EXPERIMENTAL STUDY OF THE BROWNIAN MOTION

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1. Introduction

The Brownian motion is the spontaneous fluctuation occured in nature and plays important role in the precision measurement.

It is theoretically shown that the mean squared fluctuation of a physical quantity x, denoted by $\langle x^2 \rangle$, is given as

$$\langle x^2 \rangle = k_B T \frac{dx}{dX}, \tag{1}$$

where x and X are an extensive and the relevant intensive quantities, respectively, k_B is th Boltzmann constant and T is the thermodynamic temperature.[1)]

The Brownian motion limits the precision of measurement and is regarded as the essential source of error, but, on the other hand, it carries informations on the physical nature of the object which brings about it, as is seen in Eq.(1). Therefore we can obtain information by analyzing the Brownian motion experimentally. Perrin determined the Avogadro constant from the mean squared value of the observed Brownian motion of fine particles floating on liquids and Kappler did it from that of the suspended mirror.[2)]

Further information is obtained applying modern statistical analysis with the aid of computer as shown later.

In any case, it is essential to show that the observed fluctuation is truly spontaneous, and the elaborate techniques are required to do it.

The experimental set up to measure the Brownian motion of the suspended mirror, together with the algorithm of the spectral analysis is shown in this paper.

2. The Brownian Motion of the Suspended Mirror

The suspended mirror is a simple model of a mechanical system which has wide application to metrology. The Brownian motion of the suspended mirror is described

$$I\ddot{\theta} + \tau\theta = F, \tag{2}$$

where I is the moment of inertia of the mirror and τ is the torsion constant of the suspension wire. F is divided into two parts: systematic and random. The systematic part is denoted $-2\alpha\dot{\theta}$, where α is the damping coefficient, being the moment due to the collision by the molecules of gas around the mirror. The random part appears because of the statistical nature of the molecular movement and is assumed to obey the normal distribution under the assumption of the molecular flow.

Since F has the statistical nature, Eq.(2) is a stochastic differential equation and is usually called the Langevin equation.

The mean square of the deflection is given according to the law of equipartition of energy

$$\langle\theta^2\rangle = \frac{k_B T}{\tau} . \tag{3}$$

This is also obtained from Eq.(1) using $dx/dX=1/\tau$.

When the mean free path of the gas molecules is far larger than the size of the mirror, the damping coefficient α is given

$$\alpha = 4p\sqrt{\frac{m}{2\pi k_B T}}\,\frac{IS}{M} , \tag{4}$$

where m is the mass of the molecules, p is the gas pressure and I,S and M are the moment of inertia, the surface area and the mass of the mirror, respectively.

It is understood from Eq.(4) that p can be absolutely determined since every parameter appearing in Eq.(4) can be physically measurable or precisely known.

3. Measurement of the Brownian Motion of the Suspended Mirror

It is most important and also difficult to eliminate external vibration exerted to the suspension system when we measure the Brownian motion of the suspended mirror, since it is very sensitive to the mechanical disturbance. The measurement system was designed by taking account of the matter. It was carried out in Tokyo where the spectrum of the ground vibration had dominant broad peak above 2Hz and decreased with frequency, so the natural frequency of the suspension system was chosen to be about 5Hz.

The natural angular frequency ω_o is expressed as $\omega_o = \tau / I$, and Eq.(3) is rewritten

$$\langle\theta^2\rangle = \frac{k_B T}{I\omega_o^2} \tag{5}$$

from which we can estimate the order of $\langle\theta^2\rangle$. I depends on the size of the suspended mirror which is determined taking account of the condition of measuring the deflection. The order was estimated to be 0.1μrad in our case.

The suspended mirror was fixed in a closed vessel with a window to observe the deflection optically. The vessel is evacuated in order to make the experiment in vacuum.

3.1 The Suspension System

The suspension system used for the measurement is shown in Fig.1. The mirror of hexagonal shape is suspended from both sides by a fibre of 20 μm diameter instead of suspending from one side as Kappler did in order to facilitate the manipulation.

A hook was attached to measure the moment of inertia; small wires of known but different moment of inertia were hung one after another and the period of oscillation was measured each time. The moment of inertia I was determined using the well known formula

$$I = \frac{T_2^2 I_1 - T_1^2 I_2}{T_1^2 - T_2^2} \quad . \tag{6}$$

Three small bars were also attached to the mirror. They are the most intriguing pieces to make the measurement of the Brownian motion successful; when the center of gravity of the mirror is out of the axis of suspension, the horizontal vibration of the frame brings about the rotational vibration by mode coupling and superposes to the deflection due to the random collision of gas molecules thus blurs the Brownian motion. Therefore it is essential to adjust the center of gravity to come on the axis of suspension.

Placing the frame with the axis of suspension horizontal and observing the deflection of the mirror as the frame was rotated about this axis due to its center of gravity being off the axis, one of the three bars was bent by small flame so that the deflection gets small. Finally the amount of the off-axis was measured using the device specially designed. Repeating the above mentioned process, it was made as small as 0.16 μm.

3.2 Measurement of the Deflection of the Mirror

In order to measure the Brownian motion of the suspended mirror, the small angle measuring device was set up.[3] It was stable in the order of 0.1nrad and sensitive enough to measure the Brownian motion, and had the dynamic range to measure the deflection of as large as 10 times the expected mean value which was possible to occur statistically. It was self-calibrated each time at the beginning of the experiment.

3.3 The Vibration Isolation

The vibration isolation is the key technique in the measurement of the Brownian motion.

Horizontal vibration should be isolated as good as possible since the axis of suspension was set vertical. In order to have the effective isolation, the experimental devices set on the table were suspended from the ceiling altogether by three Nylon ropes of 2.7m long. The lengths were made adjustable to keep the table horizontal. An oil damper was fixed underneath the table.

The natural frequency of the isolation system was as low as 0.4Hz, and the rms amplitude of the horizontal vabration was about 0.04μm in midnight, which made the experiment at the order of 0.1Pa possible.

4. Statistical Analysis of the Brownian Motion

Since the Brownian motion is of statistical nature, it is characterized by the power spectral density function. The suspended mirror is a 2nd order system, in other words, a system with the restoring moment, and the power spectrum is expected to have a peak around the natural frequency.

The width depends on the damping characteristics which is related to the pressure or the number density of the molecules in the vessel. Therefore it is expected to obtain information about the gas pressure through the satistical analysis of the Brownian motion.

4.1 Power Spectrum of the Brownian Motion

The deflection of the mirror is interpreted as the response of the suspended system to the collision of the molecules as the input. According to the theory of random process, the power spectrum of the output S_{yy} of the system whoses response fuction is $G(\omega)$ is given

$$S_{yy} = |G(\omega)|^2 S_{xx} \tag{7}$$

where S_{xx} is the power spectrum of the input to the system.

In our case, $S_{xx} = \sigma^2$=constant under the assumption of the random independent collision, and $G(\omega)$ is

$$G(\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + j2\zeta\omega_0\omega}, \tag{8}$$

then the power spectrum of the deflection of the mirror, $S_{\theta\theta}$ is

$$S_{\theta\theta} = \frac{\omega_0^4 \sigma^2}{\omega^4 - 2(1-2\zeta^2)\omega_0^2\omega^2 + \omega_0^4}. \tag{9}$$

The autocorrelation function $\Phi(\tau)$ is derived from $S_{\theta\theta}$ as

$$\Phi(\tau) = \exp(-\zeta\omega_0\tau)\left\{\cos\sqrt{1-\zeta^2}\,\omega_0\tau + (\zeta/\sqrt{1-\zeta^2})\sin\sqrt{1-\zeta^2}\,\omega_0\tau\right\}. \tag{10}$$

4.2 Estimation of the Power Spectrum and the Damping Factor

The observed record of the deflection of the mirror is regarded as a sample of the Brownian motion and the power spectrum calculated from the record must show statistical variation.

The most popular algorithm used for the estimation of power spectrum is called the Blackman-Tukey algorithm; the autocorrelation function is calculated using the observed record through a window fuction in order to avoid the effect of trancating the data, then the power spectrum is obtained as the Fourier transform of the autocorrelation function.

This is usable for the random data which gives rise to a broad spectrum. The power spectrum of the Brownian motion of the suspended mirror is expected to have a sharp peak around the natural frequency, and the Blackman-Tukey algorithm fails to apply, in other words , the power spectrum calculated will give flatten spectrum.

A new algorithm based on the modelling in time domain has shown to give reliable estimation of the power spectrum of the Brownian motion.

Let x_n; n=1,2,....,N be the sampled data from the observed record with a constant interval ΔT, then x_n will satisfy following difference equation if they represent the Brownian motion:

$$x_n + a_1 x_{n-1} + a_2 x_{n-2} = \varepsilon_n + b_1 \varepsilon_{n-1}, \tag{11}$$

where ε_n ; n=1,2,.....,N are the samples from the independent Gaussian process, and

$$a_1 = -2\exp(-\zeta\omega_0\Delta T),$$

$$a_2 = \exp(-2\zeta\omega_0\Delta T),$$

and

$$b_1 = \exp(-\zeta\omega_0\Delta T)\sin\sqrt{1-\zeta^2}\,\omega_0\Delta T/\sqrt{1-\zeta^2}\,\omega_0\Delta T.$$

The power spectrum of x_n is calculated applying the algorithm proposed by Akaike. Investigating the observed record, we have found some spurious behavior together with the drift, so we have not adopted Eq. (11) as the model but instead the following model

$$x_n = \sum_{m=1}^{M} c_m x_{n-m} + \varepsilon_n, \tag{12}$$

since the algorithm has been established to get the optimal M under an information criterion.[4] Physical meaning of the coefficients are not clear, however, and the result of the analysis should be investigated to ascertain the observation of the Brownian motion.
The power spectrum calculated by the above mentioned algorithm is shown in Fig. 2 around the natural frequency. Fitting Eq.(9) to the calculated power spectrum by the method of the least square, the estimated values of ζ and σ^2 are given for each observed data.

5. Experimental Results

Fig.3 shows the frequency distribution of one of the observed data and the Gaussian distribution. It is seen that the observed data approximately obey the Gaussian distribution as the Brownian motion does. The mean square deflection values have been compared with $k_B T/$ and those data which have shown large discrepancy have been discarded.

The experiments have been performed in the pressure down to 0.1Pa and the estimated values of ζ have been plotted against the pressure as is shown in Fig.4. ζ is seen linearly dependent on the pressure lower than 1Pa and the least square fitting to a curve has yielded a straight line

$$\zeta = 0.16\,p\ (\mathrm{Torr}). \tag{13}$$

The proportional constant 0.16 has been compared with the theoretical value derived from Eq.(4) and the good agreement has been confirmed. Thus we can assert that the Brownian motion has been observed.

ζ seems to be almost independent of the pressure higher than 10Pa, and it is interpreted qualitatively that the effect of the increase of number of molecules pertaining the moment transfer cancels out that of the decrease of their mean free path.

6. Conclusion

The Brownian motion of the suspended mirror has been observed and it has been shown that the damping coefficient was proportional to the pressure in the molecular flow region which was expcted from the theoretical analysis.[5] The absolute measurement of pressure is possible through the spectral analysis of the Brownian motion of the suspended mirror.[6] It is most important to have the suspension system with the center of gravity on the axis of suspension in order to measure the Brownian motion.

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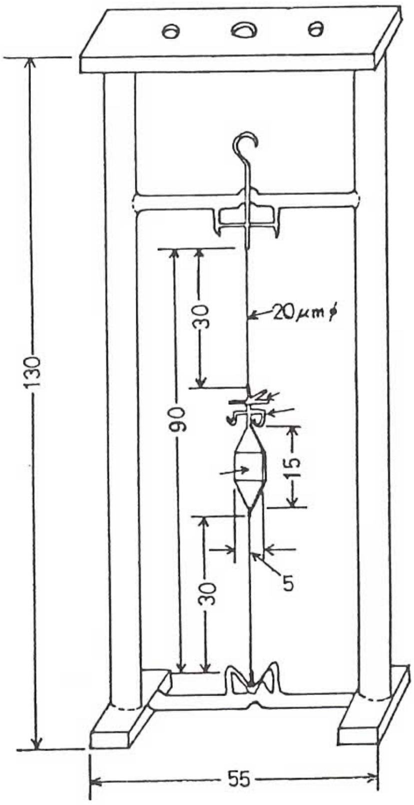

Fig.1 Suspension system made of fuzed quartz

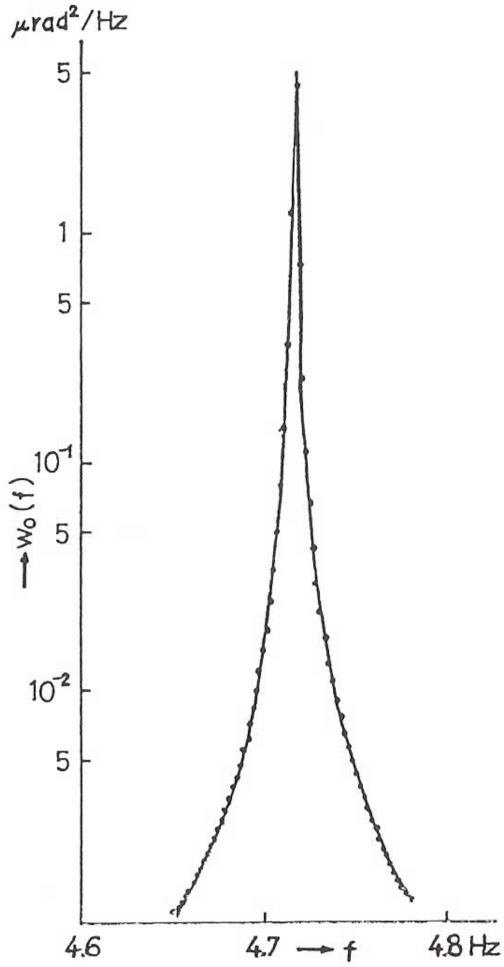

Fig.2 Power spectrum around the natural frequency

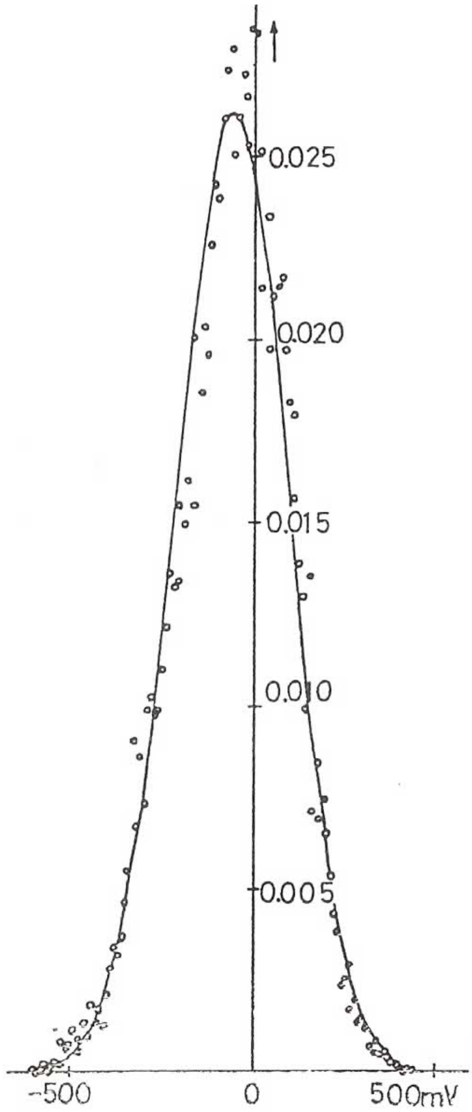

Fig.3 Frequency distribution of the sampled data

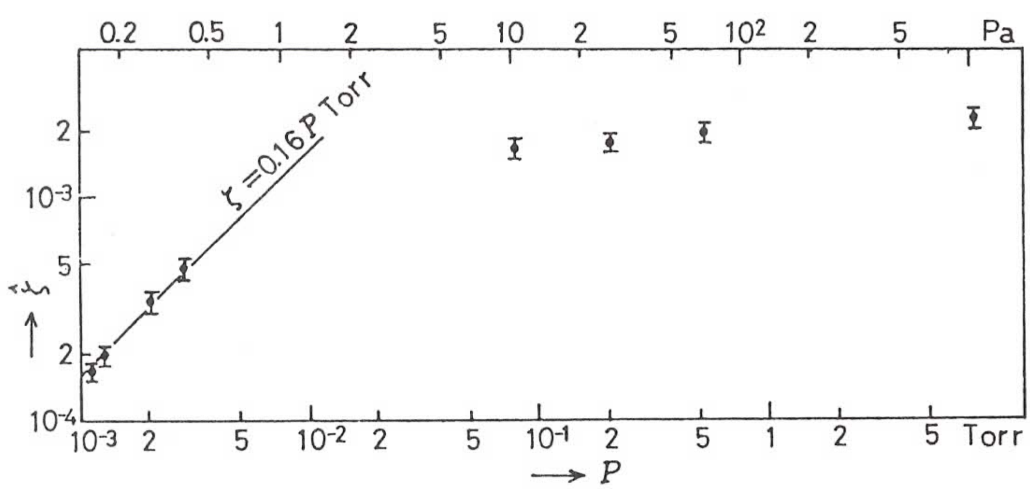

Fig.4 Pressure dependence of the damping ratio ζ $(=2\alpha/\sqrt{\tau I})$