Chapter 2

Group Structure

To be able to use groups in physics, or mathematics, we need to know what are the important features distinguishing one group from another. This is under the heading of 'GROUP STRUCTURE' which we will begin to discuss in this chapter. Once the structure is understood, we need to know how to use this knowledge to exploit the symmetry in physics. For that we need a knowledge of GROUP REPRESENTATION, a topic which will be discussed in the subsequent chapters.

2.1 Classes

The most important structure of a group is its CLASSES. Before plunging into what that is, let us first dispense with some definitions.

1. A group is `abelian` if $gg' = g'g$ for all $g, g' \in G$. Otherwise it is `nonabelian`. For example, the Z_n group and the $SO(2)$ is abelian, but the $SO(3)$ group is non-abelian.

2. The `center` of G, denoted by $Z(G)$, is the abelian subgroup which commutes with every elements of G. The center always contains the unit element e.

3. The most useful tool to specify a group is its class structure. A `class` (or 'conjugacy class') is a collection of elements invariant under SIMILARITY TRANSFORMATION. Formally, ρ is a class if and only if $g\rho g^{-1} \subset \rho$ for every $g \in G$. Here are some salient properties of classes.

(a) The INTERSECTION of two classes is a class. Thus, G can be covered by a unique set of non-overlapping irreducible classes ρ_a so that $g\rho_a g^{-1} = \rho_a$ for every $g \in G$. The total number of classes will be denoted by $|\mathcal{C}|$. If we need to specify which group $|C|$ refers to, we write $|\mathcal{C}|_G$.

Proof : For every $g \in G$, $g\rho_a g^{-1} \subset \rho_a$. The right hand side cannot be a *proper* subset, for otherwise there are two distinct elements r_a and r'_a in ρ_a so that $gr_a g^{-1} = gr_b g^{-1}$, but that implies $r_a = r_b$, contradictory to the original assumption. Hence $g\rho_a g^{-1} = \rho_a$ for every $g \in G$. ■

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The number of objects in ρ_a will be denoted by $|\rho_a|$. As g runs over G, $g\rho_a g^{-1}$ runs over ρ_a a number of times, hence $|G|$ must be divisible by $|\rho_a|$ (*prove this !*) .

Unless otherwise specified, when we say a class we mean an irreducible class from now on.

(b) $\rho_1 := (e)$ is always a class; every element in $Z(G)$ also forms a class.

(c) Each element of an ABELIAN GROUP is a class. Hence $|G| = |\mathcal{C}|$ in that case.

(d) If one element in ρ_c is inside $\rho_a\rho_b$, then the whole class ρ_c is (*prove this !*) . Hence $\rho_a\rho_b = \sum_c \gamma_{abc}\rho_c$ for some integers γ_{abc}, known as `structure constants`. They are somewhat analogous to the structure constants in the commutation relations of a Lie algebra which we will deal with later. The group structure constants can be computed elaborately from the group multiplication table, but they can also be computed from the 'SIMPLE CHARACTERS' of the group. See Chap. 6 for that.

(e) The `class sum` operator $c_a := \sum_{g\in\rho_a} g$ commutes with every $y \in G$, so that in some sense it can be treated just like a number. Unlike members of the center of the group, a class sum operator is *not* a member of the group, but it is a member of the 'group algebra' to be discussed later.

Proof : $yc_a y^{-1} = c_a$ because $y\rho_a y^{-1} = \rho_a$ ■

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(f) It follows from $\rho_a\rho_b = \sum_c \gamma_{abc}\rho_c$ that (*prove this !*)

$$c_a c_b = \sum_c \gamma_{abc} c_c. \tag{2.1}$$

4. Starting from a group, we can form a `group algebra` by allowing addition and multiplication by numbers.

2.2 Subgroups

The next important tool in analyzing the structure of a group is its SUBGROUPS, especially the kind known as NORMAL SUBGROUPS. A group that cannot be broken down any further in terms of normal subgroups is called SIMPLE. Simple groups are liked PRIME NUMBERS. All integers can be factored into prime numbers; similarly every finite group in some sense can be broken down into simple groups (JORDAN-HOLDER THEOREM). However, there is a difference. Integers can be reconstructed by multiplying the component prime numbers, but it may not be possible to reconstruct the group G from the component simple groups. In that sense a simple group is not as 'complete' as a prime number, but in another sense it is more complete because all simple finite groups are now classified and known, whereas large prime numbers are not known.

For Lie groups, whose local structure can be cast in terms of LIE ALGEBRAS, the classification of SIMPLE LIE ALGEBRAS is also completely known. We will discuss these simple Lie algebras systematically in Chapter 9.

SIMPLE LIE ALGEBRAS consist of 4 `regular series` (A_n, B_n, C_n, D_n), and 5 `exceptional algebras` $(E_6, E_7, E_8, F_4, G_2)$. FINITE SIMPLE GROUPS consist of $Z_p, A_n (n \geq 5)$, plus 16 A, B, C, D, E, F, G groups over *finite fields*, and 26 `sporadic groups`, which are the analog of the exceptional Lie algebras. The largest sporadic group is known as the `Monster group`, whose order is

$$80,801,742,479,451,287,588,645,990,496,171,075,700,575,436,800,000,000,$$

roughly 8×10^{52}. As a comparison, the number of protons contained in the EARTH is 3.6×10^{51}. Despite its monstrous size and difficulty to understand, the Monster group is studied quite intensely because it seems to contain many deep connections with other branches of mathematics.

With this general introduction, let us now look at some of the details.

1. If $H \subset G$ is a group, then it is called a `subgroup` of G. A `proper subgroup` is one in which $H \neq G$. The following symbols are also used to designate respectively the former and the latter: $H \leq G,\ H < G$.

2. If a group G has a subgroup H, then G can be broken down into copies of H in the following way. The set of elements gH for some $g \in G$ is called a `left coset`; the set of elements Hg is called a `right coset`. Unless $g = e$, generally a coset is not a subgroup. Clearly the number of elements in a coset is equal to $|H|$. Two (left or right) cosets overlap only if they are identical: if $g_1 h_1 = g_2 h_2$, then $g_1 = g_2(h_2 h_1^{-1})$, so $g_1 \in g_2 H$. Similarly, $g_2 \in g_1 H$, hence $g_1 H = g_2 H$. The same is true for right cosets. Thus G can be covered by non-overlapping left cosets of H, or by its non-overlapping right cosets. Hence, $|G|$ MUST BE DIVISIBLE BY $|H|$.

3. If $g^m = e$, then m is called the `order` of the element g. g and its powers form an abelian subgroup, hence m divides $|G|$. It follows then that every element of a finite group has a finite order.

4. $[g, h] := ghg^{-1}h^{-1}$ is called the `commutator` of g and h. The product of two commutators are not necessarily a commutator. The `commutator subgroup` of G, or the `derived subgroup` of G, is the group generated by all the commutators of G. It is usually denoted by G' or $G^{(1)}$. The commutator subgroup of the commutator subgroup is denoted as $G'' := G^{(2)}$, etc. If $G^{(k)} = e$ for some k, then the group is called `solvable`.

5. All abelian groups are solvable.

6. S_2, S_3, S_4 are solvable.

`Proof` : S_2 is solvable because it is abelian. For S_3 and S_4, the main observation is that the commutator $[g, h] = ghg^{-1}h^{-1}$ is ALWAYS an even permutation.

If $g = (12)$ and $h = (23)$, then $[g, h] = (123)(123) = (132)$. Similarly, if $g = (13)$ and $h = (32)$, then $[g, h] = (132)$, hence the commutator subgroup of S_3 must contain $A_3 = Z_3$. It must be exactly that because commutators must be even. Since Z_3 is abelian, its commutator subgroup is $\{e\}$, hence the second derived group of S_3 is e, making S_3 solvable.

The proof for $G = S_4$ is similar, but more complicated. By imitating the construction of S_3, we know that the generator of G' contains the 3-cycles. Since (123)(234)=(21)(34), it must also include the 2^2-cycles like (12)(34). As G' contains only the even permutations, G' must be identical to A_4.

Next, we will show that the derived group of A_4 is the `Klein group` K_4, consisting of $e, (12)(34), (13)(24), (14)(23)$. By the way, this group is isomorphic to $Z_2 \times Z_2$ (*prove this !*) .

To show that, we simply consider all possibilities. If g, h are both 2^2-cycles, then since all 2^2-cycles commute, $[g, h] = e$. If g is a 3-cycle and h is a 2^2 cycle, then ghg^{-1} is still a 2^2-cycle, so $(ghg^{-1})h^{-1} \in K_4$. The same can be seen to be true if g is a 2^2-cycle and h a 3-cycle when we write $[g, h] = g(hg^{-1}h^{-1})$. Lastly, suppose both g and h are 3-cycles. Since there are only 4 letters to permute, g and h either contain the same 3 letters, e.g., $g = (123), h = (213)$, or they have one letter different, e.g., $g = (123), h = (214)$. In the former case, g and h commute, so $[g, h] = e$. In the latter case, let a, b be the shared letters. If $g = (a, b, \times)$, then $h = (a, b, \circ)$ or $h = (b, a, \circ)$, where $\times \neq \circ$. In the former case, $[g, h] = (b, \times, \circ)(\circ, b, a)$ is a 2^2-cycle, according to our previous rule. In the latter case, $[g, h] = (\times, b, \circ)(\circ, a, b)$ is still a 2^2-cycle. This exhausts all possibilities, so we conclude that $(A_4)' = K_4$. Lastly, since K_4 is abelian, we get $(K_4)' = e$, hence $(S_4)'' = e$, and S_4 is solvable. ■

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7. HISTORICALLY, the name originated from the study of explicit solutions of algebraic equations. We know that algebraic equations of degrees 2, 3, 4 have explicit solutions expressed in terms of radicals, and those with degree 5 and higher do not. According to Galois, whether an algebraic equation of degree n is always explicitly solvable or not is related to whether the symmetry group S_n of its roots is a solvable

group or not. Hence the name. It turns out that the groups S_2, S_3, S_4 are solvable, but not S_n for $n \geq 5$.

8. The GROUP COMMUTATOR $[g,h] = ghg^{-1}h^{-1}$ is related to the commutator used in QUANTUM MECHANICS in the following way.

   Let t_1 and t_2 be some operators, and $[t_1, t_2] := t_1t_2 - t_2t_1$ be their commutator in the sense of quantum mechanics. Let $g = e^{i\xi t_1}$ and $h = e^{i\epsilon t_2}$, where ξ, ϵ are two real parameters. Then $g^{-1} = e^{-i\xi t_1}$ and $h^{-1} = e^{-i\epsilon t_2}$. When $\xi \ll 1$, $\epsilon \ll 1$, we can expand g and h in power series to the second order,

$$g \simeq 1 + i\xi t_1 - \xi^2 t_1^2/2 + O(\xi^3),\ h \simeq 1 + i\epsilon t_2 - \epsilon^2 {t^2}_2/2 + O(\epsilon^3),$$
$$g^{-1} \simeq 1 - i\xi t_1 - \xi^2 t_1^2/2 + O(\xi^3),\ h^{-1} \simeq 1 - i\epsilon t_2 - \epsilon^2 t_2^2/2 + O(\epsilon^3).$$

   Then up to the second order, one can verify that

$$ghg^{-1}h^{-1} \simeq 1 - \xi\epsilon[t_1, t_2],$$

   and this is the relation between the two 'commutators'.

9. A subgroup N such that $gNg^{-1} \subset N$ for all $g \in G$ is called an invariant or a `normal subgroup`. In that case $gNg^{-1} = N$ (*prove this*). We sometimes write $N \trianglelefteq G$ if N is a normal subgroup of G, and $N \triangleleft G$ if it is a proper normal subgroup.

   (a) Any subgroup of an abelian group is a normal subgroup (*prove this !*) .

   (b) A_n is a maximal normal subgroup of S_n (*prove this !*) .

10. A group without a proper normal subgroup (except the trivial group consisting of the identity e) is called `simple`.

    (a) For example, the CYCLIC GROUP Z_p for prime p is simple because it has no proper subgroup, hence no normal subgroup.

    (b) If G is simple, then $G' = G$ or $G' = (e)$. Thus simple groups and solvable groups occupy two ends of a spectrum.

    `Proof` : If G is abelian, then $[g,h] = e$ and hence $G' = e$. If N is a normal subgroup of G, then $gNg^{-1} = N$ implies that every

$n \in N$ can be written as $n = gn'g^{-1}$ for some $n' \in N$. If G is non-abelian and simple, then $N = G$, so every $n \in G$ can be written as $n = gn'g^{-1}$. That means every $m \equiv nn'^{-1} \in G$ can be written in the form $gn'g^{-1}n'^{-1} \in G'$, hence $G \subset G'$. Since G' is alwasy a subgroup of G, this implies $G = G'$. ■

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11. The left cosets of a normal subgroup coincide with the right cosets (*prove this*). In this case, the cosets themselves form a group as follows. Let $O_i = Nx_i$ $(i = 1, \cdots, k)$ be the right cosets that cover G. We can multiply two such cosets to get another because N is normal: $O_iO_j = (Nx_i)(Nx_j) = Nx_iNx_i^{-1}x_ix_j = NNx_ix_j = Nx_ix_j$, which is the right coset generated by x_ix_j. In the same way we can also show that the inverse of a coset is a coset if we take the 'identity coset' to be $Ne = N$. This group H of cosets is called the `factor group`, or the `quotient group`, and is denoted as $Q = G/N$.

    Note that although $Q = G/N$ is a group, it is NOT necessarily isomorphic to a subgroup of G because $Nx_1Nx_2 = Nx_3$ does not imply $x_1x_2 = x_3$. All that it says is $x_1x_2 = nx_3$ for some $n \in N$.

    Here are two examples, the first showing that G/N is isomorphic to a subgroup of G, and the second showing that G/N is not isomorphic to a subgroup of G. **[6.02]**

    (a) The group K_4 consisting of $e, (12)(34), (13)(24), (14)(23)$ is a normal subgroup of A_4. In fact, it is abelian and isomorphic to $Z_2 \times Z_2$ (*prove this !*) . The quotient group A_4/K_4 consists of the three cosets $K_4, (123)K_4, (132)K_4$. It is obviously isomorphic to the subgroup $Z_3 \subset A_4$ consisting of $e, (123), (132)$.

    (b) The group Z_4 consists of the four numbers $1, i, -1, -i$. It has one and only one proper subgroup Z_2 consisting of the number 1 and -1, which is necessarily normal because Z_4 is abelian. The quotient group Z_4/Z_2 consists of the two cosets $Z_2 = -Z_2$ and $iZ_2 = -iZ_2$, so it is isomorphic to Z_2. However, this quotient group is not isomorphic to a subgroup of Z_4 because the two numbers $(1, i)$ does not form a group, neither does $(-1, i), (1, -i)$, nor $(-1, -i)$.

    Note further than given two groups N and Q, there is more than one way to construct a group G so that N is a normal subgroup of G and Q is isomorphic to G/N. One way is to take G to be the `direct product` $N \times Q$, but it is often possible to construct also 'TWISTED' PRODUCTS known as semi-direct products (see item (13) of this section). How general G can be so that

12. `Jordan Holder theorem:` if G is not simple, then it can be decomposed into a sequence of maximal normal subgroups: $G = N_n \supset N_{n-1} \supset \cdots \supset N_0 = 1$, where N_i is a maximal normal subgroup of N_{i+1}. This series is not unique, but the length n is, and the SIMPLE quotient groups $H_{i+1} = N_{i+1}/N_i$ are unique up to permutations (the JH theorem).

    However, given a set of simple groups $H_{n+1}, \cdots, H_1 = N_1$, N_i CANNOT be uniquely obtained because of the possibility of taking 'twisted products'.

    The breakdown of a group this way is a bit like breaking down an integer into prime factors, with simple groups being the analog of prime numbers. In that sense, simple groups are the fundamental building blocks of groups just like prime numbers are the fundamental building blocks of integers.

    For example, $360 > 180 > 90 > 45 > 15 > 5 > 1$ is like the JH series, with the prime quotients 2, 2, 2, 3, 3, 5 the analog of the simple quotient groups.

13. A 1-1 mapping of G to G preserving the products is called an `automorphism` of G. In other words, an automorphism is an isomorphism of a group onto itself. If φ is an automorphism of G, then by definition $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$, so $\varphi(e) = e$ and $\varphi(g^{-1}) = \varphi(g)^{-1}$. The set of all automorphisms forms a group denoted by Aut(G).

    The automorphism $\varphi_q(G) := qGq^{-1}$, with $q \in G$, is called an `inner automorphism`. It clearly satisfies $\varphi_q(\varphi_{q'}(g)) = \varphi_{qq'}(g)$. Other automorphisms are called `outer automorphisms`. The automorphisms of an abelian group is all outer.

    If Q is a group, we can define a set of automorphisms φ_q on G for every $q \in Q$ such that $\varphi_q(\varphi_{q'}(g)) = \varphi_{qq'}(g)$. This implies $\varphi_e(g) = g$. I will simply refer this to be Q-AUTOMORPHISM OF G, but please note that this is not an official name. If $Q \subset G$, it may just be an inner automorphism with $q \in Q$. If $Q \not\subset G$, then qnq^{-1} does not make sense, so we cannot write it that way.

14. `Direct product:` The direct product $G = N \times Q$ of two non-overlapping groups N and Q is a group consisting of elements (n, q) so that $(n, q) \cdot (n', q') = (nn', qq')$. Its order is $|G| = |N| \cdot |Q|$.

15. `Semi-direct product:`

    A semi-direct product $G = N \rtimes_{\varphi} Q$ is a 'twisted' product: n' is 'twisted' by a q-automorphism of N before a direct product is taken. Formally, $(n, q) \cdot (n', q') = (n\varphi_q(n'), qq')$. The order of G is still $|N| \cdot |Q|$, but the semi-direct product depends on the automorphisms φ used.

    Note that the product is not symmetric between 1 and 2, hence G can be non-abelian even if both N and Q are abelian. Many non-abelian groups can be built up this way from the abelian cyclic groups Z_n. In fact, of the 93 groups of order less than 32, 88 of them can be built up that way.

    Using the properties of Q-automorphisms, it follows that

$$\begin{aligned} (n, q)^{-1} &= (\varphi_{q^{-1}}(n^{-1}), q^{-1}) \\ [(n_1, q_1) \cdot (n_2, q_2)] \cdot (n_3, q_3) &= (n_1, q_1) \cdot [(n_2, q_2) \cdot (n_3, q_3)] \,. \end{aligned}$$

    This shows that semi-direct product does form a group. Here are some other properties of a semi-direct product:

    (a) N is a normal subgroup of G because

$$\begin{aligned} (n, q)(n', e)(\varphi_{q^{-1}}(n^{-1}), q^{-1}) &= (n, q)(n'\varphi_{q^{-1}}(n^{-1}), q^{-1}) \\ &= (n\varphi_q(n'\varphi_{q^{-1}}(n^{-1})), e) \\ &= (n\varphi_q(n')\varphi_q(\varphi_{q^{-1}}(n^{-1}))), e) \\ &= (n\varphi_q(n')n^{-1}, e) \in N. \end{aligned}$$

    (b) The cosets generating G/N is of the form $g \cdot N$, where g can be taken to be elements of the form (e, q). Thus G/N is isomorphic to Q.

    (c) If $\varphi_q(n) = n$, then $(n_1, q_1)(n_2, q_2) = (n_1 n_2, q_1 q_2)$ is just the `direct product`.

(d) If Q and N are two non-overlapping subgroups of G and if the Q-automorphism is just an inner automorphism of G, $\varphi_q(n) = qnq^{-1}$, then (n, q) can be identified with nq, because

$$(n_1, q_1)(n_2, q_2) = (n_1 q_1 n_2 q_1^{-1}, q_1 q_2) = n_1 q_1 n_2 q_1^{-1} q_1 q_2 = n_1 q_1 n_2 q_2.$$

(e) **An example.** Consider $G = S_3$, the symmetric group of three objects, with order $3! = 6$. The 3-cycles generate a normal subgroup $N = A_3 \cong Z_3$, and the quotient group $Q = G/N \cong Z_2$, where Z_2 is generated by (12). In fact, the two cosets with respect to $N = A_3 = [e, (123), (132)]$ are $Ne = N$ and $N(12) = [(12), (13), (23)]$. Thus every element of S_3 is of the form $nq := (n, q)$, where $n = e, (123)$, or (132), and $q = e$ or (12). The product of two elements is $nqn'n' = (n\varphi_q(n'), qq')$, where $\varphi_e(n') = n'$ and $\varphi_{(12)}(n')$ is obtained from n' by interchanging 1 and 2. With this φ, $S_3 = Z_3 \rtimes_\varphi Z_2$.

Note that the direct product $Z_3 \times Z_2 = Z_6$ is a different group.