Chapter 9

Lie algebra

9.1 Lie group and Lie algebra

1. *from group to algebra:*

   Let $g_0 \in \mathcal{G}$ be a member of a Lie group $\mathcal{G}$, and N_0 a neighborhood of g_0. $g_0^{-1}N_0$ is then a neighborhood of the identity $e := 1$. Therefore, the structure of the group in any neighborhood N_0 is identical to the structure of the group near the identity, so most properties of the group is already revealed in its structure near the identity.

   Near the identity of an n-dimensional Lie group, we saw in §1.3 that a group element can be expressed in the form $g = 1 + i\vec{\xi}\cdot\vec{t} + O(\xi^2)$, where $\vec{t} = (t_1, t_2, \cdots, t_n)$ is the infinitesimal generator and $|\xi_i| \ll 1$. If $h = 1 + i\vec{\epsilon}\cdot\vec{t} + O(\epsilon^2)$ is another group element, we saw in §2.2 that the infinitesimal generator of $ghg^{-1}h^{-1}$ is proportional to $[\vec{\xi}\cdot\vec{t}, \vec{\eta}\cdot\vec{t}]$. Hence $[t_i, t_j]$ must be a linear combination of t_k,

   $$[t_i, t_j] = ic_{ijk}t_k. \tag{9.1}$$

   The constants c_{ijk} are called the `structure constants`; one n-dimensional group differs from another because their structure constants are different. The i in front is there to make c_{ijk} real when the generators t_i are hermitian. The antisymmetry of the commutator implies that $c_{ijk} = -c_{jik}$.

   In order to ensure

   $$[t_i, [t_j, t_k]] = [[t_i, t_j], t_k] + [t_j, [t_i, t_k]], \tag{9.2}$$

an equality known as the `Jacobi identity`, the structure constants must satisfy the relation

$$c_{jkl}c_{lim} + c_{ijl}c_{lkm} + c_{kil}c_{ljm} = 0. \tag{9.3}$$

2. *from algebra to group:*

   An n-dimensional `Lie algebra` is defined to be a set of linear operators t_i $(i = 1, \cdots, n)$ closed under commutation as in (9.1), that satisfies the Jabobi identity (9.2). Given a Lie algebra, one can use it as infinitesimal generators to construct group elements near the identity, then by repeated multiplication push the group elements further and further away from the identity, and eventually integrate it out to cover the whole group. It can be shown that this is feasible but we will not go into the details. The closure under commutation is to ensure that if $g \in G,\ h \in G$, then the commutator $ghg^{-1}h^{-1}$ is also in G. The Jacobi identity is there to make sure of associativity of group multiplication.

   t_i are usually matrices, but they can be differential operators.

   Character in the lower-case is usually used to denote a Lie algebra. Thus $\boldsymbol{su(n)}$ is the Lie algebra for the Lie group $SU(n)$. Unless otherwise specified, henceforth g shall stand for a Lie algebra.

   Since the Lie algebra reveals only the local structure of a group, it is possible that two different groups with different global structures may share the same algebra. The Lie algebra determines the differential equation from which to integrate out to reach the rest of the group, but the resulting solution may depend on the boundary condition. For example, $SO(3)$ and $SU(2)$ both share the same angular momentum Lie algebra $su(2)$, but their global topological structures are different, as we saw in §7.2.

3. If the generators of a Lie algebra g' coincides with some of the generators of another Lie algebra g, then g' is said to be a `subalgebra` of g. This relation is denoted as $\mathsf{g}' \subset \mathsf{g}$.

4. If $[t_i, t_j] = 0$ for all t_i, t_j in g, then g is said to be `abelian`.

9.2 Adjoint representation

1. If $d \times d$ matrices T_i satisfy the same commutation relations as t_i, $[T_i, T_j] = ic_{ijk}T_k$, then $\{T_i\}$ is said to form a d-dimensional `representation` of $\mathfrak{g}$. The most important representation is the n-dimensional representation called the *adjoint representation*, which is the counterpart of regular representation for a finite group.

2. The linear mapping $ad(a)$ defined by $ad(a)b := [a, b]$ gives rise to an n-dimensional representation of a Lie algebra known as the `adjoint representation`. This means, if $[a, b] = c$, then $ad([a, b]) = [ad(a), ad(b)] = ad(c)$.

   Proof : $[ad(a), ad(b)]d = [a, [b, d]] - [b, [a, d]] = [[a, b], d] = ad([a, b])d = ad(c)d$. ■
   .

   Explicitly, if $ad(t_i)t_j = ad(t_i)_{kj}t_k$, then $[ad(t_i)]_{kj} = ic_{ijk} = -ic_{jik}$. Note the index order of the matrix element.

   Adjoint representation is crucial in the Lie algebra theory because it converts its study into a problem in linear algebra representations. A very useful tool in studying the adjoint representation is the *Killing form*:

3. The bilinear symmetric form $K_{ab} := (a, b) := \mathrm{Tr}[ad(a)ad(b)]$ is called the `Killing form`. Jacobi identity shows that $([a, b], c) = (a, [b, c])$. Killing form is independent of the choice of basis, because it is defined through a trace, hence it indeed reflects the intrinsic structure of a Lie algebra.

   If we do choose a basis, then $(t_i, t_j) = ad(t_i)_{pq}ad(t_j)_{qp} = -c_{iqp}c_{jpq}$.

9.3 Killing form as a classification tool

1. Since K is symmetric, it can be diagonalized by an orthogonal matrix O so that $O^TKO := D = \mathrm{diag}(k_i)$ is diagonal. If we let $t'_i = O_{pi}t_p$, $[t'_i, t'_j] = ic'_{ijl}t_l$, then $k_i\delta_{ij} = \mathrm{Tr}(ad(t'_i)ad(t'_j))$ and $[ad(t'_i), ad(t'_j)] = ic'_{ijl}ad(t'_l)$. Multiply both sides by $ad(t'_p)$ and take the trace, we get

$ik_p c'_{ijp} = -ik_p c'_{jip} = \mathrm{Tr}(ad(t'_p)[ad(t'_i), ad(t'_j)]) = \mathrm{Tr}([ad(t'_p), ad(t'_i)]ad(t'_j))$ $= ik_j c'_{pij} = -ik_j c'_{ipj}$. To summarize, we have

$$k_p c'_{ijp} = k_j c'_{ipj}, \quad c'_{ijp} = -c'_{jip}. \tag{9.4}$$

We can further simplify the form by dividing t'_i by $\sqrt{|k_i|}$ if $k_i \neq 0$. From now on we shall assume that to be done and will refer to basis of this kind as `orthonormal`. This means

$$(\tau_i, \tau_j) = \kappa_i \delta_{ij}, \tag{9.5}$$

where $\tau_i = t'_i$ if $k_i = 0$, $\tau_i = t'_i/\sqrt{|k_i|}$ if $k_i \neq 0$, and $\kappa_i = +1, 0$, or -1. We will refer to the index i as positive, zero, or negative depending on whether κ_i is $1, 0$, or -1. Using (9.4), one can conclude that

> *In an orthonormal basis, $c_{ijk} = -c_{jik}$ is zero if the sign of j or i is zero but the sign of k is not. Otherwise, $c_{ijk} = \mp c_{ikj}$ if the signs of j and k are same/different.*

2. If the Killing form is positive definite, then (a, b) defines a proper scalar product. This defines a class of very important algebra known as **`semi-simple algebra`**, a class of algebra which we will study in much more detail later.

3. If the generators t_i are `hermitian` (so that the Lie group G is compact), then c_{ijk} are real. Moreover, $\langle t_i, t_j \rangle := \mathrm{tr}(t_i t_j)$ is a positive definite real symmetric form, so we can choose basis and normalization so that $\mathrm{tr}(t_i t_j) = \delta_{ij}$. Then $-ic_{jik} = ic_{ijk} = \mathrm{tr}([t_i, t_j]t_k) = \mathrm{tr}(t_i, [t_j, t_k]) = ic_{jki}$, so the structure coefficient is antisymmetric in the last two indices, showing that the Killing form (t_i, t_j) is positive definite. Hence the Lie algebras of all compact Lie groups are all semi-simple.

Example

In $\boldsymbol{su(2)}$, $t_i = \sigma_i$, the Pauli matrices, and $t_i = t_i^\dagger$. The structure constants are $c_{ijk} = \epsilon_{ijk}$, so the Killing form $(\sigma_i, \sigma_j) = +\epsilon_{ipq}\epsilon_{jpq}$ is positive definite.

9.4 Semi-simple and solvable algebras

1. There are two main types of Lie algebras: semi-simple, and solvable. Levi's theorem states that any real Lie algebra is a direct product of a semi-simple and a solvable algebra.

2. a subset $\mathbf{n} \subset \mathtt{g}$ is `invariant` if $[t, \mathbf{n}] \subset \mathbf{n}$ for every $t \in \mathtt{g}$. $\mathbf{n}$ is also called an `ideal`. Since $[\mathbf{n}, \mathbf{n}] \subset \mathbf{n}$, $\mathbf{n}$ is automatically a subalgebra. In fact, it is the subalgebra of a normal subgroup $N \subset G$. This is so because $gNg^{-1} \subset N$ for every $g \in G$. If $g = \exp(i\xi t)$, then for $|\xi| \ll 1$, $g \simeq 1 + i\xi t$, so $gNg^{-1} \simeq N + i\xi[t, N] \subset N$ implies that $[t, N] \subset N$. If $\mathbf{n}$ is the Lie algebra of N, then it also implies $[t, \mathbf{n}] \subset \mathbf{n}$.

   With this, we can define simple Lie algebra and solvable Lie algebra to be the Lie-algebraic equivalent of simple group and solvable group.

3. A `simple Lie algebra` is one without a proper ideal. The commutator subalgebra $\mathtt{g}^{(1)} := [\mathtt{g}, \mathtt{g}]$ is always an ideal, so if $\mathtt{g}$ is simple, then $\mathtt{g}^{(1)} = \mathtt{g}$. In particular, a simple algebra is always non-abelian.

   Example: $su(2)$.

4. A `semi-simple Lie algebra` is one without an abelian ideal.

5. Let $\mathtt{g} = \mathtt{g}_1 \oplus \mathtt{g}_2$, where $\mathtt{g}_i$ are simple algebras. In other words, an element $x \in \mathtt{g}$ looks like (x_1, x_2), $x_i \in \mathtt{g}_i$, and $[x, y] = ([x_1, y_1], [x_2, y_2])$. Then $\mathtt{g}$ is not simple, because $(\mathtt{g}_1,0)$ and $(0,\mathtt{g}_2)$ are proper ideals. However, these ideals are not abelian because $\mathtt{g}_i$ are simple. Hence, the algebra is semi-simple.

6. Let $\mathtt{g}^{(0)} = \mathtt{g}$, $\mathtt{g}^{(k)} = [\mathtt{g}^{(k-1)}, \mathtt{g}^{(k-1)}]$, then a `solvable` algebra is one in which $\mathtt{g}^{(k)} = 0$ for some finite k. If $[\mathtt{g}, [\mathtt{g}, \cdots, [\mathtt{g}, \mathtt{g}] \cdots]] = 0$ when the chain is sufficiently long, then the algebra is nilpotent. A nilpotent algebra is always solvable but not necessarily the other way around. A solvable algebra cannot be semi-simple because $\mathtt{g}^{(k-1)}$ is an abelian ideal.

   Note: every $\mathtt{g}^{(k)}$ is an ideal. This can be proven by induction as follows. First, $\mathtt{g}^{(1)}$ is an ideal. If $\mathtt{g}^{(k-1)}$ is, then for every $a \in \mathtt{g}$, $[a, \mathtt{g}^{(k)}] = [a, [\mathtt{g}^{(k-1)}, \mathtt{g}^{(k-1)}]] = [[a, \mathtt{g}^{(k-1)}], \mathtt{g}^{(k-1)}] + [\mathtt{g}^{(k-1)}, [a, \mathtt{g}^{(k-1)}]] \subset [\mathtt{g}^{(k-1)}, \mathtt{g}^{(k-1)}] = \mathtt{g}^{(k)}$, by induction.

Example

The `Heisenberg algebra` of canonical commutation relations, $[q_i, p_j] = i\hbar\delta_{ij}$, is nilpotent, hence solvable.

7. We will now state an important theorem without proof:

   `Cartan's criterion for semi-simple algebra:` $\mathtt{g}$ is semi-simple if and only if its Killing form is non-degenerate. That means, (a, b) defines a proper inner product in $\mathtt{g}$.

8. Let t_i $(i = 1 \cdots n)$ be a basis of a semi-simple Lie algebra. Using the Killing form inner product, and the Schmidt orthogonalization process, we can assume them to be orthonormal, $(t_i, t_j) = \delta_{ij}$. With this basis, the structure coefficients $c_{ijk} = -c_{jik}$ is completely antisymmetrical in all three indices.

   `Proof` : Multiply both sides of $[ad(t_i), ad(t_j)] = ic_{ijk}ad(t_k)$ by $ad(t_l)$ and take the trace, we get

   $$ic_{ijl} = \mathrm{tr}\,(ad(t_l)[ad(t_i), ad(t_j)]) = \mathrm{tr}\,([ad(t_l), ad(t_i)]ad(t_j)) = ic_{lij} = -ic_{ilj}$$

   ■

   .

9.5 $su(2)$

The simplest simple Lie algebra is $su(2)$, the angular momentum algebra which we are familiar with in quantum mechanics. The three generators t_i are the angular momentum operators J_i satisfying the commutation relation

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y. \tag{9.6}$$

It turns out that the treatment of all other simple algebras is very similar to that of $su(2)$, so let us first review what we know about $su(2)$.

Putting $J_x \pm iJ_y = J_\pm$, we obtain an equivalent commutation relation

$$(a): \quad [J_+, J_-] = 2J_z, \qquad (b): \quad [J_z, J_\pm] = \pm J_\pm \tag{9.7}$$

which is better suited for analysis. Note that J_x, J_y, J_z are hermitian, but J_+ and J_- are hermitian conjugates of each other.

Let $|m\rangle$ be an eigenfunction of J_z with eigenvalue m: $J_z|m\rangle = m|m\rangle$. Then $J_\pm|m\rangle$ is proportional to $|m\pm 1\rangle$ according to (9.7)(b), hence $J_\pm$ serves to raise/lower the quantum number m by 1 unit, and they are known as `creation` and `annihilation` operators, or, `raising` and `lowering` operators. Assume the eigenfunctions to be normalized, $\langle m|m\rangle = 1$. We can choose the phase of $|m+1\rangle$ so that $\langle m+1|J_+|m\rangle := N_{m+1}$ is real and positive. Then because $J_+^\dagger = J_-$, we also have $\langle m|J_-|m+1\rangle = N_{m+1}$.

Suppose the angular momentum in question is finite, so that m is bounded from above by some number j. In that case $J_+|j\rangle = 0$, and (9.7)(a) implies $N_j = \sqrt{2j}$. More generally, $\langle m|[J_+, J_-]|m\rangle = N_m^2 - N_{m+1}^2 = \langle m|2J_z|m\rangle = 2m$ gives an recursion relation to compute all N_m for $m < j$. For example, $N_{j-1}^2 = 2j + 2(j-1)$, $N_{j-2}^2 = 2j + 2(j-1) + 2(j-2)$, etc. The general formula is $N_m^2 = (j+m)(j-m+1)$, hence

$$J_\pm|m\rangle = \sqrt{(j\pm m)(j\mp m-1)}|m\pm 1\rangle,$$

and in particular, $J_-|-j\rangle = 0$. Since we reach $m=-j$ from $m=j$ in $2j$ steps, $2j$ must be an integer.

The state $|m\rangle$ is often written as $|j,m\rangle$ to make sure what j it belongs to. It can be checked directly that $\vec{J}^2 = J_z^2 + (J_+J_- + J_-J_+)/2$, and that

$$\vec{J}^2|j,m\rangle = j(j+1)|j,m\rangle.$$

The states $|j,-j\rangle, \cdots, |j,j\rangle$ form a $(2j+1)$-dimensional `representation` of the angular momentum operators.

By the way, this analysis relies on the hermiticity of J_i. This property is shared by all semi-simple Lie algebras, a property which we shall use from now on. That does not mean that every generator is necessarily hermitian, because even in $su(2)$, $J_\pm$ are not. However, it does mean that if a generator is not hermitian, then its hermitian conjugate is also a generator.

9.6 Roots of semi-simple algebras

In this section we generalize the analysis of $su(2)$ to all semi-simple Lie algebras.

1. A maximal *abelian* subalgebra of $\mathtt{g}$ is called a `Cartan subalgebra`, to be denoted by $\mathtt{h}$. Since $\mathtt{g}$ is semi-simple, we can assume the basis of $\mathtt{h}$ to consist of hermitian operators.

2. The dimension l of the vector-space of $\mathtt{h}$ is called the `rank` of $\mathtt{g}$. This is the number of simultaneously commuting hermitian operators in the algebra.

   For $su(2)$, $l = 1$, and the Cartan algebra is generated by J_z.

3. Since $\mathtt{h}$ is abelian, all the $ad(h)$'s for $h \in \mathtt{h}$ commute, so they have simultaneous eigenvectors e_α with eigenvalues $\alpha(h)$ for $h \in \mathtt{h}$:

$$ad(h)e_\alpha = \alpha(h)e_\alpha, \;\; e_\alpha \in \mathtt{g}. \tag{9.8}$$

   the eigenvalue $\alpha(h)$ is called a `root`.

   Taking the hermitian conjugate of (9.8), keeping in mind that $h = h^\dagger$, we see that $e_\alpha^\dagger$ is also an eigenvector of $ad(h)$ with eigenvalue $-\alpha(h)$. We may hence write $e_\alpha^\dagger = e_{-\alpha}$, and conclude that non-zero roots come in pairs: α and $= \alpha$.

   For $su(2)$, the roots are J_+, with eigenvalue $\alpha_+ = 1$, and J_-, with eigenvalue $\alpha_- = -1$; positive root for creation and negative root for annihilation operators.

4. All the eigenvectors with the same $\alpha(h)$ form a subspace of $\mathtt{g}$, designated as L_α. If $e_\alpha \in L_\alpha, e_\beta \in L_\beta$, then

$$ad(h)[e_\alpha, e_\beta] = [ad(h)e_\alpha, \epsilon_\beta] + [e_\alpha, ad(h)e_\beta] = [\alpha(h) + \beta(h)][e_\alpha, e_\beta],$$

   hence $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ and $(\alpha + \beta)(h) = \alpha(h) + \beta(h)$. In particular, $\mathtt{h} \subset L_0$. Actually, it can be shown that $\mathtt{h} = L_0$. That means, the origin of the root diagram is a root, and that

   > `if` α, β `are roots, then` $\alpha + \beta$ `is a either root, or` $[e_\alpha, e_\beta] = 0$.

   For $su(2)$, $-1+1 = 0, -1+0 = -1, +1+0 = +1$ correspond respectively to the commutators $[J_z, J_\pm] = \pm J_\pm$, and $[J_+, J_-] = 2J_3$.

5. If $a \in L_\alpha$ and $b \in L_\beta$, with $\alpha \neq -\beta$, then $(a, b) = 0$. This is so because $([a, h], b) = -\alpha(a, b) = (a, [h, b]) = \beta(a, b)$

   For $su(2)$, this implies $(+1, 0) = 0$ and $(+1, -1) \neq 0$. We can see directly why that is true. On the one hand, $(+1, 0) = \mathrm{Tr}(ad(J_+)ad(J_z))$. On the other hand, $ad(J_i)$ as a linear operator on $\mathtt{g}$ is isomorphic to the quantum-mechanical operator $J)i$ on the Hilbert space states $|jm\rangle$ with $j = 1$. Hence $(J_+, J_z) = \mathrm{Tr}(ad(J_+)ad(J_z)) = \sum_{m=-1,0,+1}(J_+)_{mm}m = 0$ because J_+ does not have diagonal matrix elements.

9.7 Geometry of the Cartan algebra

The Cartan algebra is abelian, has dimension l, so it can be treated as an ordinary l-dimensional vector space. The inner product of this vector space can be taken to be the Killing form, because, for a semi-simple algebra,

1. the Killing form restricted to $\mathtt{h}$ is non-degenerate.

   `Proof` : Otherwise, there is a $h' \in \mathtt{h}$ so that $(h', h) = 0 \;\forall\; h \in \mathtt{h}$. But $(h', a) = 0$ as well for all $a \in L_\alpha$ when $\alpha \neq 0$. This is not allowed because the Killing form on $\mathtt{g}$ is non-degenerate. ■

   .

   ---

2. With this inner product, we can choose an orthonormal basis $h_i \in \mathtt{h}$ so that $(h_i, h_j) = \delta_{ij}$. Then $\alpha(h_i)$ will be the ith component of the root α. A diagram plotting all the roots is called a `root diagram`; the axes of this diagram are h_i so the root components are $\alpha(h_i)$. It would be more intuitive to write it simply as α_i, but unfortunately that symbol is usually used for something else which we will encounter below.

   For $su(2)$, $l = 1$, the root diagram consists of the three points $-1, 0, +1$ on the real line.

   The members of $\mathtt{h}$ are zero roots, but often by roots one means non-zero roots.

3. Roots can be ordered by first ordering $\alpha(h_1)$, then $\alpha(h_2)$, then $\alpha(h_3)$, etc. A root is called a `positive root` if the first non-zero $\alpha(h_i)$ is positive, a `negative root` if the first non-zero $\alpha(h_i)$ is negative. Positive roots and negative roots come in pairs; the only root that is neither

positive nor negative are roots with $\alpha(h_i) = 0$ for all i, and these are roots that belong to L_0. The set of all positive roots will be denoted by Δ^+, the set of negative roots by Δ^-.

4. A positive root that cannot be expressed as a sum of two other positive roots is called a `simple root`. There are l simple roots, to be denoted by α_i, which are located at the boundary of the convex cone of positive roots. Please do not confuse α_i with $\alpha(h_i)$.

Examples

Fig. 9.1 shows the root diagram of all the rank-2 semi-simple Lie algebras, whose origin will be discussed later in the Chapter. The Lie algebras $su(2), su(3), so(3)$ are here named A_1, A_2, B_2 respectively. G_2 is not the Lie algebra of any classical group, so it is called an `exceptional algebra`. The two simple roots in each case are denoted by α and β, and the positive roots are all those roots between the two simple roots. $A_1 \times A_1$, being the direct product of two simple Lie algebras A_1, is semi-simple.

Note that

(a) the angle between roots are confined to $90°, 60°, 45°, 30°$, or multiples thereof, and that the angles are different for these four cases;

(b) for A_2, whose angle is $60°$, the root lengths are all the same, but for B_2 and G_2, with $45°$ and $30°$ angles, the root lengths are not all identical;

(c) the angle between the simple roots is always $\geq 90°$.

It turns out that these are generic features valid for all semi-simple Lie algebras, as we will see later.

5. The simultaneous eigenvalues $\lambda(h)$ of all $h \in \mathtt{h}$ for a state $|\lambda\rangle$ is called a `weight`. A weight in the adjoint representation is just a root.

6. The non-degeneracy of the Killing form allows us to associate a unique $t_\alpha \in \mathtt{h}$ with every root α, so that $(t_\alpha, h_i) = \alpha(h_i)$, or equivalently, $(t_a, h) = \alpha(h)$ for any $h \in \mathtt{h}$. We can expand t_α in terms of h_i, then $t_\alpha :=$

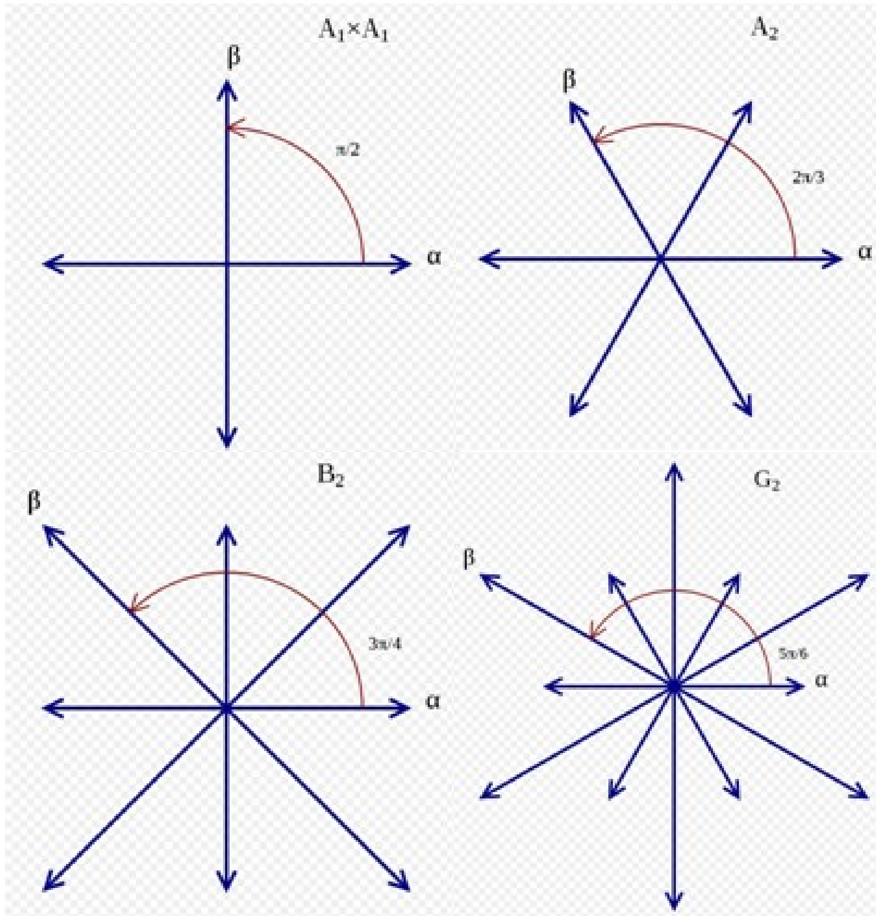

Figure 9.1: $l = 2$ semi-simple Lie algebras

$\sum_j (t_\alpha, h_j)h_j = \sum_j \alpha(h_j)h_j$, so $(t_\alpha, t_\beta) = (t_\alpha, h_i)(h_i, t_\beta) = \alpha(h_i)\beta(h_i) = (\alpha, \beta)$.

Note: This association between α and $t_\alpha \in \mathtt{h}$ is true for any l-dimensional vector α, not necessarily a root. In particular, we can associate a weight λ with a $t_\lambda = \sum_i \lambda(h_i)h_i \in \mathtt{h}$ so that $(t_\lambda, h) = \lambda(h)$ for all $h \in \mathtt{h}$. This also means $(t_\lambda, t_\alpha) = \sum_i \lambda(h_i)\alpha(k_i) = (\lambda, \alpha)$.

7. To each L_α, we can associate a unique $e_\alpha \in L_\alpha$, $f_\alpha \in L_{-\alpha}$, and $h_\alpha \in \mathtt{h} = L_0$, such that $f_\alpha = e_\alpha^\dagger$, and that

$$h_\alpha = [e_\alpha, f_\alpha],\ [h_a, e_\alpha] = 2e_\alpha,\ [h_\alpha, f_\alpha] = -2f_\alpha. \tag{9.9}$$

The corresponding $su(2)$ group will be denoted as $su(2)_\alpha$. Note that

(a) the normalization of the generators here, $h_\alpha \cong \sigma_3, e_\alpha \cong \sigma_+, f_\alpha \cong \sigma_-$, are twice that of the angular momentum operators, so that the eigenvalues of h_α are integers k, rather than half integers $j = k/2$ for J_z;

(b) it implies dim $(L_\alpha) = 1$, and hence $[L_\alpha, L_\beta] = L_{\alpha+\beta}$;

(c) it turns out that $h_\alpha = 2t_\alpha/|\alpha|^2$, hence $(h_\alpha, h_\alpha) = 4/|\alpha|^2$.

Proof : Pick any $e \in L_\alpha, f \in L_{-\alpha}$, then $[e, f] \in L_0$, so the first step is to find a $h \in L_0$ so that $[e, f] = h$. It should come as no surprise that h turns out to be proportional to the unique t_α that is associated with α, as discussed above.

To see that, let h' be any element in L_0. Then $(h', [e, f]) = ([h', e], f) = \alpha(h')(e, f) = (h', (e, f)t_\alpha)$. Since Killing is non-degenerate on L_0, this implies $[e, f] = (e, f)t_\alpha$. Now $[t_\alpha, e] = \alpha(t_\alpha)e = |\alpha|^2 e$, $[t_\alpha, f] = -\alpha(t_\alpha)f = -|\alpha|^2 f$. Let $h_\alpha = 2t_\alpha/|\alpha|^2$, then $[h_\alpha, e] = 2e, [h_\alpha, f] = -2f$, and $[e, f] = h_\alpha\{(e, f)|\alpha|^2/2\}$, so a rescaling of e to e_α and f to f_α to make $\{\cdots\} = 1$ will finish the job. The scaling is unique if we insist on $f_\alpha = e_\alpha^\dagger$.

To see the uniqueness, suppose e'_α is another highest root in L_α which is not a multiple of e_α, then the two $su(2)$ algebras $g = (e_\alpha, h_\alpha, f_\alpha)$ and $g' = (e'_\alpha, h'_\alpha, f'_\alpha)$ form a spin-1 representations of each other. When g is considered as weights of algebra g', we have

$$[e'_\alpha, e_\alpha] = 0, [h'_\alpha, e_\alpha] = 2e_\alpha, [f'_\alpha, e_\alpha] = -h_\alpha, [f'_\alpha, h_\alpha] = 2f_\alpha, [e'_\alpha, f_\alpha] = h_\alpha$$

When g' is considered as weights of algebra g, then

$$[e_\alpha, e'_\alpha] = 0, [h_\alpha, e'_\alpha] = 2e'_\alpha, [f_\alpha, e'_\alpha] = -h'_\alpha, [f_\alpha, h'_\alpha] = 2f'_\alpha, [e_\alpha, f'_\alpha] = h'_\alpha$$

This shows $g = g'$. Essentially the same calculation shows that $n\alpha$ cannot be a root. Hence, dim $(L_\alpha) = 1$.

We have shown preciously that $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$, but since dim $(L_{\alpha+\beta}) = 1$, $[L_\alpha, L_\beta] = L_{\alpha+\beta}$. ■

.

---

8. `Cartesian generators`: The $su(2)$ generators can be taken to be J_1, J_2, J_3, or, $J_\pm = J_1 \pm iJ_2$ and J_3. I shall refer to the former as the `Cartesian basis` and the latter as the `spherical basis`. In this language, the generators based on roots in a semi-simple Lie algebra are in the spherical basis. We shall now proceed to construct the equivalent operators in the Cartesian basis.

   Let $T_{\alpha 1} = |\alpha|^2(e_\alpha + f_\alpha)/4$, $T_{\alpha 2} = -i|\alpha|^2(e_\alpha - f_\alpha)/4$, and $T_{\alpha 3} = |\alpha|^4 h_\alpha/8$. Then $T_{\alpha i}^\dagger = T_{\alpha i}$, and $[T_{\alpha i}, T_{\alpha,j}] = i\epsilon_{ijk}T_{\alpha k}$. In short, every $T_{\alpha i}$ behaves like J_i if $|\alpha|^2 = 2$, which is the case for $su(2)$. Note that $4(T_{\alpha 1}^2 + T_{\alpha 2}^2) = |\alpha|^2(e_\alpha f_\alpha + f_\alpha e_\alpha)/2 = |\alpha|^2 f_\alpha e_\alpha + |\alpha|^2 h_\alpha/2 = |\alpha|^2(e_\alpha f_\alpha + f_\alpha e_\alpha)/2 = |\alpha|^2 f_\alpha e_\alpha + t_\alpha$.

   A Lie algebra of rank l and dimension n has $n - l$ roots and l diagonal generators in the Cartan subalgebra. In the spherical basis, the generators are e_α, f_α for every $\alpha \in \Delta^+$, and h_i for $1 \le i \le l$. Note that it is not h_α, for otherwise there would be too many of them. In the Cartesian basis, the generators would be $T_{\alpha 1}, T_{\alpha 2}$ and $h_i/2$. It sometimes convenient to write them together as $T_p = T_p^\dagger$, for $1 \le p \le n$.

   The advantage of the Cartesian basis is that all the generators are hermitian, hence the corresponding structure constants are totally antisymmetric in its three indices (§9.3(3)).

9. `Casimir operators`: For $su(2)$, $J^2 = \sum_{i=1}^3 J_i^2$ commutes with every J_i, so by Schur's lemma it is a constant in every irreducible representation. Its generalization to a semi-simple algebra is the (quadratic) Casimir operator

$$C_2 := 4\sum_{p=1}^n T_p^2 = 4\sum_{\alpha\in\Delta^+}(T_{\alpha 1}^2 + T_{\alpha 2}^2) + \sum_{i=1}^l h_i^2 = \sum_{\alpha\in\Delta^+}[|\alpha|^2 f_\alpha e_\alpha + t_\alpha] + \sum_{i=1}^l h_i^2. \tag{9.10}$$

`Proof` :

We have to show that $[C_2, T_q] = 0$. Since T_p are hermitian, the structure constant is completely antisymmetric in its three indices (§9.3(3)).

Hence

$$[C_2, T_q] = 4 \sum_{p,r=1}^{n} \left(T_p T_r c^r_{pq} + T_r T_p c^r_{pq} \right) = 0. \quad \blacksquare$$

.

10. Let $R_\alpha = \{e_\beta \in \mathtt{g} | [e_\alpha, e_\beta] = 0\}$ be the set of `highest roots` of $su(2)_\alpha$. By Jacobi identity, if $e_\beta, e_\gamma \in R_\alpha$, then $[e_\beta, e_\gamma] \in R_\alpha$, so R_a is a subalgebra of $\mathtt{g}$. Clearly $e_\alpha \in R_\alpha$.

    If $e_\beta \in R_\alpha$, and $ad(h_\alpha)e_\beta = ke_\beta$, then §9.5 shows that k is an integer (remember that $h_\alpha = 2J_z$).

11. Recall that $(\alpha, \beta) = (t_\alpha, t_\beta) = \beta(t_a)$ and $h_\alpha = 2t_\alpha/|\alpha|^2$. Hence $(\alpha, \beta) = |\alpha|\beta(h_\alpha)/2$, and $[h_\alpha, e_\beta] = \beta(h_\alpha)e_\beta = [2(\alpha, \beta)/|\alpha|^2]e_\beta := \langle\beta|\alpha\rangle e_\beta$. The quantity $k = \langle\beta|\alpha\rangle$ is an integer called the `Cartan integer`. Note that it is linear in β but not in α. β can be a root or a weight of $\mathtt{g}$.

    Since $\mathtt{h}$ is a l-dimensional Euclidean space, we may express (α, β) in terms of the angle θ between the two roots, $(\alpha, \beta) = |\alpha|\,|\beta| \cos\theta$. Since $k = \langle\beta|\alpha\rangle = 2\cos\theta|\beta|/|\alpha|$ and $k' = \langle\alpha|\beta\rangle = 2\cos\theta|\alpha|/|\beta|$ are both integers, the allowed values of $\cos\theta = \pm\sqrt{kk'}/2$ are $0, \pm 1/2, \pm\sqrt{2}/2, \pm\sqrt{3}/2$, and the corresponding allowed angles are $90°, 60°(120°), 45°(135°), 30°(150°)$.

    If $\cos\theta = 0$, the lengths of $|\beta|$ and $|\alpha|$ are not related. Otherwise, the ratio is $|\beta|/|\alpha| = k/2\cos\theta$. For $\theta = 60°(120°)$, this ratio is 1. For $\theta = 45°(135°)$, this ratio is $\sqrt{2}$ or $1/\sqrt{2}$. For $\theta = 30°(150°)$, this ratio is $\sqrt{3}$ or $1/\sqrt{3}$.

12. $\alpha^\vee := 2\alpha/(\alpha, \alpha)$ is called a `coroot`. The Cartan integer $\langle\beta|\alpha\rangle$ is therefore equal to $(\beta, \alpha^\vee)$. If Δ is the system of roots, then $\Delta^\vee$ is the system of coroots, and $(\Delta^\vee)^\vee = \Delta$.

13. The cosine of the angle between two simple roots α_i and α_j is always negative, hence such an angle is always between 90° and 180°. Moreover, $\alpha_i - \alpha_j$ cannot be a root, hence $[e_{\alpha_i}, f_{\alpha_j}] = 0$.

    `Proof` : $\beta := \alpha_i - \alpha_j$ cannot be a root. Otherwise it is either positive or negative. In the first case, $\alpha_i = \alpha_j + \beta$ so α_i cannot be simple. In the second case, $-\beta$ is positive so $\alpha_j = \alpha_i - \beta$ cannot be simple. $\blacksquare$

    .

14. Fig. 9.2 shows all possible connections between pairs of simple roots. Each circle represents a simple root. The number of lines connecting two circles indicates the angle between the two simple roots: 0, 1, 2, 3 lines correspond respectively to $90°, 120°, 135°, 150°$. In the case of two or three lines, the ratio of the lengths of the two roots is $\sqrt{2}$ or $\sqrt{3}$ respectively. The arrow then points to the shorter root. The three-line connection occurs only in the algebra G_2, and that is explicitly indicated in the Figure. From left to right, the other three possibilities correspond respectively to the algebras $A_1 \times A_1$, A_2, and B_2 of Fig. 9.1.

    We shall see in the next section that every semi-simple Lie algebra of rank l can be represented by a l-circle diagram like these. Moreover, the diagram is always connected, but the lines connecting the circles never form a loop. This classification of semi-simple Lie algebra is due to `Cartan`, and these l-circle diagrams are called `Dynkin diagrams`.

Figure 9.2: Connection of pairs of simple roots

9.8 Cartan matrix

The $l \times l$ asymmetric matrix $A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ form from the simple roots α_i is known as the `Cartan matrix`. Diagonal entries are always $A_{ii} = 2$, and a non-diagonal entry is either $0, -1, -2$, or -3, with the constraint that $A_{ij}A_{ji} = 4\cos^2\theta < 4$. Pictorially, one draws $A_{ij}A_{ji}$ lines between nodes i and j. If $|A_{ij}| < |A_{ji}|$, meaning $|\alpha_j| < |\alpha_i|$, then an arrow is drawn from node i to node j. In other words, an arrow always points to the shorter root. These diagrams are shown in Fig. 9.2.

Properties of the Cartan matrix

1. there are at most $l - 1$ pairs of vertices connected by lines;

2. there are no loops;

3. there are at most three edges issuing from any node;

4. the only connected diagram containing a triple bond is the rank-two diagram G_2;

5. if two nodes are connected by a single line, then we may shrink the line and merge the two nodes to get another legitimate Cartan matrix (or Dynkin diagram);

6. a diagram cannot have two double bonds, nor two T-junctions of single bonds;

7. if the double bond occurs once in a linear chain, then it must be at the beginning (B_l), at the end (C_l), or right in the middle for a four-node chain (F_4);

8. if a diagram contains a linear chain of single bonds without any T-junction, then the Lie algebra is known as A_l.

9. if a diagram containing single bonds has one T-function, and the three branches emerging from the T-junction has $p-1, q-1, r-1$ nodes, so that $l = p+q+r-2$. Suppose further that $p \geq q \geq r$. Then the allowed diagrams must satisfy $p^{-1}+q^{-1}+r^{-1} > 1$, which has solutions $(p, q, r) = (p, 2, 2)$ (D_l) and (3/4/5,3,3) ($\mathsf{E}_6/\mathsf{E}_7/\mathsf{E}_8$).

For a reference, see p. 172 of Ref. [9].

Proof :

1. Let $x = \sum_{i=1}^{l} \alpha_i/|\alpha_i|$. Then $0 < |x|^2 = l + 2\sum_{i<j}(\alpha_i, \alpha_j)/|\alpha_i||\alpha_j| = l - \sum_{i<j}\sqrt{A_{ij}A_{ji}}$. Since each non-zero summand above must be 1, $\sqrt{2}$, or $\sqrt{3}$, there must be at most $l-1$ pairs of non-connected edges.

2. A connected diagram of l nodes with one loop has at least l lines.

3. Let $\beta_1, \cdots, \beta_r$ be the nodes (simple roots) connected to the node α. Then $(\beta_i, \beta_j) = 0$ for $i \neq j$, or else a loop is present. Since all the simple roots are linearly independent, α must have a non-zero component orthogonal to all the β_i's. Hence $|\alpha|^2 > \sum_i (\alpha, \beta_i)^2/|\beta_i|^2 = |\alpha|^2 \sum_i A_{\alpha i}A_{i,\alpha}/4$. Therefore $\sum_i A_{\alpha i}A_{i\alpha} \leq 3$.

4. Otherwise there is at least one node with four edges emerging from it.

5. Let $K_{ij} = (\alpha_i, \alpha_j) = A_{i,j}(\alpha_i, \alpha_i)/2$ be the positive definite Killing form associated with a Cartan matrix A, or a Dynkin diagram. Positive definiteness means $\sum K_{ij}x_ix_j > 0$ if at least one $x_i \neq 0$. Suppose nodes a and b are connected by a single line. Then $K_{aa} = K_{bb} = -2K_{ab}$, so that $x_a^2K_{aa} + x_b^2K_{bb} + 2x_ax_bK_{ab} = x_a^2K_{aa}$ if we set $x_a = x_b$. Now merge nodes a, b into a new node c, and use i, j from now on to denote nodes that are neither a nor b. Then the new symmetric matrix K' that corresponds to the new Dynkin diagram is $K'_{ij} = K_{ij}$, $K'_{cc} = K_{aa} = K_{bb}$, and $K'_{ic} = K_{ia} + K_{ib} = K'_{ci}$. Note that if $K_{ia} \neq 0$, meaning i and a are connected, then we must have $K_{ib} = 0$ or else there will be a loop in the Dynkin diagram. If we set $x'_c = x_a = x_b$ and $x'_i = x_i$, then $x^TKx = x'^TK'x'$, showing that K' is positive definite if K is. This proves that the new diagram is a legitimate Dynkin diagram if unmerged one is.

6. If there are two double bonds, then by shrinking all the single bonds between the two double bonds would result in a node with four lines, which is not allowed. The same is true if there are two T-junctions of single bonds.

7. Suppose we enumerate the l nodes from left to right, suppose the double bond occurs between nodes p and $p+1$, and suppose its arrow points from right to left. Then we may assume $|\alpha_i|^2 = 1$ for $1 \leq i \leq p$, and $|\alpha_i|^2 = 2$ for $p+1 \leq i \leq l$. Relabel the $q = l - p$ nodes from right to left as $\beta_1, \cdots, \beta_q$. Let $\alpha = \sum_{i=1}^p i\alpha_i$, $\beta = \sum_{i=1}^q i\beta_i$. Then

$$\begin{aligned}
(\alpha,\alpha) &= \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} 2i(i+1)/2 = p^2 - \sum_{i=1}^{p-1} i = p^2 - p(p-1)/2 = p(p+1)/2, \\
(\beta,\beta) &= \sum_{j=1}^{q} 2j^2 - \sum_{j=1}^{q-1} 2j(j+1) = q(q+1), \\
(\alpha,\beta) &= pq(\alpha_p, \beta_q) = -pq.
\end{aligned} \tag{9.11}$$

Since α and β cannot be parallel, Schwarz inequality requires $(pq)^2 < (\alpha,\alpha)(\beta,\beta) = pq(p+1)(q+1)/2$, or $2pq < (p+1)(q+1) \Rightarrow pq <$

$p+q+1$. The only allowed solutions are $(p,q) = (1,n),(n,1),(2,2)$, for an arbitrary $n \geq 1$. The desired conclusion then follows.

8. This is the definition of the Lie algebra A_l.

9. Let $\alpha = \sum_{i=1}^{p-1} i\alpha_i, \beta = \sum_{j=1}^{q-1} j\beta_j, \gamma = \sum_{k=1}^{r-1} k\gamma_k$, and δ be the simple root at the junction. Then $(\alpha,\alpha) = p(p-1)(\delta,\delta)/2$, and similarly for (β,β) and (γ,γ). Moreover, $(\alpha,\delta) = -(p-1)(\delta,\delta)/2$, etc. Now $\alpha/|\alpha|, \beta/|\beta|, \gamma/|\gamma|, \epsilon$ form an orthonormal set if ϵ is a unit vector orthogonal to the other three. Since δ is independent of α,β,γ, $(\delta,\epsilon) \neq 0$, hence $|\delta|^2$ is largest than the square of its components along α,β,γ. This works out to be

$$\begin{aligned}
1 &> \left(\frac{(\alpha,\delta)}{|\alpha||\delta|}\right)^2 + \left(\frac{(\beta,\delta)}{|\beta||\delta|}\right)^2 + \left(\frac{(\gamma,\delta)}{|\gamma||\delta|}\right)^2 \\
&= \left(\frac{p-1}{2}\right)^2 \frac{2}{p(p-1)} + \left(\frac{q-1}{2}\right)^2 \frac{2}{q(q-1)} + \left(\frac{r-1}{2}\right)^2 \frac{2}{r(r-1)} \\
&= \frac{p-1}{2p} + \frac{q-1}{2q} + \frac{r-1}{2r} \quad \Rightarrow \\
1 &< p^{-1} + q^{-1} + r^{-1}.
\end{aligned}$$

The only solutions are $(p,q,r) = (2,2,r)$, whose algebra is called D_l, and (2,3,3/4/5), whose algebras are called $E_6/E_7/E_8$ respectively. ■

.

9.9 Dynkin diagrams

Given the properties of the Cartan matrix in the last section, the only allowed connected Dynkin diagrams for a semi-simple algebra are summarized below. Disconnected diagram represents the direct product of algebras associated with its connected components.

It is known that the four infinite series, A_l, B_l, C_l, D_l, correspond to the Lie algebra of `classical groups`: $A_l = sl(l+1), B_l = so(2l+1), C_l = sp(2l), D_l = so(2l)$. The other five, G_2, F_4, E_6, E_7, E_8 are called `exceptional Lie algebras`

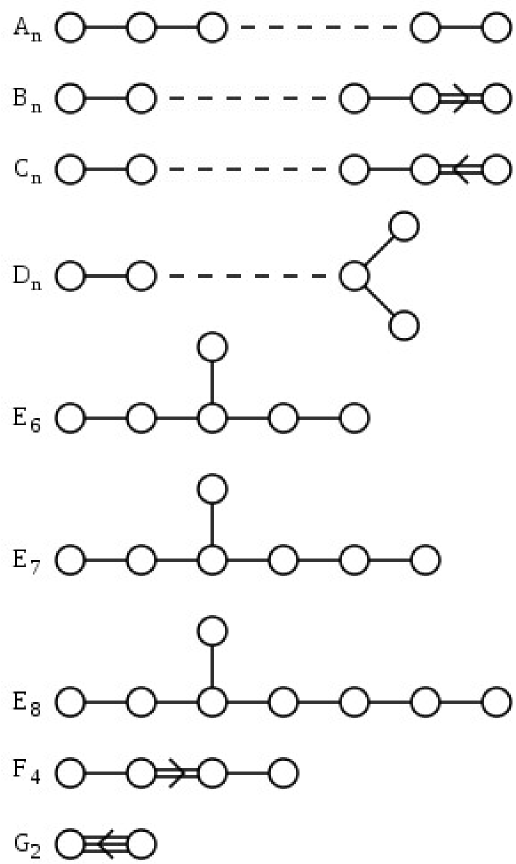

Figure 9.3: Dynkin diagrams for simple Lie algebras

9.10 Classical Lie algebras

The roots of an algebra must obey the following two rules: (i), if α is a root, then $n\alpha$ is a root if and only if $n = \pm 1$; (ii), if α, β are roots, and $(\alpha, \beta) \neq 0$, then $|\cos\theta| = |(\alpha, \beta)|/|\alpha||\beta|$ should be equal to the ratio $|\alpha|/|\beta|$, or $|\beta|/|\alpha|$, and these must have the values $0, 1/\sqrt{2}, 1/2, \sqrt{3}/\sqrt{2}$; (iii), the ratio $\sqrt{3}/\sqrt{2}$ occurs only in G_2, hence irrelevant for classical Lie algebras.

We shall refer to these rules as the `root rules` in the rest of this section. These rules will help us to find out what classical algebras A_l, B_l, C_l, D_l correspond to.

Note that roots must satisfy the root rule, but a vector satisfying the root rule may not be a root. To be sure, we have to work out the whole algebra.

To simplify writing, we shall not distinguish a root α from its counterpart t_α in $\mathtt{h}$.

For simplicity in writing, it is useful to introduce the matrices E_{ij}, whose element is 1 in the (i,j) position, and 0 everywhere else. Then $E_{ij}E_{pq} = \delta_{jp}E_{iq}$, $(E_{ij}, E_{pq}) = \mathrm{Tr}(E_{ij}E_{pq}) = \delta_{jp}\delta_{ki}$.

9.10.1 $su(l+1)$

We know from §7.2 that the generators are $(l+1)\times(l+1)$ hermitian traceless matrices, and there are $(l+1)^2-1 = l(l+2)$ of them. Let $h_i = E_{ii} - E_{i+1,i+1}$. Then the l h_i's span the Cartan subalgebra $\mathtt{h}$ with rank l. Its commutation relation with E_{pq} is

$$[h_i, E_{pq}] = (\delta_{ip} - \delta_{iq} - \delta_{i+1,p} + \delta_{i+1,q})E_{pq} := \alpha_{pq}E_{pq},$$

hence E_{pq} ($p \neq q \leq l+1$) constitute the $l(l+1)$ root (creation or annihilation) operators. Note that such operators are traceless, but not hermitian, so they are analogous to $J_\pm$. However, since $E_{pq} = E_{qp}^\dagger$, we can make two hermitian operators out of the linear combination of E_{pq} and E_{qp}. Analogous to σ_+, let us define E_{pq} for $1 \leq p < q \leq l+1$ to be the creation operators, and α_{pq} for $p < q$ to be the positive roots. The corresponding operator in $\mathtt{h}$ is $t_{\alpha_{pq}} := t_{pq} = h_p - h_q$.

The simple roots are $\alpha_1 = \alpha_{12}, \alpha_2 = \alpha_{23}, \cdots, \alpha_l = \alpha_{l,l+1}$. The dot products of the simple roots are $(\alpha_i, \alpha_j) = 2$ if $i = j$, $= -1$ if $j = i \pm 1$, $= 0$ otherwise. Hence the Dynkin diagram of $su(l+1)$ is A_l.

As an illustration, let us look at the roots of $A_2 = su(3)$ in Fig. 9.1. Here $\alpha_1 = \alpha, \alpha_2 = \beta$ are simple roots, and the only other positive root is $\alpha_{13} = \alpha + \beta$.

9.10.2 $so(2l)$

We know from §7.1 that the number of generators of $so(n)$ is $n(n-1)/2 = l(2l-1)$, and these generators are imaginary anti-symmetric $2l \times 2l$ matrices. These can be taken to be the matrices $F_{pq} = -i(E_{pq} - E_{qp}) = -F_{qp}$ for $1 \leq p < q \leq 2l$. The l matrices $h_i = F_{2i-1,2i}$ which commute with one another span $\mathtt{h}$. The commutation of h_i with F_{pq} is

$$\begin{aligned}
[h_i, F_{pq}] &= -[E_{2i-1,2i} - E_{2i,2i-1}, E_{pq} - E_{qp}] \\
&= i\left(-\delta_{2i,p}F_{2i-1,q} - \delta_{2i,q}F_{p,2i-1} + \delta_{2i-1,p}F_{2i,q} + \delta_{2i-1,q}F_{p,2i}\right).
\end{aligned}$$

In particular, with $j < k$,

$$\begin{aligned}
[h_i, F_{2j-1,2k-1}] &= i\left(\delta_{ij}F_{2j,2k-1} + \delta_{ik}F_{2j-1,2k}\right), \\
[h_i, F_{2j-1,2k}] &= i\left(\delta_{ij}F_{2j,2k} - \delta_{ik}F_{2j-1,2k-1}\right), \\
[h_i, F_{2j,2k-1}] &= i\left(-\delta_{ij}F_{2j-1,2k-1} + \delta_{ik}F_{2j,2k}\right), \\
[h_i, F_{2j,2k}] &= i\left(-\delta_{ij}F_{2j-1,2k} - \delta_{ik}F_{2j,2k-1}\right).
\end{aligned}$$

We can write these in a matrix form

$$[h_i, F_{jk}] = CF_{jk}, \quad F_{jk} = \begin{pmatrix} F_{2j-1,2k-1} \\ F_{2j-1,2k} \\ F_{2j,2k-1} \\ F_{2j,2k} \end{pmatrix}, \quad C = \delta_{ij}A + \delta_{ik}B, \tag{9.12}$$

where $A = i\begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$, $B = i\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}$. These two matrices commute, so they have 4 simultaneous eigenvectors v^a,

$$v^1 = \begin{pmatrix} 1 \\ i \\ i \\ -1 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 1 \\ i \\ -i \\ 1 \end{pmatrix}, \quad v^3 = \begin{pmatrix} 1 \\ -i \\ i \\ 1 \end{pmatrix}, \quad v^4 = \begin{pmatrix} 1 \\ -i \\ -i \\ -1 \end{pmatrix},$$

with eigenvalues of C to be

$$\lambda^1 = -\delta_{ij} - \delta_{ik},\ \lambda^2 = \delta_{ij} - \delta_{ik},\ \lambda^3 = -\delta_{ij} + \delta_{ik},\ \lambda^4 = \delta_{ij} + \delta_{ik}. \tag{9.13}$$

Multiplying both sides of (9.12) by v^{aT}, defining $G^a_{jk} = v^{aT}\cdot F_{jk}$, and making using of the antisymmetry of C, we arrive at

$$[h_i, G^a_{jk}] = -\lambda^a G^a_{jk}. \tag{9.14}$$

Since $j < k$, the positive roots are $\alpha_{jk}(h_i) = -\lambda^1$ with the creation operator G^1_{jk}, and $\alpha'_{jk} = -\lambda^3$ with the creation operator G^3_{jk}. The corresponding annihilation operators are G^4_{jk} and G^2_{jk} respectively. The simple roots are $\alpha_1 = \alpha_{12},\ \alpha_2 = \alpha_{23},\ \cdots,\ \alpha_{l-1} = \alpha_{l-1,l}$,and $\alpha_l = \alpha'_{l-1,l}$. It is easy to verify that all the positive roots are positive combinations of the simple roots.

The dot products of the simple roots are have norm 2, and for $i < j < l$, $(\alpha_i, \alpha_j) = -\delta_{j,i+1}$. Moreover, $(\alpha_j, \alpha_l) \neq 0$ only for $j = l - 2$, in which case it is -1. Therefore, the Dynkin diagram formed by these simple roots are D_l.

9.10.3 $so(2l+1)$

The number of generators is now the $(2l+1)l$ imaginary antisymmetric $(2l+1) \times (2l+1)$ matrices $F_{pq} = -i(E_{pq} - E_{qp})$, with $1 \leq p < q \leq 2l+1$. As in the previous case, $h_i = F_{2i-1,2i}$ spans the Cartan algebra $\mathtt{h}$ so this algebra is of rank l, leaving behind l^2 positive roots. Other than the roots α_{jk} and α'_{jk} $(1 \leq j < k \leq l)$ in $so(2l)$, which are also positive roots here corresponding to the creation operators G^1_{jk} and G^3_{jk}, there should be l more positive roots. They are $\beta_j = \delta_{ij}$, corresponding to the creation operator $H_j = F_{2j-1,2l+1} + iF_{2j,2l+1}$, because

$$[h_i, H_j] = \delta_{ij} H_j.$$

The first $(l-1)$ simple roots are identical with those in $so(2l)$, namely, $\alpha_i = \alpha_{i,i+1}$, and the last one is $\alpha_l = \beta_l$. Since $(\alpha_l, \alpha_l) = 1$, $(\alpha_{l-1}, \alpha_{l-1}) = 2, (\alpha_{l-1}, \alpha_l) = -1$, the Dynkin diagram for these simple roots is B_l.

9.10.4 $sp(2l)$

The symplectic group (§7.3) is defined by $2l \times 2l$ matrices M satisfying $M^T \Omega M = \Omega$, where $\Omega = \begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}$. Near the identity, $M \simeq \mathbf{1}_{2l} + i\vec{\xi}\vec{t} + \cdots$, hence the Lie algebra $sp(2l)$ is defined by the generators satisfying

$$t_i^T \Omega + \Omega t_i = 0. \tag{9.15}$$

Unitarity of A implies $\mathbf{1} = M^\dagger M \simeq \mathbf{1} + i\vec{\xi}\cdot(\vec{t}^\dagger - \vec{t}) + \cdots$, hence $t_i^\dagger = t_i$. Unit determinant implies $1 = \det(M) = \exp(\mathrm{Tr}(\ln(A)) \simeq \exp\Big(\mathrm{Tr}(i\vec{\xi}\cdot\vec{t} + \cdots)\Big)$, hence $\mathrm{Tr}(t_i) = 0$.

Let us write a generator t as $t = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are $l \times l$ matrices. To satisfy (9.15), we need $A = -D^T, B = B^T, C = C^T$. The traceless condition is automatically satisfied, and the unitarity condition implies that A, D are hermitian, and $B^\dagger = B^* = C$.

The number of real parameters needed to specify a complex symmetric matrix B is $l(l+1)$. The number of real parameters needed to specify a hermitian matrix A is l^2, hence the dimension of $sp(2l)$ is $2l^2 + l$. The Cartan algebra $\mathtt{h}$ is spanned by $h_i = E_{ii} - E_{l+i,l+i}$, $1 \leq i \leq l$, hence the number of positive roots is l^2. They are $\alpha_{jk}, \beta_{jk} (j < k)$, and γ_j, with the corresponding creation operators to be $A_{jk} = E_{jk} - E_{l+j,l+k}$, $B_{jk} = E_{l+j,l+k} + E_{l+k,l+j}$, $C_{jk} = 2E_{l+j,l+k}$:

$$\begin{aligned} [h_i, A_{jk}] &= \alpha_{jk} A_{jk}, \quad [h_i, B_{jk}] = \beta_{jk} B_{jk}, \quad [h_i, C_j] = \gamma_j C_j, \\ \alpha_{jk}(h_i) &= \delta_{ij} - \delta_{ik}, \quad \beta_{jk}(h_i) = \delta_{ij} + \delta_{ik}, \quad \gamma_j = 2\delta_{ij}. \end{aligned} \tag{9.16}$$

The simple roots are $\alpha_i = \alpha_{i,i+1}$ for $1 \le i < l$, and $\alpha_l = \gamma_l$. The norm of the roots α_i is 2 for $1 \le i < l$, and that of α_l is 4. Moreover, $(\alpha_{l-1}, \alpha_l) = -2$, hence the Dynkin diagram is C_l.

9.11 Isomorphism of low-order algebras

The Dynkin diagrams of certain low-order algebras are indistinguishable, hence these algebras are identical. For example, $B_2 = so(5) = C_2 = sp(4), D_2 = so(4) = A_1 \times A_1 = su(2) \times su(2), D_3 = so(6) = A_3 = su(4)$.

9.11.1 Remarks

1. Only single bonds occur in algebras A_l, D_l, E_l. These algebras are said to be `simply laced`.

2. It is convenient to re-scale $h_\alpha, e_\alpha, f_\alpha$ in such a way that the root lengths are $|\alpha|^2 = 2$ for all simply laced algebras, and the standard commutation relation (9.9) are maintained. It is also convenient to adopt this convention for B_l and C_l so that the roots linked by single bonds have the same norm.

9.12 Weyl and Coxeter groups

1. The root system of any semi-simple algebra has a very regular structure, reminding us of the symmetry of a molecule or a crystal. The symmetry group of the root system is called a `Weyl group`. It is generated by r_i, reflection about the hyperplane perpendicular to the simple root α_i. If the (acute) angle between $-\alpha_i$ and α_j is $\theta_{ij} = \pi/m_{ij}$, then $m_{ij} = 2, 3, 4, 6$ respectively, if α_i and α_j are connected with no, one, two, and three bonds. Since $r_i r_j$ is a rotation by an angle 2θ in the

plane of these two vectors, $(r_i r_j)^{m_{ij}} = 1$, and the presentation of the Weyl group is

$$W = \{r_1, r_2, \cdots, r_l | r_i^2 = (r_i r_j)^{m_{ij}} = 1\}. \tag{9.17}$$

2. Analytically, the Weyl reflection is $r_i : (l) = \lambda - 2(\lambda, \alpha_i)\alpha_i/(\alpha_i.\alpha_i) = \lambda - \langle\lambda|\alpha_i\rangle\alpha_i$. Remember from §9.7(9) (??) that $\langle\lambda|\alpha_i\rangle$ is the eigenvalue of h_{α_i} for the state $|\lambda\rangle$, hence the corresponding eigenvalue of $r_i(\lambda)$ is $\langle\lambda|\alpha_i\rangle - \langle\lambda|\alpha_i\rangle\langle\alpha_i|\alpha_i\rangle = -\langle\lambda|\alpha_i\rangle$, which is what we expect a reflection should do. This implies, in particular, that the multiplicity of λ is the same as the multiplicity of $r_i(\lambda)$. The states in a Weyl orbit therefore all have the same multiplicities.

3. Reflection should preserve scalar products, so $(\lambda, \mu) = (r_i(\lambda), r_i(\mu))$. This can be analytically checked using the reflection formula in §9.12(2).

4. The convex cone-like regions bounded by adjacent roots are known as `Weyl chambers`. The elements of W permutes the Weyl chambers.

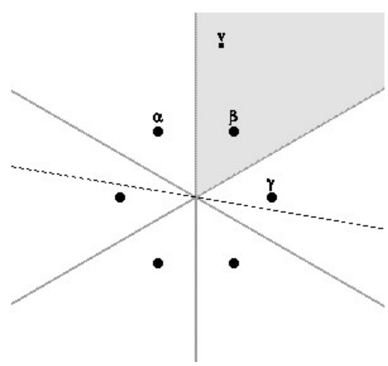

Figure 9.4: Weyl chambers of A_2

5. Let r_i be a Weyl reflection. Then r_i changes α_i to $-\alpha_i$, but permutes all other positive roots in $\Delta^+ - \{\alpha_i\}$.

`Proof` : If α is a positive root, then $\alpha = \sum_i n_i \alpha_i$ with $n_i \geq 0$. Since $r_i(\alpha)$ differs from α only in the α_i component, it follows that $r_i(\alpha)$ must have some positive n_j when $\alpha \in \Delta^+ - \{\alpha_i\}$. If $r_i(\alpha)$ is a root, then it must be a positive root because a root is either positive or negative, with nothing in between. Now, $r_i(\alpha)$ is obtained from α by using f_i to crank it down $\langle\alpha|\alpha_i\rangle$ times, namely, the state with an h_i quantum number opposite to that of α, so it is indeed a root. ■

.

---

6. The simple roots and Weyl reflections of classical Lie algebras are summarized below. For the exceptional algebras, see http://en.wikipedia.org/wiki/Root_system

   (a) A_l:
   $\alpha_i = e_i - e_{i+1}, \quad |\Delta| = l(l+1).$
   $r_i: \quad e_i \mapsto e_{i+1},\ e_{i+1} \mapsto e_i,\ e_j \mapsto e_j (j \neq i, i+1), \quad W = S_{l+1}$

   (b) B_l:
   $\alpha_i = e_i - e_{i+1}$ $(i \leq l-1)$ (long), $\alpha_l = e_l$ (short).
   $|\Delta| = 2l^2, \quad |\Delta^<| = 2l$ (short).
   $r_i:\ e_i \leftrightarrow e_{i+1}$ $(i \leq l-1)$, $r_l:\ e_l \leftrightarrow -e_l$. $|W| = 2^l l!$.

   (c) C_l:
   $\alpha_i = e_i - e_{i+1}$ $(i \leq l-1)$ (short), $\alpha_l = 2e_l$ (long).
   $|\Delta| = 2l^2, \quad |\Delta^>| = 2l$ (long).
   $r_i:\ e_i \leftrightarrow e_{i+1}$ $(i \leq l-1)$, $r_l:\ e_l \leftrightarrow -e_l$. $|W| = 2^l l!$.

   (d) D_l:
   $\alpha_i = e_i - e_{i+1}$ $(i \leq l-1)$, $\alpha_l = e_{l-1} + e_l$. $|\Delta| = 2l(l-1)$.
   $r_i:\ e_i \leftrightarrow e_{i+1}$ $(i \leq l-1)$, $r_l:\ e_l \leftrightarrow -e_l$. $|W| = 2^{l-1} l!$.

7. More generally, the group defined by a presentation of the kind in '1.' above, for any positive integers m_{ij}, is called a `Coxeter group`. Other than the Weyl groups, we already know that the dihedral group D_n is a Coxeter group, with $r_1 = r_1, r_2 = r_2, m_{12} = n$. See §3.3(1).

   It can be shown that the only other finite Coxeter groups are H_3 and H_4, given by

$$\begin{aligned} H_3 &= \{r_1, r_2, r_3 | r_i^2 = (r_1 r_2)^5 = (r_2 r_3)^3 = (r_1 r_3)^2 = 1\}, \\ H_4 &= \{r_1, r_2, r_3, r_4 | r_i^2 = (r_1 r_2)^5 = (r_2 r_3)^3 = (r_3 r_4)^3 = (r_a r_b)^2 = 1\}, \end{aligned} \tag{9.18}$$

   where $r_a r_b$ are products that have not occured before in the relations.

9.13 Extended Dynkin Diagrams

An `extended Dynkin diagrams` has one node added to the ordinary Dynkin diagrams as shown in the following diagram.

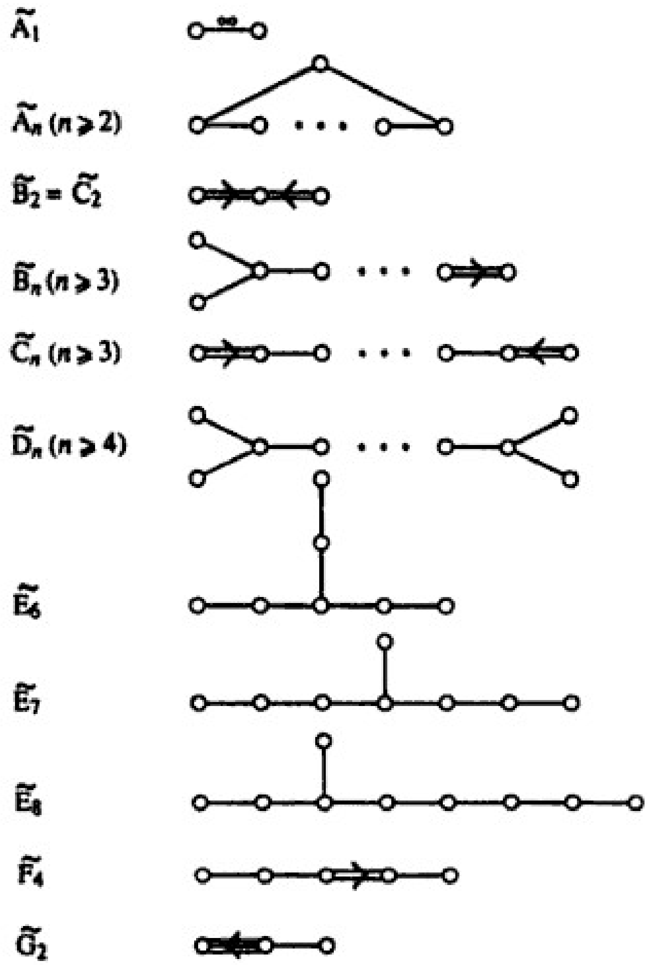

Figure 9.5: Extended Dynkin diagrams

If X is a Dynkin diagram for an algebra, then we use the notation $\tilde{X}$ to denote the extended Dynkin diagram.

One use of the extended diagram is to use it to find some subalgebras of a given algebra. The rule is as follows: remove one node from the extended diagram. Then the resulting algebra is a subalgebra of the original unextended Dynkin diagram.

We can get no information on A_l because no matter which node of $\tilde{A}_l$ we remove, we get back to A_l. This is not the case for the other algebras:

1. $B_2 = C_2 = so(5) \supset A_1 \times A_1 = su(2) \times su(2) = so(4)$.

2. $B_l \supset A_1 \times A_1 \times B_{l-2},\ A_3 \times B_{l-3},\ D_k \times B_{l-k},\ D_{l-1} \times A_1$.

3. $C_l \supset A_1 \times C_{l-1},\ C_k \times C_{l-k}$.

4. $D_l \supset A_1 \times D_{l=1},\ D_k \times D_{l-k}$.

5. $E_6 \supset A_1 \times A_5,\ A_2 \times A_2 \times A_2$.

6. $E_7 \supset A_7,\ A_1 \times D_6,\ A_2 \times A_5$.

7. $E_8 \supset A_8,\ D_8,\ A_1 \times A_2 \times A_5,\ A_4 \times A_4, D_5 \times A_3,\ E_6 \times A_2,\ E_7 \times A_1$.

8. $F_4 \supset A_1 \times C_3, A_2 \times A_2, A_3 \times A_1$.

9. $G_2 \supset A_1 \times A_1$.

9.14 Weights

1. The eigenvalue $\lambda(h)$ of $h \in \mathtt{h}$ in a representation is called a `weight`. In particular, if α is a root, then $\lambda(h_\alpha)$ is an eigenvalue of $h_\alpha \in su(2)_\alpha$, and we know that it has to be an integer. Since $h_\alpha = 2t_\alpha/(\alpha, a)$, we conclude that

$$\langle\lambda|\alpha\rangle := \lambda(h_\alpha) = \frac{2\lambda(t_\alpha)}{(\alpha,\alpha)} = \frac{2(t_\lambda, t_\alpha)}{(\alpha,\alpha)} = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \tag{9.19}$$

   has to be an integer.

2. The l weights λ_i obeying $\langle\lambda_i|\alpha_j\rangle = \delta_{ij}$ are called the `fundamental weights`. Every weight is an integer linear combination of the fundamental weights.

3. Integer combinations of α_i constitute points of the `root lattice`. Integer combinations of λ_i constitute points of the `weight lattice`. The weights of the adjoint representation are the roots, hence the root lattice is a sublattice of the weight lattice. For simply laced algebras with $|\alpha|^2 = 2$, the weight lattice is simply the reciprocal lattice of the root lattice.

4. The weight λ_0 in an irreducible representation is called the `highest weight` if none of $\lambda_0 + \alpha_i$ are weights. That means $e_{\alpha_i}|\lambda_0\rangle = 0$ for all i. Every state in the IR can be obtained from $|\lambda_0\rangle$ by applying repeatedly the $su(2)_\alpha$ annihilation operators of the positive roots α. The highest weight of the adjoint representation is called the `highest root`.

5. If λ_0 is the highest weight of an IR, then the value of the Casimir operator (§9.7(8,9)) for this IR can be computed from the highest weight state to be

$$\begin{aligned} C_2|\lambda_0\rangle &= [\sum_{\alpha\in\Delta^+} t_\alpha + \sum_{i=1}^{l} h_i^2]|\lambda_0\rangle = [\sum_{\alpha\in\Delta^+}(\lambda_0,\alpha) + \sum_{i=1}^{l}(\lambda_0)_i^2]|\lambda_0\rangle \\ &= (\lambda_0 + 2\rho, \lambda_0)|\lambda_0\rangle, \end{aligned} \tag{9.20}$$

where $\rho = \sum_{\alpha\in\Delta^+}\alpha/2$ is called the `Weyl vector`. See §9.16(3) for more discussion of it.

6. If $\lambda_0 = \sum_{m=1}^{l} m_i\lambda_i$ is the highest weight of an IR, then this IR is often labeled by $(m_1, m_2, \cdots, m_l)$, where $m_i \geq 0$ are integers.

7. If λ be a highest weight and α_i a simple root. Then $m = \langle\lambda|\alpha_i\rangle$ is an integer, and $r_i(\lambda) = \lambda - m\alpha_i$.

`Proof` : The h_i quantum number of λ is $m = \langle\lambda|\alpha_i\rangle$, an integer. The h_i quantum number of $r_i(\lambda)$ $-m$, corresponding to a weight vector $\lambda - m\alpha_i$. ■

.

---

8. Dimensions of IR are listed in Appendix I of Ref. [7], and in Ref. [17].

9. If the Young tableau labeling an IR of $SU(n)$ is $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n-1})$, then the corresponding IR of $su(n)$ is given by $m_i = \lambda_i - \lambda_{i+1}$, if we take $\lambda_n = 0$. Complex conjugate representations are given by the conjugate tableau. In Lie algebra, they are given by the representations $(m_l, m_{l-1}, \cdots, m_2, m_1)$. Adjoint representations are $(1, 0, \cdots, 0, 1)$; they are self-conjugate.

9.15 $su(3)$ representations and quark model

PLEASE REVERSE λ_1 AND λ_2 IN THIS SECTION, AS WELL AS α_1 AND α_2.

Let us illustrate all these with $su(3) = A_2$ shown in Fig. 9.5. Orange dots are roots, and purple dots are weights. Simple roots are α_1 and α_2, fundamental weights are λ_1 and λ_2; they are shown embedded with a white $\times$.

If we take $h_1 \in \mathtt{h}$ along the x-axis and $h_2 \in \mathtt{h}$ along the y-axis, then the quantum number $\alpha(h_1)$ is called the (third component of) `isospin` in physics, I_3, and $\alpha(h_2)$ is called hypercharge, Y. Hypercharge is normalized so that $\alpha_2(Y) = +1$, and I_3 is normalized so that $\alpha_1(I_3) = 1$.

Since $2(\lambda_2, \alpha_2)/|\alpha_2|^2 = 1$ and the angle between λ_2 and α_2 is 30°, it is easy to show from trigonometry that $Y(\lambda_2) = 2/3$ and $Y(\lambda_1) = 1/3$.

As discussed in Example 2 of §8.10.4, if we ignore the mass difference between the u, d, s quarks, then the quarks possess both a color symmetry $SU(3)_c$ and a flavor symmetry $SU(3)_F$, both are $SU(3)$ groups. The quark model of hadrons was briefly discussed there, but the quantum numbers of the quarks and hadrons can best be seen in the $su(3)$ weight diagram shown in Figs. 9.5 and 9.6.

An IR is labeled by (m_1, m_2). The following representations are shown in Fig. 9.5:

1. the quark triplet $\mathbf{3} = (1, 0)$ is the inverted purple triangle with highest weight λ_1;

2. the antiquark triplet $\mathbf{3}^* = (0, 1)$ is the purple triangle with highest weight λ_2;

3. the octet baryon (Fig. 9.6, left) or the meson octet $\mathbf{8} = (1, 1)$ is shown with the orange dots. The highest root $\alpha_1 + \alpha_2$ is indicated by a white $+$;

4. the decuplet baryon (Fig. 9.6, right) $\mathbf{10} = (3, 0)$ made up of three quarks is shown with dark dots, and its highest weight is indicated by a while $+$.

5. In Fig. 9.6, the name of the baryons are filled in. S is called `strangeness` and Q is the electric charge. They are related to I_3 and Y by $S = Y - B$ where $B = 1$ is the `baryonic number`, and $Q = I_3 + Y/2$;

6. at the time when the decuplet classification was proposed by Gell-Mann (circa 1963), Ω^- was missing. Its subsequent discovery got him a Nobel prize, and furthr confirmation to the correctness of the quark model.

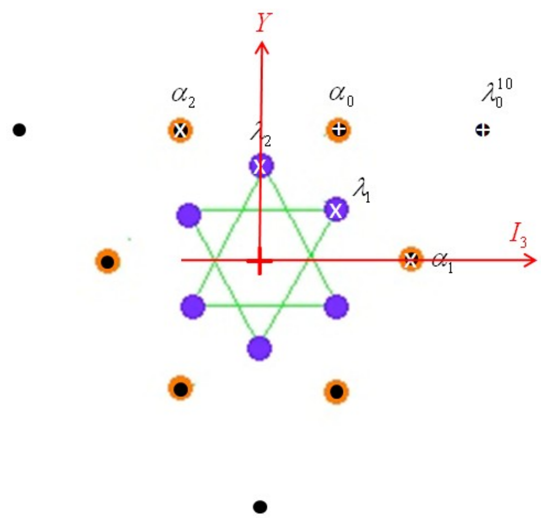

Figure 9.6: A_2 root and weight diagrams

9.16 Weyl character formula

1. Let G be a Lie group with a Lie algebra $\mathtt{g}$ and its Cartan subalgebra $\mathtt{h}$. The weights $\lambda(h_i)$ of $\mathtt{g}$ are the simultaneous quantum numbers of h_i $(1 \leq i \leq l)$. If λ_0 is the highest weight of an mth dimensional IR, then the representation H_i of of h_i are $m \times m$ diagonal matrices, with entries $\lambda(h_i)$.

2. The `maximal tori` T of G in this representation is the abelian subgroup generated by the H_i's, namely, diagonal matrices of the form $M(\theta) := M(\theta_1, \cdots, \theta_l) = \exp(i\sum_{j=1}^{l} \theta_j H_j)$. Its character is

$$\chi_{\lambda_0}(\theta) = \sum_{\lambda} \exp(i\vec{\theta}\cdot\vec{\lambda}), \tag{9.21}$$

where $\vec{\lambda} = (\lambda(H_1), \cdots, \lambda(H_l))$, and the sum is taken over all weights, with multiplicity taken into account, and $\vec{\lambda} = \vec{0}$ included. For groups

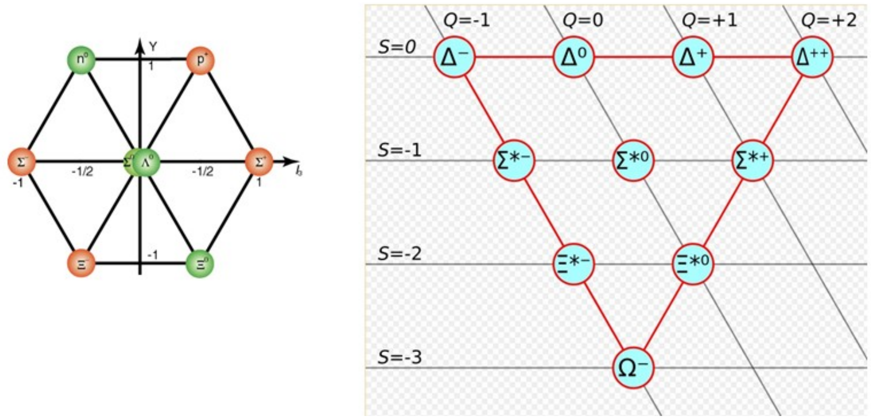

Figure 9.7: Baryon octet and decuplet

like $SU(n)$ and $SO(n)$ whose matrices are diagonalizable, (9.21) gives the character of every group member whose diagonalized form is $M(\theta)$.

3. The problem with (9.21) is that it is hard to know what the weights λ and their multiplicities are. The explicit `Weyl character formula` solves that problem,

$$\chi_\lambda(\theta) = \frac{\sum_{w\in W}(-)^w \exp(iw(\lambda+\rho)\cdot\theta)}{e^{i\rho.\theta}\prod_{\alpha\in\Delta^+}(1-e^{-i\alpha.\theta})}, \tag{9.22}$$

where $(-)^w = \pm 1$ if w is a product of an even/odd number of r_i. In this formula, λ is the *highest weight*, and $\rho = \sum_{i=1}^{l}\lambda_i$ is the `Weyl vector`. This formula is often written in the abstract form

$$\chi_\lambda = \frac{\sum_{w\in W}(-)^w \exp(w(\lambda+\rho))}{e^{\rho}\prod_{\alpha\in\Delta^+}(1-e^{\alpha})}. \tag{9.23}$$

Note that the Weyl vector has the following properties:

$$\rho = \sum_{\alpha\in\Delta^+}\alpha/2, \quad r_i(\rho) = \rho - \alpha_i. \tag{9.24}$$

`Proof` : $r_i(\rho) = \rho - \langle\rho|\alpha_i\rangle\alpha_i = \rho - \alpha_i$ because $\langle\lambda_j|\alpha_i\rangle = \delta_{ij}$.

To show that ρ is half the sum of all positive roots, first note that such a sum also has the same reflection property under r_i. To see this, write the half sum as $\tau = \beta + \alpha_i/2$, where β is half the sum of all positive roots except α_i. Now $r_i(\alpha_i) = -\alpha_i$, and from §9.12(5), $r_i(\beta) = (\beta)$, hence $r_i(\tau) = \tau - \alpha_i$. Consequently $\rho - \tau$ is invariant under all reflections r_i, so it must be zero. ■

.

Before proceeding to prove this formula, let us first make some observations to motivate it.

(a) Characters should be invariant under Weyl reflections: $\chi_{w(\lambda)} = \chi_\lambda$ for every $w \in W$. To see that, consider a simple reflection $w = r_i$. The h_{α_i} quantum number of λ is $\langle\lambda|\alpha_i\rangle$, and the h_{α_i} quantum number of $r_i(\lambda) = \lambda - \langle\lambda|\alpha_i\rangle\alpha_i$ is $\langle r_i(\lambda)|\alpha_i\rangle = -\langle\lambda|\alpha_i\rangle$. Thus $|r_i(\lambda)\rangle$ can be obtained from $|\lambda\rangle$ by using f_{α_i} to crank it down, so it is also a weight.

This is an important property because it will be used to obtain indirectly all the weights and their multiplicities.

(b) To be consistent with this property, one can show that both the numerator and the denominator are Weyl-invariant up to a sign:

i. using (9.24), the denominator can be written as $D = \prod_{\alpha\in\Delta^+}(e^{\alpha/2} - e^{-\alpha/2})$. Recall from §9.12(4) that r_i maps α_i to $-\alpha_i$ but all other positive roots into positive roots, it is clear that r_i maps D into $-D$, hence $w(D) = (-)^w D$ for all $w \in W$.

ii. The numerator N clearly has that property as well, because W forms a group.

(c) Let us now look to su(2) for a guide. With $\lambda(J_3) = j$, $\lambda = \{j, j-1, \cdots, -j\}$, its character is

$$\chi_\lambda(\theta) = \sum_{m=-j}^{j} e^{im\theta} = \frac{e^{i(j+1/2)\theta} - e^{-i(j+1/2)\theta}}{e^{i\theta/2} - e^{-i\theta/2}}, \tag{9.25}$$

and its dimension is

$$\dim\,(V_\lambda) = \chi_\lambda(0) = 2j + 1. \tag{9.26}$$

The abstract form of (9.25) is

$$\chi_\lambda = \frac{e^{\lambda+\rho} - e^{-(\lambda+\rho)}}{e^\rho(1-e^{-\alpha})}, \tag{9.27}$$

because its only positive root is $\alpha = 1$, its Weyl vector is $\rho = 1/2$, and the dominant weight is $\lambda = j$. In this form, (9.27) is precisely equal to (9.23), so the Weyl character formula is proved for $su(2)$.

(d) The number of weights in an IR is generally not equal to the order $|W|$ of the Weyl group. For example, in $su(n)$, $|W| = |S_n| = n!$, but the number of weights ν in an IR depends on what λ is, so generally $\nu \neq |W|$. One might therefore wonder how possibly can (9.23) give rise to the character. The example of $su(2)$ shows us how. By expanding the denomintor of (9.27) into power series, one gets

$$\begin{aligned} \chi_\lambda &= (1 + e^{-\alpha} + e^{-2\alpha} + \cdots)(e^\lambda - e^{-(\lambda+2\rho)}) \\ &= e^\lambda + e^{\lambda-\alpha} + e^{\lambda-2\alpha} + \cdots + e^{-\lambda}. \end{aligned} \tag{9.28}$$

This expansion of the denominator that produces an infinite number of terms, so the total number of terms is no longer controlled by $|W|$. Moreover, the unwanted terms $e^{-(\lambda+m\alpha)}$ for any $m > 0$ all get cancelled out, so we are finally left with the $2j + 1$ terms needed for the character.

Another way of saying the same thing is that we can factorize the numerator into

$$e^{\lambda+\rho}(1-e^{-2(\lambda+\rho)}) = e^{\lambda+\rho}(1-e^{-\alpha})(1+e^{-\alpha}+e^{-2\alpha}+\cdots+e^{-2(\lambda+\rho)+\alpha}),$$

because both 2λ and 2ρ are integer multiples of α. Hence we are left with the last expression in (9.28) after dividing it by the denominator of (9.27) because $2\rho = \alpha$.

(e) A similar mechanism is at work for the general Weyl character formula. If ξ is a linear combination of e^μ of weights μ, and $w(\xi) = (-)^w\xi$, then ξ is divisible by the denominator D of (9.23). This is so because for every $\alpha \in \Delta^+$, $r_\alpha(\mu) = \mu - k\alpha$ for some integer $k = 2\langle\mu|\alpha\rangle$, hence ξ must contain the combination $e^\mu - e^{\mu-k\alpha}$.

This however can be factorized into $e^\mu(1-e^{-\alpha})(1+e^{-a}+e^{-2\alpha}+\cdots+e^{-(k-1)\alpha})$. In particular, since N has the same property of ξ, the numerator N in (9.23) is divisible by its denominator D, so χ_λ is given by a finite series.

9.17 Proof of the Weyl character formula

A `Verma module` $M(\lambda)$ with a highest weight λ is the set of all states obtainable from λ by operating products of f_α ($\alpha \in \Delta^+$) on $|\lambda\rangle$. It is a vector space much larger than the IR $L(\lambda)$ with highest weight λ. While the character χ_λ of $L(\lambda)$ is relatively difficult to obtain because we need to know its lower boundaries and the multiplicity of states, the character of $M(\lambda)$ is simple and is equal to

$$\chi_{M(\lambda)} = e^\lambda \prod_{\alpha\in\Delta^+}(1+e^{-\alpha}+e^{-2\alpha}+\cdots) = \frac{e^\lambda}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})}. \qquad (9.29)$$

The vector space $M(\lambda)$ can be decomposed into sums of irreducible spaces $L(\lambda')$, with $\lambda' \le \lambda$. Conversely, $L(\lambda)$ can also be written as sums and differences of $M(\lambda')$ with $\lambda' \le \lambda$. What distinguishes $L(\lambda)$ from other states in $M(\lambda)$ is that every state in $L(\lambda)$ has the same Casimir number $(\lambda+\rho)^2-\rho^2$. We may therefore write

$$L(\lambda) = \sum_{\lambda'\le\lambda,\ (\rho+\lambda')^2=(\rho+\lambda)^2} c_{\lambda'} M(\lambda'), \qquad (9.30)$$

with $c_\lambda = 1$. Now take the formal character from both sides of (9.30), and multiply them by

$$D = e^\rho \prod_{\alpha\in\Delta^+}(1-e^{-\alpha}) = \prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2}).$$

Using (9.29), the result is

$$D\chi_\lambda = \sum_{\lambda'\le\lambda,\ (\rho+\lambda')^2=(\rho+\lambda)^2} c_{\lambda'} e^{\rho+\lambda'}. \qquad (9.31)$$

It follows from §9.12(5) that $w(D) = (-)^w D$ for every $w \in W$. Since $\chi_{w(\lambda)} = \chi_\lambda$, the right hand side of (9.31) must also be skew symmetric under all $w \in W$, hence it must of the form

$$\sum_{\lambda' \leq \lambda,\ (\rho+\lambda')^2 = (\rho+\lambda)^2} c_{\lambda'} \sum_{w \in W} (-)^w e^{w(\rho+\lambda')}. \tag{9.32}$$

Finally, note that $(w(\rho+\lambda'))^2 = (\rho+\lambda')^2$ if $\lambda' = \lambda$, but this cannot be true if $\lambda' < \lambda$. Hence we conclude that

$$\chi_\lambda = \frac{\sum_{w \in W} (-)^w e^{w(\rho+\lambda')}}{D} = \frac{\sum_{w \in W} (-)^w e^{w(\rho+\lambda)}}{e^\rho \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}, \tag{9.33}$$

thus proving (9.23). ■

9.18 An example in $su(3)$

For $su(3)$, $\rho = \alpha_1 + \alpha_2 := \alpha_0$, $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_0\}$, hence

$$D = e^\rho (1 - e^{-\alpha_1})(1 - e^{-\alpha_2})(1 - e^{-\alpha_0}). \tag{9.34}$$

The Weyl group is $W = S_3$, generated by the Weyl reflections r_0, r_1, and r_2, respectively about the bluish dotted lines perpendicular to α_0, α_1 and α_2, as seen in the following figure. $r_0 = r_1 r_2 r_1$. Other than the identity, the two other even elements of $W = S_3$ are $r_1 r_2$ and $r_2 r_1$, rotation for $+120°$ and $-120°$, respectively.

The roots are given by the orange dots.

quark representation in su(3)

I shall use Fig. 9.8 to illustrate the derivation and the use of the Weyl character formula.

1. First, use (9.23) to compute the character χ_{λ_1} of the quark representation.

   The quark weights are the blue dots forming the inverted triangle with highest weight λ_1. The three weights are $\lambda_1 = r_2(\lambda_1), \lambda_1' = r_1(\lambda_1) = r_1 r_2(\lambda_1) = \lambda_1 - \alpha_1$, and $\lambda_1'' = r_0(\lambda_1) = r_0 r_2(\lambda_1) = \lambda_1 - \rho$.

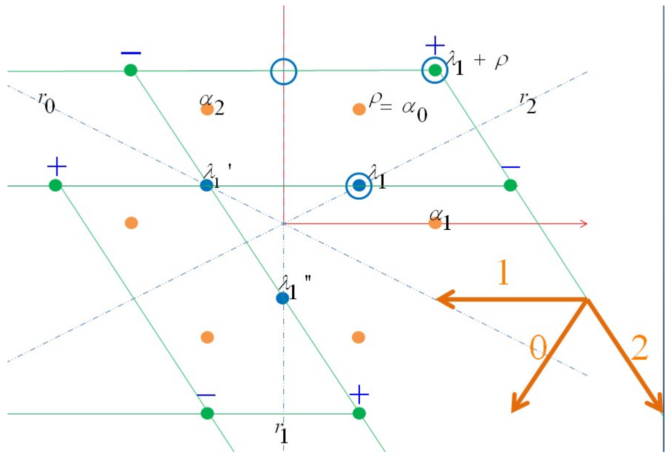

Figure 9.8: An illustration of the Weyl character formula

The numerator of the character formula is

$$\begin{aligned}
N &= e^{\lambda_1+\rho} + e^{\lambda_1-2\alpha_1} + e^{\lambda_1-\rho-\alpha_2} - e^{\lambda_1-\alpha_1+\alpha_2} - e^{\lambda_1+\alpha_1} - e^{\lambda_1-2\rho} \\
&= e^{\lambda_1+\rho}(1 + e^{-3\alpha_1-\alpha_2} + e^{-2\alpha_1-3\alpha_2} - e^{-2\alpha_1} - e^{-\alpha_2} - e^{-3\alpha_1-3\alpha_2}) \\
&= e^{\lambda_1+\rho}(1 - e^{-\alpha_2})(1 + e^{-3\alpha_1-2\alpha_2} + e^{-3\alpha_1-\alpha_2} - e^{-2\alpha_1-2\alpha_2} \\
&\qquad -e^{-2\alpha_1-\alpha_2} - e^{-2\alpha_1}) \\
&= e^{\lambda_1+\rho}(1 - e^{-\alpha_2})(1 - e^{-\alpha_1})(1 - e^{-\alpha_1-\alpha_2})(1 + e^{-\alpha_1} + e^{-\alpha_1-\alpha_2}),
\end{aligned} \tag{9.35}$$

hence

$$\chi_{\lambda_1} = \frac{N}{D} = e^{\lambda_1}(1 + e^{-\alpha_1} + e^{-\alpha_1-\alpha_2}) = e^{\lambda_1} + e^{\lambda_1'} + e^{\lambda_1''}. \tag{9.36}$$

2. Let us use Fig. 9.8 to illustrate the derivation of the character formula. The six Weyl reflection points $w(\lambda_1 + \rho)$ are given by the six green dots, each of which acting as the highest weight of a Verma module with character

$$e^{w(\lambda_1+\rho)} \prod_{\alpha\in\Delta^+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots).$$

The thick orange lines in the lower right corner indicates the direction to crank from the highest weight states to get other states in the Verma module. The blue $\pm$ sign on top of each green dot comes from the factor $(-)^w$, and indicates whether the states in this Verma module should be added or subtracted to get the states in the IR $L(\lambda_1)+|\rho\rangle$. These final states are indicated by the three big blue open circles. When shifted back by $-\rho$, we get the three solid blue quark states $\lambda_1, \lambda_1', \lambda_1''$, whose character is given by (9.23).

9.19 Weyl denominator formula

is an identity obtained by setting $\lambda = 0$ in (9.23):

$$e^\rho \prod_{\alpha\in\Delta^+}(1-e^{-\alpha}) = \sum_{w\in W}(-)^w e^{w(\rho)}. \tag{9.37}$$

1. $r_i(\rho) = \rho - \alpha_i$, hence $w(\rho)$ is of the form $\rho - \sum_i n_i^w \alpha_i$. (9.37) is then equivalent to

$$\prod_{\alpha\in\Delta^+}(1-e^{-\alpha}) = \sum_{w\in W}(-)^w e^{-\sum_i n_i^w \alpha_i}. \tag{9.38}$$

2. (9.38) is non-trivial to prove because the number of terms on the left is $2^{|\Delta^+|}$, which is not equal to the number of terms $|W|$ on the right.

3. Let us demonstrate (9.37) explicitly for A_1, A_2. For this, it is more convenient to write the left hand side as

$$\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2}). \tag{9.39}$$

   Recall from §9.10.1 and §9.12(4a) that the positive roots of A_l are $e_{ij} = e_i - e_j$, for $1 \le i < j \le l+1$, the simple roots are $\alpha_i = e_{i,i+1}$, and the Weyl group $W = S_{l+1}$ is the group permuting the e_i's.

   We shall use L and R respectively to denote the left hand side and the right hand side of (9.37).

4. A_1: $\Delta^+ = \{\alpha_1\}, \rho = \alpha_1/2$.

$$L = e^{\alpha_1/2} - e^{-\alpha_1/2},\ R = e^\rho - e^{(12)\rho} = e^{\alpha_1/2} - e^{-\alpha_1/2}. \Rightarrow L = R\ \blacksquare$$

5. A_2: $\Delta_+ = \{e_{12}, e_{23}, e_{13}\}$, $\rho = e_{13}$.

$$\begin{aligned} L &= e^{(e_{12}+e_{23}+e_{13})/2} - e^{(-e_{12}+e_{23}+e_{13})/2} - e^{(e_{12}-e_{23}+e_{13})/2} \\ &- \underline{e^{(e_{12}+e_{23}-e_{13})/2}} - e^{(-e_{12}-e_{23}-e_{13})/2} + e^{(e_{12}-e_{23}-e_{13})/2} \\ &+ e^{(-e_{12}+e_{23}-e_{13})/2} + \underline{e^{(-e_{12}-e_{23}+e_{13})/2}} \end{aligned}$$

Since the two underlined terms cancel, we are left with only 6 terms.

$$R = e^{e_{13}} - e^{e_{23}} - e^{e_{12}} - e^{e_{31}} + e^{e_{21}} + e^{e_{32}} \tag{9.40}$$

Since $R(1,2,3,4,5,6) = L(1,2,3,5,7,6)$ and $L(4) + L(8) = 0$, hence $L = R$.

9.20 Weyl dimensional formula

is obtained from (9.22) by letting $\vec{\theta} = \theta\vec{\rho}$ and setting $\theta = 0$. After some manipulation, we get

$$\dim\,(V_\lambda) = \prod_{\alpha>0} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}. \tag{9.41}$$

By $\alpha > 0$, we mean $\alpha \in \Delta^+$.

Proof : Starting from the denominator identity, replace α by $(\alpha, \theta\xi)$, then set $\theta = 0$. In this way we get the identity $\prod_{\alpha>0}(\alpha,\xi) = \sum_w (-)^w (w\rho, \xi)$. Now set $\theta = 0$ in (9.22) and use this formula, we get

$$\chi_\lambda(0) = \dim\,(V_\lambda) = \frac{\sum_w (-)^w (\lambda+\rho, w\rho)}{\prod_{\alpha>0}(\alpha,\rho)} = \prod_{\alpha>0}\frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}. \blacksquare$$

.

Let us apply this formula to $su(2)$ and $su(3)$.

1. A_1: there is only one positive root, $\alpha = \alpha_1$, and $\rho = \alpha/2$. The highest weight is $\lambda = j\alpha$, where j is an integer or a half integer. Hence $\dim\,(V_j) = 2(j+1/2) = 2j+1$.

2. A_2: the positive roots are $\alpha_1, \alpha_2, \beta = \alpha_1 + \alpha_2$, with $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$, and $(\alpha_1, \alpha_2) = -1$. Moreover, $\rho = \beta$. If λ_1, λ_2 are the fundamental weights, then $(\lambda_i, \alpha_j) = \delta_{ij}$, and the highest weight is given by $\lambda = m\lambda_1 + n\lambda_2$. Thus

$$\begin{aligned} \dim\,(V_{m,n}) &= \frac{(\lambda+\rho,\alpha_1)(\lambda+\rho,\alpha_2)(\lambda+\rho,\beta)}{(\rho,\alpha_1)(\rho,\alpha_2)(\rho,\beta)} \\ &= \frac{(m+1)(n+1)(m+n+2)}{2}. \end{aligned} \tag{9.42}$$

In particular, the dimensions of (1,0) and (0,1) are 3, (2,0) and (0,1) are 6, (3,0) and (0,3) are 10, (1,1) is 8, and (2,2) is 27.

9.21 Kac-Moody algebra

A free field $\phi(x,t)$ is `1+1 dimension` satisfies the wave equation

$$(\partial_t^2 - \partial_x^2)\phi(x,t) = 0, \tag{9.43}$$

whose general solution is of the form $f(x,t) = L(x+t) + R(x-t)$. The wave packets L and R propagates respectively to the left and to the right, without changing their shape. This is a special feature of $1+1$ dimensions; in higher dimensions, the wave packet always spreads and changes shape as time goes on. Imagine now the wave packet carries a charge. In this case not only the total charge carried by each packet is conserved, but the amount of charge in any fixed interval of the packet is also conserved. In other words, charge density itself is also conserved: $\partial_t j^0(x,t) = 0$ for any x. This is still true even if the charge is non-abelian. To emphasize this fact, we write it as $\partial_t \vec{j}^0(x,t) = 0$, where the arrow indicates isotopic spin, $SU(3)$ charge, etc.

If the Lie algebra for the nonabelian charge is $\mathtt{g}$, then with this greatly enlarged conservation, $\mathtt{g}$ is also greatly enlarged. The resluting algebra $\tilde{\mathtt{g}}$ is known as a `Kac-Moody algebra`. This is the kind of symmetry that a string theory possesses.

Simple Lie algebras can be classified by $l \times l$ Cartan matrices A_{ij}, with diagonal entries equal to 2, and non-diagonal entries equal to $0, -1, -2, -3$. Moreover, if $A_{ij} = 0$, then $A_{ji} = 0$, and, the rank of A is always l. Kac-Moody algebra can also be classified by $n \times n$ `generalized Cartan matrices` of rank l, whose diagonal entries are 2, and non-diagonal entries arbitrary non-positive numbers, such that $A_{ij} = 0$ implies $A_{ji} = 0$. The number of simple roots in this case is n.

I will not go into the details of Kac-Mooday algebra, except to refer you to the book by Victor Kac: 'Infinite dimensional Lie algebras' (Cambridge University Press).