Uniformly Accelerated Reference Frames and Equivalence Principle

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I. Introduction

"Let now K be an inertial system. Masses which sufficiently far from each other and from other bodies are then with respect to K, free from acceleration. We shall also refer these masses to a system of co-ordinates K', uniformly accelerated with respect to K. Relatively to K' all the masses have equal and parallel accelerations; with respect to K' they behave just as a gravitational field were present and K' were unaccelerated."

— A. Einstein, The Meaning of Relativity

How to define K' ? Can K' be identified with as an unaccelerated system in a $\left(\begin{smallmatrix} \text{static} \\ \text{uniform} \end{smallmatrix} \right)$ gravitational field?

Einstein has argued that “the laws of physics in a laboratory under uniform gravity is identical to those in a laboratory undergoing an acceleration.”

— einstein.stanford.edu/STEP/

After ignored the curvature, “gravity is completely indistinguishable from any other acceleration.”

$$\text{gravity} \Longleftrightarrow \text{acceleration of ref. system}$$

Really? In what sense?

Uniformly accelerated reference frame

Møller frame:

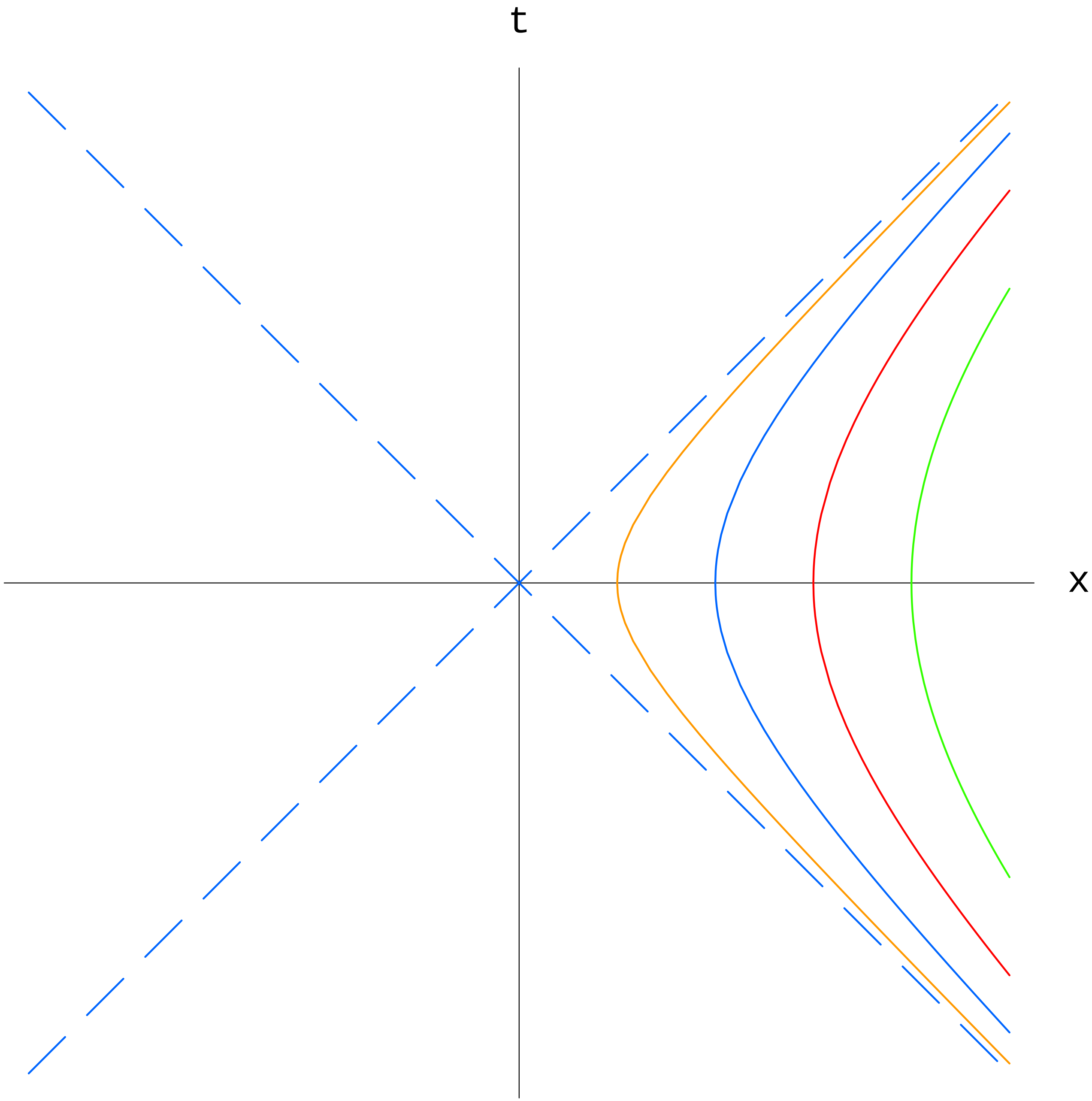
$$ds^2 = \left(1 + \frac{g\tilde{x}}{c^2}\right)^2 c^2 d\tilde{t}^2 - d\tilde{x}^2 - d\tilde{y}^2 - d\tilde{z}^2 \qquad (1)$$

By setting $\zeta = \frac{c^2}{g}\left(1 + \frac{g\tilde{x}}{c^2}\right)$, $\eta = \frac{g}{c^2}\sqrt{\frac{G_N \hbar}{c^3}}\tilde{t}$, one may obtain Rindler frame.

The proper acceleration of an observer at $\tilde{x} = const.$ is

$$a = \frac{g}{1 + g\tilde{x}/c^2} = a(\tilde{x}) \qquad (2)$$

It depends on the position.


t
x

However,

all points in an accelerated elevator

should have

the same acceleration!

II. A New Accelerated Frame

The trajectories of a set of accelerated observers with the same acceleration a passing points $(0, \bar{x}, y, z)$ are

$$x(t) = \bar{x} + \frac{c^2}{a}\left(\cosh\frac{as}{c^2} - 1\right), \qquad t = \frac{c}{a}\sinh\frac{as}{c^2}. \quad (3)$$

Choose new coordinates: $\bar{t} := s/c, \bar{x}, \bar{y} = y, \bar{z} = z$

New accelerated frame

$$\boxed{ds^2 = c^2 d\bar{t}^2 - 2c\sinh\frac{a\bar{t}}{c} d\bar{t} d\bar{x} - d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2} \quad (4)$$

An arbitrary ‘static’ observer in the new frame with $\bar{x}, \bar{y}, \bar{z} = consts.$ has 4-velocity

$$U^\mu = c(1, 0, 0, 0), \tag{5}$$

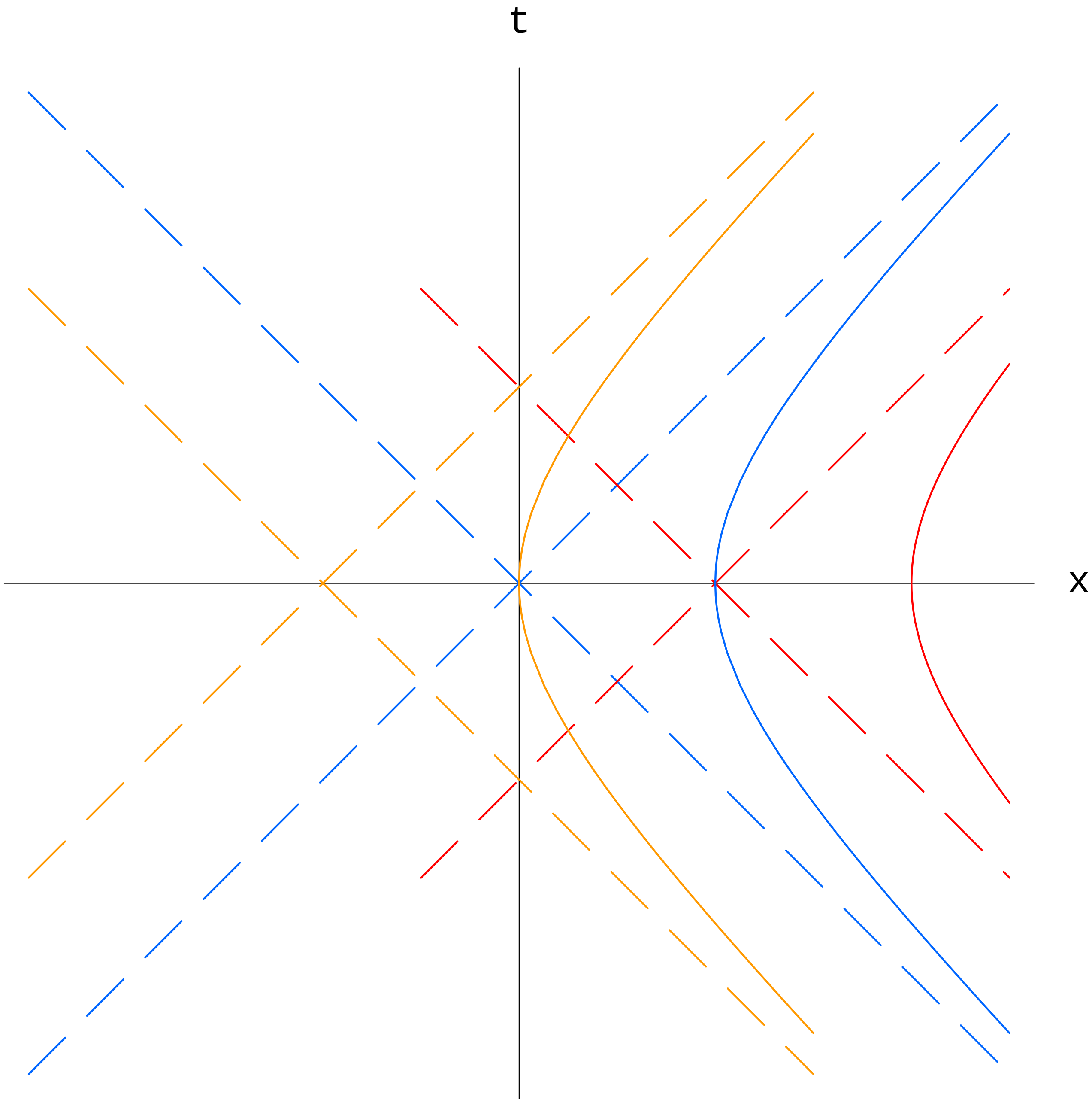
and 4-acceleration

$$a^\mu = U^\nu \nabla_\nu U^\mu = a \tanh \frac{a\bar{t}}{c} \delta_0^\mu + \frac{a}{\cosh \frac{a\bar{t}}{c}} \delta_1^\mu. \tag{6}$$

The magnitude of the acceleration is

$$(-g_{\mu\nu} a^\mu a^\nu)^{1/2} = a, \tag{7}$$

which is independent of the position of a ‘static’ observer in the spacetime.


t
x

Properties of the new frame:

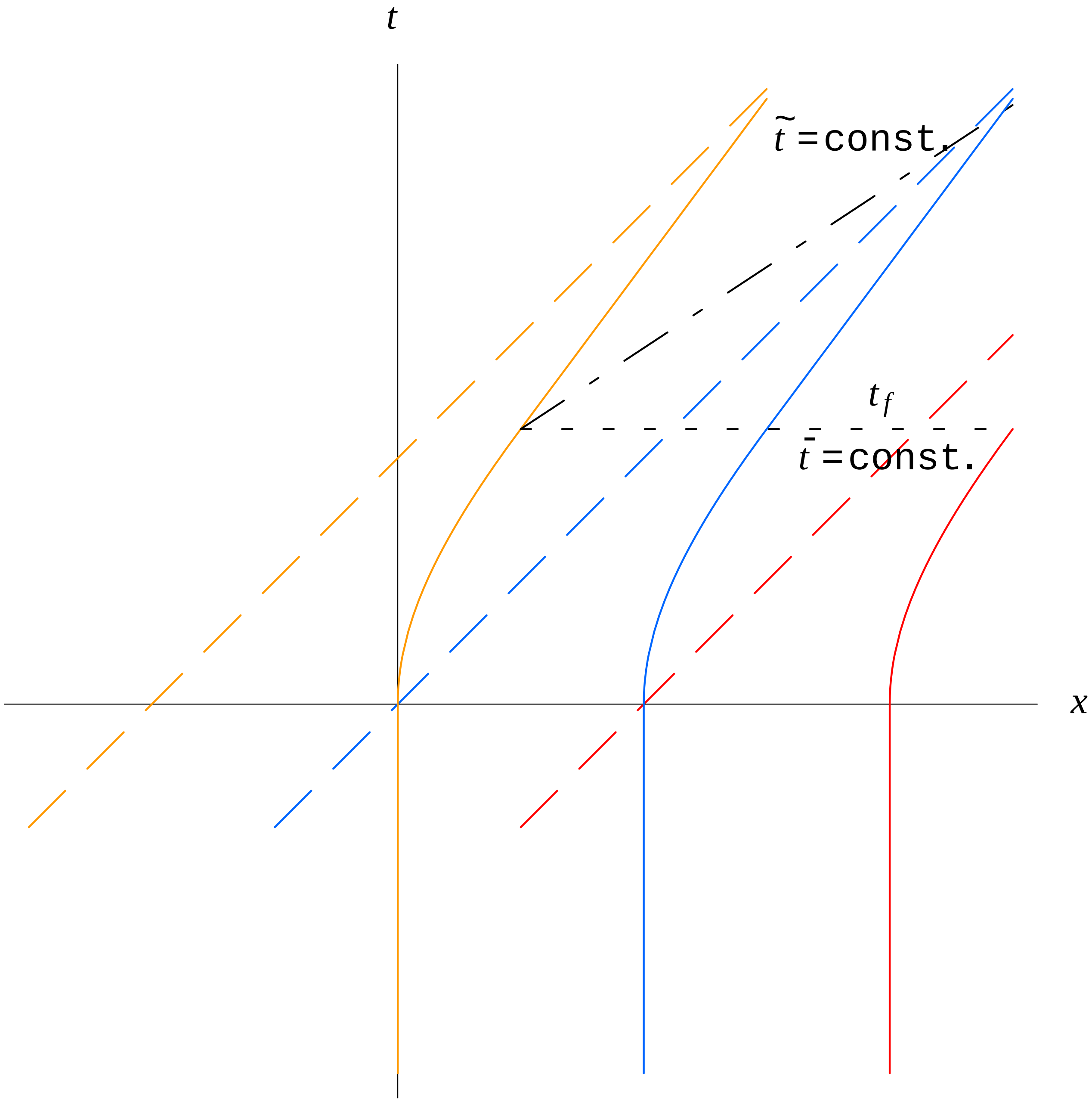
1. Each ‘static’ observer has the same acceleration.
2. At the time origin, the concepts of space and time are the same as those of inertial observers.
3. The accelerated frame differs from an inertial frame only by a non-diagonal term.
4. The frame is not a stationary one.
5. Each ‘static’ observer has his own horizon.
6. The entire Minkowski spacetime may be suffused by the trajectories of a set of observers.

7. The new frame is easily used to study the physics in the cabin which is first at rest, as an inertial frame, and begins to be uniformly accelerated at $t = 0$, and finally settles down an inertial frame at $t = t_f$ with the final velocity

$$v_f = at_f \left(1 + \frac{a^2 t_f^2}{c^2}\right)^{-1/2} \approx at_f \qquad (8)$$

with respect to the rest inertial frame.

The piecewise uniformly accelerated frame is more realistic.



t
x
$\tilde{t}$ = const.
t_f
$\bar{t}$ = const.

III. Effects to Distinguish the Two Frames

Arrival Time

In Møller frame

Suppose the light source is fixed at $\tilde{x}_1$ and the receiver is fixed at $\tilde{x}_2 (> \tilde{x}_1)$. The time for a photon traveling from $\tilde{x}_1$ to $\tilde{x}_2$ (or returning from $\tilde{x}_2$ to $\tilde{x}_1$) is

$$\Delta \tilde{t} = \frac{\tilde{x}_2 - \tilde{x}_1}{c}. \tag{9}$$

In the new frame

Suppose the light source is fixed at $\bar{x}_1$ and the receiver is fixed at $\bar{x}_2$. When $0 < \bar{x}_2 - \bar{x}_1 \lll c^2/a$, the time for a photon traveling from $\bar{x}_1$ to $\bar{x}_2$

$$\Delta\bar{t}_{1\to 2} = \bar{t}_2 - \bar{t}_1 \approx \frac{\bar{x}_2 - \bar{x}_1}{c} e^{a\bar{t}_2/c} \approx \frac{\bar{x}_2 - \bar{x}_1}{c}(1 + \frac{a\bar{t}_2}{c}) \quad (10)$$

while the time for a photon returning from $\bar{x}_2$ is

$$\Delta\bar{t}_{2\to 1} = \bar{t}_3 - \bar{t}_2 \approx \frac{\bar{x}_2 - \bar{x}_1}{c} e^{-a\bar{t}_2/c} \approx \frac{\bar{x}_2 - \bar{x}_1}{c}(1 - \frac{a\bar{t}_2}{c}) \quad (11)$$

The round-trip time is approximately by

$$\bar{t}_3 - \bar{t}_1 \approx \frac{2(\bar{x}_2 - \bar{x}_1)}{c} \cosh \frac{a\bar{t}_2}{c}. \quad (12)$$

The time for going out and coming back is different!

The difference depends on the time of the experiment!

If a rocket moves at 30m/s^2 for 400s ($a_{\text{sp.-shut.}} = 29.4\text{m/s}^2$, $a\bar{t} \approx 12\text{km/s} \gtrsim 11.2$ km/s — the 2nd cosmic velocity), the difference in arrival time will be 40 ppm.

If $\bar{x}_2 - \bar{x}_1 = 1\text{m}$, the difference in arrival time will be about $1.3 \times 10^{-13}\text{s}$.

— It is obviously measurable!

Redshift

In Møller frame

The redshift at $\tilde{x}_2$ is

$$z = \frac{g(\tilde{x}_2 - \tilde{x}_1)}{c^2}. \tag{13}$$

In the new frame

The redshift observed by the 'static' observer at $\bar{x}_2$ is

$$z \approx \frac{a(\bar{x}_2 - \bar{x}_1)}{c^2} e^{a\bar{t}_1/c} \approx \frac{a(\bar{x}_2 - \bar{x}_1)}{c^2}(1 + \frac{a\bar{t}_1}{c}). \tag{14}$$

The redshift of the light emitted from $\bar{x}_2$ and observed at $\bar{x}_1$ is

$$z \approx -\frac{a(\bar{x}_2 - \bar{x}_1)}{c^2} e^{-a\bar{t}_1/c} \approx -\frac{a(\bar{x}_2 - \bar{x}_1)}{c^2}(1 - \frac{a\bar{t}_1}{c}). \quad (15)$$

The redshift is different from time to time!

The redshift is getting larger and larger, while the blueshift is getting smaller and smaller.

If a rocket moves at 30m/s^2 for 400s, the redshift difference will be 4×10^{-5}.

Note: Y. Cheng et al (*CPL* **22**(2005)2530) Rhodium Mössbauer effect can be used to detect the gravitational redshift for $\Delta h = 30$nm on the Earth.

Hence, the above effect might be testable.

IV. CM in Accelerated Frames

The second law of mechanics in the new frame is

$$\begin{aligned} m_0 c \frac{dU^0_{\rm acc}}{ds} &= F^0_{\rm acc} - m_0 \frac{a}{c^2} (U^0_{\rm acc})^2 \tanh \frac{a\bar{t}}{c} \\ m_0 c \frac{dU^1_{\rm acc}}{ds} &= F^1_{\rm acc} - m_0 \frac{a}{c^2} (U^0_{\rm acc})^2 {\rm sech} \frac{a\bar{t}}{c} \\ m_0 c \frac{dU^2_{\rm acc}}{ds} &= F^2_{\rm acc} \\ m_0 c \frac{dU^3_{\rm acc}}{ds} &= F^3_{\rm acc}, \end{aligned} \tag{16}$$

where the subscript acc represents the quantity is measured in the accelerated frame.

Newtonian approximation

The 2nd law reduces to

$$\boxed{m_0 \frac{dU^i_{\rm acc}}{d\bar{t}} = F^i_{\rm acc} - m_0 a \delta^i_1}. \tag{17}$$

It's the standard form of the NM in an accelerated frame.

In addition, Eq. (3) reduces to

$$\boxed{t = \bar{t}, \qquad x = \bar{x} + \frac{1}{2} a \bar{t}^2}. \tag{18}$$

They are the standard relations between coordinates in an inertial and a uniformly accelerated system.

In contrast, the 2nd law in Møller frame reduces to

$$m_0 c \frac{dU^i_{\mathrm{M}}}{ds} = F^i_{\mathrm{M}} - \frac{m_0 g}{1 + g\tilde{x}/c^2}\delta^i_1 = F^i_{\mathrm{M}} - m_0 a(\tilde{x})\delta^i_1.$$

under the Newtonian approximation. Here, the subscript M represents the quantity measured in Møller frame.

It is not the standard form of the 2nd law of NM in an accelerated frame even though the acceleration of constant-$\tilde{x}$ trajectory is $g/(1 + g\tilde{x}/c^2)$.

The second law in the Rindler metric does not reduce to the standard form in an accelerated frame either.

Hamiltonian of a massive particle

The Lagrangian for a relativistic massive particle is

$$L_t = -mc^2 \left(1 - 2\frac{\dot{\bar{x}}}{c}\sinh\frac{at}{c} - \frac{\dot{\bar{x}}^2}{c^2} - \frac{\dot{\bar{y}}^2}{c^2} - \frac{\dot{\bar{z}}^2}{c^2}\right)^{1/2} \quad (19)$$

The canonical momentum are

$$\left\{\begin{array}{c} p_{\bar{x}} \\ p_{\bar{y}} \\ p_{\bar{z}} \end{array}\right\} = \left\{\begin{array}{c} \dfrac{\partial L_t}{\partial \dot{\bar{x}}} \\ \dfrac{\partial L_t}{\partial \dot{\bar{y}}} \\ \dfrac{\partial L_t}{\partial \dot{\bar{z}}} \end{array}\right\} = \left\{\begin{array}{c} m\dot{\bar{x}} + mc\sinh\dfrac{at}{c} \\ m\dot{\bar{y}} \\ m\dot{\bar{z}} \end{array}\right\} \times$$

$$\left(1 - 2\frac{\dot{\bar{x}}}{c}\sinh\frac{at}{c} - \frac{\dot{\bar{x}}^2}{c^2} - \frac{\dot{\bar{y}}^2}{c^2} - \frac{\dot{\bar{z}}^2}{c^2}\right)^{-1/2} \quad (20)$$

The Hamiltonian is

$$
\begin{aligned}
H &= \dot{\bar{x}} p_{\bar{x}} + \dot{\bar{y}} p_{\bar{y}} + \dot{\bar{z}} p_{\bar{z}} - L \\
&= (m^2c^4 + p_{\bar{x}}^2 c^2 + p_{\bar{y}}^2 c^2 + p_{\bar{z}}^2 c^2)^{1/2} \cosh \frac{at}{c} - c p_{\bar{x}} \sinh \frac{at}{c}
\end{aligned}
\tag{21}
$$

It satisfies

$$
\begin{aligned}
&(H \mathrm{sech} \frac{at}{c} + c p_{\bar{x}} \tanh \frac{at}{c})^2 \\
&= (m^2c^4 + p_{\bar{x}}^2 c^2 + p_{\bar{y}}^2 c^2 + p_{\bar{z}}^2 c^2).
\end{aligned}
\tag{22}
$$

The Poisson equations give rise to

$$\dot{p}_{\bar{x}} = 0, \qquad \dot{p}_{\bar{y}} = 0, \qquad \dot{p}_{\bar{z}} = 0.$$

It shows that the momentum is conserved in the accelerated frame. It reflects spatial translation invariance of the metric.

The Hamiltonian is not conserved in the frame because

$$\dot{H} = a(m^2c^2 + p_{\bar{x}}^2 + p_{\bar{y}}^2 + p_{\bar{z}}^2)^{1/2} \sinh \frac{at}{c} - ap_{\bar{x}} \cosh \frac{at}{c}.$$

It is the result of the violation of time-translation invariance in the frame.

However, the following function is conserved,

$$\mathcal{H} = H \text{sech} \frac{at}{c} + cp_{\bar{x}} \tanh \frac{at}{c}. \tag{23}$$

The non-relativistic approximation of the Hamiltonian:

$$H_{\mathrm{nr}} := H - mc^2 = \frac{1}{2m}(p_{\bar{x}}^2 + p_{\bar{y}}^2 + p_{\bar{z}}^2) + \frac{1}{2}ma^2t^2 - atp_{\bar{x}}. \tag{24}$$

$$\dot{H}_{\mathrm{nr}} = \dot{H} = ma^2t - ap_{\bar{x}}. \tag{25}$$

Obviously, the non-relativistic Hamiltonian is not conserved either.

The conserved quantity is

$$\mathcal{H}_{\mathrm{nr}} = H_{\mathrm{nr}} - \frac{1}{2}ma^2t^2 + atp_{\bar{x}} \tag{26}$$

V. QM in Accelerated Frames

Klein-Gordon equation with an external field

$$(\Box + \frac{m^2c^2}{\hbar^2} + \frac{2mW_\mu U^\mu}{\hbar^2 c})\Psi = 0, \tag{27}$$

where W_μ is 4-EM vector due to external field and U^μ the 4-velocity of an obs.

For a static observer $U^\mu = \{c, 0, 0, 0\}$ in Minkowski frame, the external field becomes a potential $W_\mu U^\mu = cV$. Let

$$\Psi = e^{-imc^2t/\hbar}\Phi(t, \boldsymbol{r}) \tag{28}$$

and take the non-relativistic approximation, one obtains

$$i\hbar\frac{\partial}{\partial t}\Phi(t, \boldsymbol{r}) = \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V(t, \boldsymbol{r})\right]\Phi(t, \boldsymbol{r}). \tag{29}$$

Similarly, for a static observer $U^{\mu} = \{c, 0, 0, 0\}$ in Møller frame, the external field becomes a potential $W_{\mu}U^{\mu} = c\tilde{V}(\tilde{t}, \tilde{\boldsymbol{r}})$. Let

$$\Psi = e^{-imc^2\tilde{t}/\hbar}\Phi(\tilde{t}, \tilde{\boldsymbol{r}}) \tag{30}$$

and take the non-relativistic approximation, one obtains

$$i\hbar\frac{\partial}{\partial\tilde{t}}\Phi(\tilde{t}, \tilde{\boldsymbol{r}}) = \left[-\frac{\hbar^2}{2m}\tilde{\boldsymbol{\nabla}}^2 + mg\tilde{x} + \tilde{V}\right]\Phi(\tilde{t}, \tilde{\boldsymbol{r}}) \tag{31}$$

$$= \left[-\frac{\hbar^2}{2m}\tilde{\boldsymbol{\nabla}}^2 + \frac{ma\tilde{x}}{1 - \frac{a\tilde{x}}{c^2}} + \tilde{V}\right]\Phi(\tilde{t}, \tilde{\boldsymbol{r}}). \tag{32}$$

On the other hand, the Schrödinger equation in a static uniform gravitational field is

$$i\hbar\frac{\partial}{\partial t}\Phi(t,\boldsymbol{r}) = \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + m_G g x + V\right]\Phi(t,\boldsymbol{r}) \quad (33)$$

which has the same form as Eq.(31), but the inertial m is replaced by gravitational mass m_G and g is replaced by the gravitational acceleration.

Therefore, only when $a\tilde{x}/c^2 \ll 1$, we may conclude from $m = m_G$ that the equivalence principle is valid at quantum level (because the 2nd term on RHS of Eq.(32) deviates from the linear form).

For a ‘static’ observer $U^\mu = \{c, 0, 0, 0\}$ in the new frame, the external field becomes a potential $W_\mu U^\mu = c\bar{V}$. Let

$$\Psi = e^{-imc^2\bar{t}/\hbar}\Phi(\bar{t}, \bar{\boldsymbol{r}}) \tag{34}$$

and take the non-relativistic approximation, one obtains

$$\boxed{i\hbar\frac{\partial\Phi(\bar{t}, \bar{\boldsymbol{r}})}{\partial\bar{t}} = \left[-\frac{\hbar^2}{2m}\bar{\boldsymbol{\nabla}}^2 + i\hbar a\bar{t}\frac{\partial}{\partial\bar{x}} + \frac{1}{2}ma^2\bar{t}^2 + \bar{V}\right]\Phi(\bar{t}, \bar{\boldsymbol{r}})} \tag{35}$$

This equation is dramatically different from the Schrödinger equation in a static uniform gravitational field!

Based on this equation, it is difficult to conclude that the equivalence principle is valid at quantum level even when $m = m_G$.

However, if we set

$$\Phi(\bar{t}, \bar{\boldsymbol{r}}) = e^{ima\bar{t}\bar{x}/\hbar}\psi(\bar{t}, \bar{\boldsymbol{r}}), \tag{36}$$

the equation becomes

$$i\hbar\frac{\partial\psi(\bar{t}, \bar{\boldsymbol{r}})}{\partial\bar{t}} = \left(-\frac{\hbar^2}{2m}\bar{\boldsymbol{\nabla}}^2 + ma\bar{x} + \bar{V}\right)\psi(\bar{t}, \bar{\boldsymbol{r}}) \tag{37}$$

If we analyze experiments by use of Eq.(37) and the Schödinger equation in a static uniform gravitational field, we shall find that the principle equivalence is valid at quantum level when $m = m_G$.

VI. Thermal Property

Under the Wick rotation

$$\bar{t} \to \tau = i\bar{t}, \qquad s \to is_E \tag{38}$$

Eq. (4) becomes

$$ds_E^2 \;=\; d\tau^2 - 2c\sin\frac{a\tau}{c}d\tau d\bar{x} + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2. \tag{39}$$

The metric has a period $2\pi c/a$ in τ direction. The Green function will acquire the imaginary period $-i2\pi c/a$ automatically. Therefore, according to the Green's function theory at finite temperature, all 'static' observers in the new frame will observe the same finite Hawking temperature

$$T_H = \frac{a\hbar}{2\pi c k_B}. \tag{40}$$

VI. Conclusion

1. K'-system Einstein mentioned in The Meaning of Relativity should be the new uniformly accelerated frame.

2. The new frame is the better generalization of classical non-relativistic, linear, uniformly accelerated frame than the Møller frame (or Rindler frame).

3. The experimental data in a rocket is suggested to be analyzed in the new frame.

4. $m_I = m_G \quad \not\Leftrightarrow \quad$ Gravity $\Leftrightarrow$ acceleration

Thank You Very Much
for Your Attention!