{\chi}_0^\ell\,\sigma$	$\chi_1^k\ \chi_0^\ell\ \ \sigma$	$\chi_1^k\ \chi_0^\ell\ \bar{\chi}_0^1\,\bar{\chi}_0^2\ \sigma$
$K = 1,..,4,\ \ell = 1, 2$		$k, \ell = 1, 2$
	1 massless boson	1 massless boson
8 massless fermions	3 massive bosons	3 massive bosons

- twisted sector: $\boxed{a_{\frac{1}{2}}^K\ \chi_{\frac{1}{2}}^\ell\ T_{\vec{a}},}$ $K = 1, .., 4, \ell = 1, 2,\ \ \vec{a} \in \mathbb{F}_2^4,\quad 4 \times 2 \times 16 = 128$ states

32 massless fermions + 96 massive fermions

net number of massive states: $96 - 6 = 90 = 45 + \overline{45}$

The 90 twisted massive states of interest

128 twisted states: $a^{K}_{\frac{1}{2}}\, \chi^{\ell}_{\frac{1}{2}}\, T_{\vec{a}},\ K = 1, .., 4, \ell = 1, 2,\ \vec{a} \in \mathbb{F}_2^4$

32 twisted massless states

$$\underbrace{\left(\chi^1 j^1_- + \chi^2 j^2_-\right)}_{\text{SU(2) singlet}} T_{\vec{a}}, \qquad \underbrace{\left(\chi^1 j^2_+ - \chi^2 j^1_+\right)}_{\text{SU(2) singlet}} T_{\vec{a}}, \quad \vec{a} \in \mathbb{F}_2^4.$$

96 massive twisted states

$\{W T_{\vec{a}} \mid W \in \mathbf{3} \cup \bar{\mathbf{3}},\ \vec{a} \in \mathbb{F}_2^4\}$ where

$$\mathbf{3} := \{\chi^1 j^2_+ + \chi^2 j^1_+,\ \chi^1 j^1_+,\ \chi^2 j^2_+\}, \quad \bar{\mathbf{3}} := \{\chi^1 j^1_- - \chi^2 j^2_-,\ \chi^1 j^2_-,\ \chi^2 j^1_-\},$$

SU(2) triplet — SU(2) (anti)triplet

$W\, T_{\vec{a}}$ decompose as

$$96 = (\mathbf{3} \oplus \bar{\mathbf{3}}) \otimes \mathbf{16} = (\mathbf{3} \oplus \bar{\mathbf{3}}) \otimes \mathbf{1} \oplus \boxed{(\mathbf{3} \oplus \bar{\mathbf{3}}) \otimes \mathbf{15}} = 6 + 90$$

↑ How does this decouple from 16?

Group action on the 90 twisted massive states

Back to the symmetries of the full theory: $T/\mathbb{Z}_2 \quad T := \mathbb{R}^4/\Lambda, \quad \Lambda$ lattice

Symmetry groups induced by geometric symmetries of torus

$$G = G_t \rtimes G_T \cong \mathbb{Z}_2^4 \rtimes G_T \subset \mathbb{Z}_2^4 \rtimes A_8 \cong \mathrm{Aff}(\mathbb{F}_2^4)$$

- Translational automorphism group action: common to all $\mathcal{C} := \mathcal{T}/\mathbb{Z}_2$

16 singular points : $\vec{F}_{\vec{a}} = \frac{1}{2}\sum_{i=1}^{4} a_i \vec{\lambda}_i, \qquad \vec{\lambda}_i$ generators of lattice Λ; $\vec{a} \in \mathbb{F}_2^4$

shifts by half lattice vectors generate $G_t \simeq \mathbb{Z}_2^4$

Example : $\vec{a} = (0,1,1,0)$ and shift by $\frac{1}{2}\vec{\lambda}_1$: $\vec{F}_{\vec{a}} \to \vec{F}_{\vec{a}} + \frac{1}{2}\vec{\lambda}_1 = \frac{1}{2}(\vec{\lambda}_1 + \vec{\lambda}_2 + \vec{\lambda}_3)$

or $\vec{a} = (0,1,1,0) \to \vec{a}' = (1,1,1,0)$

permutation of labels

- $G_T \cong G'_T/\mathbb{Z}_2, \quad G'_T \subset \mathbb{Z}_4, \mathbb{Z}_6, \mathcal{O}, \mathcal{D}, \mathcal{T}, \quad G_T \subset SU(2)$

The 45 dimensional representation space

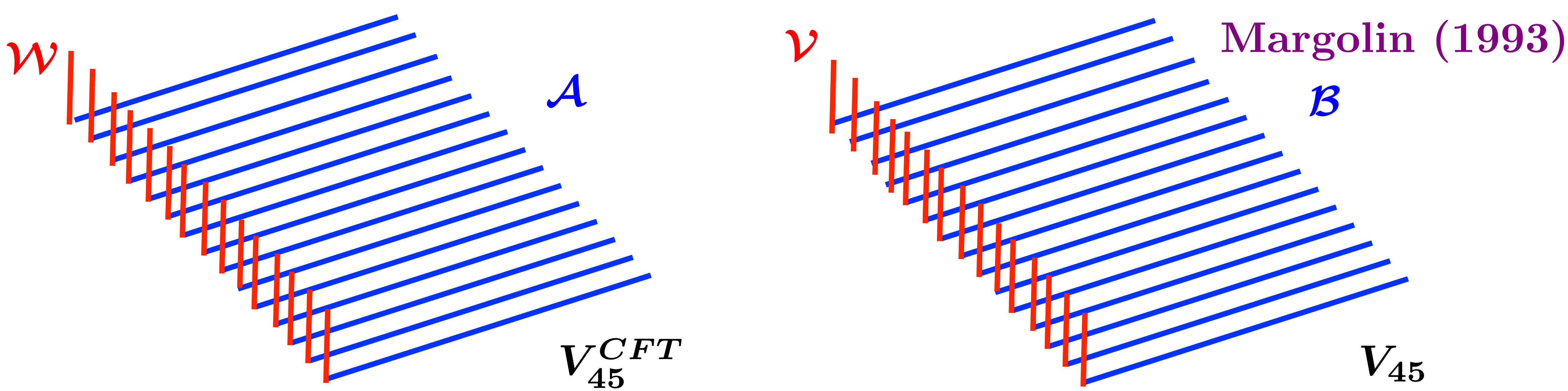
To decouple 1 from $16 = 1 \oplus 15$:

- **change base from $T_{\vec{a}}$ to $N_{\vec{a}}$, $\vec{a} \in \mathbb{F}_2^4$**
- $N_{\vec{0}} := \frac{1}{4} \sum_{\vec{a} \in \mathbb{F}_2^4} T_{\vec{a}}$ **is invariant under $G = G_t \rtimes G_T$**
- **take the orthogonal complement $\mathcal{A}$ of $N_{\vec{0}}$ in the space of twisted ground states**
- **the space**

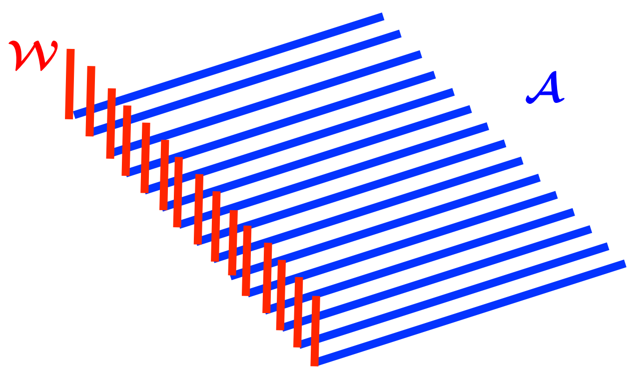
$$V_{45}^{CFT} := \mathrm{span}_{\mathbb{C}} \left\{ W A \mid W \in \{ \chi^1 j_+^2 + \chi^2 j_+^1,\ \chi^1 j_+^1,\ \chi^2 j_+^2 \},\ A \in \mathcal{A} \right\}$$

  is a 45-dimensional vector space of massive states

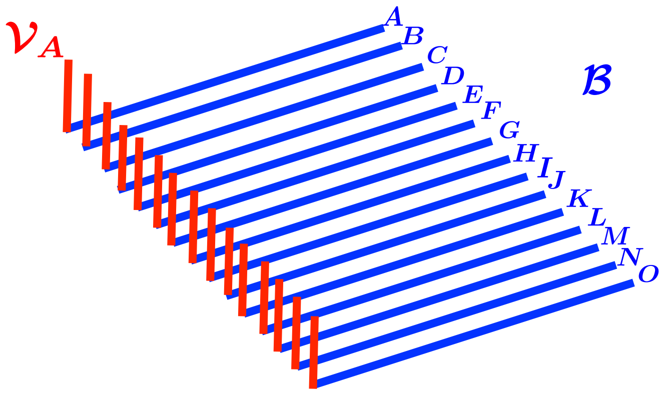
- V_{45}^{CFT} **is a tensor product $\mathcal{W} \otimes \mathcal{A}$, where $\mathcal{W}$ is the three-dimensional representation $\mathbf{3}$ of $SU(2)$, while $\mathcal{A}$ is a 15-dimensional representation of $\mathrm{Aff}(\mathbb{F}_2^4)$.**



Group action on V_{45}^{CFT}



V_{45}^{CFT}



V_{45}

$\mathbb{Z}_2^4 \rtimes A_8$ on $\mathcal{A}$ equivalent to $\mathbb{Z}_2^4 \rtimes A_8$ on $\mathcal{B}$

A_8 action on base yields a twist on fibres not observed on CFT side

$\mathbb{Z}_2^4$ action on fibres is trivial

obstruction to the action of $\mathbb{Z}_2^4 \rtimes A_8$ on CFT states, hence obstruction to the action of M_{24} on the same states

However

$\mathbb{Z}_2^4 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$, $\mathbb{Z}_2^4 \rtimes A_4$ and $\mathbb{Z}_2^4 \rtimes S_3$ do act without twist on V_{45}^{CFT}

and their action on V_{45}^{CFT} is equivalent to their action on V_{45}

These are the maximal geometric symmetry groups of $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$

Conclusions

- M_{24} does not act on the 90 states selected by the elliptic genus of K3 from the full set of states of a $\mathbb{Z}_2$-orbifold SCFT
- it has been proven by Terry Gannon that there is an action of M_{24} on the massive states governed by the elliptic genus of K3, in particular on the 90 states appearing at lowest order in the q-expansion of the massive sector of the elliptic genus
- this apparent contradiction is a sign that the M_{24} action is well hidden, that is, we have not yet identified the 90 objects, common to all $N = (4, 4)$ SCFT on K3, that enjoy an M_{24} action
- to solve the Mathieu Moonshine, we have to identify the correct states at all orders, i.e. not just 90, but 231, 770, etc
- it is likely that the solution will use ideas from different fields of mathematics (algebraic geometry, number theory, group theory). The Mathieu Moonshine is yet another example of how string theory is stimulating research between theoretical physicists and mathematicians