# Grand Unification Theory

Ling-Fong Li

## Grand Unification Theory

The standard model of $SU(3)_C \times SU(2)_L \times U(1)_Y$ are all based on gauge theory. Want to combine all these interactions as components of a single force; a theory with only one guage coupling.

$SU(5)$ Model :1974 Georgi and Glashow

A general representation of $SU(5)$ in tensor notation transforms as,

$$\psi^{ij\cdots}_{kl\cdots} \longrightarrow U^i_m U^j_n U^s_k U^t_l \cdots \psi^{mn\cdots}_{st\cdots}$$

where

$$[U]^i_m = [\exp(i\alpha^a \lambda^a/2)]^i_m$$

is a $5 \times 5$ unitary matrix and $\{\lambda^a\}$, $a = 0, 1, 2, \cdots, 23$ is a set of $5 \times 5$ hermitian traceless matrices with ,

$$T(\lambda_a \lambda_b) = 2\delta_{ab}$$

For example,

$$\lambda^3 = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}, \qquad \lambda^0 = \frac{1}{\sqrt{15}} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

To obtain the $SU(3)_C \times SU(2)_L$ content of a representation, we identify first 3 of $SU(5)$ indices as color indices and the other 2 as $SU(2)_L$ indices,

$$i = (\alpha, r), \qquad \text{with} \;\; \alpha = 1,2,3 \qquad r = 1,2 \tag{1}$$

**Fermion content**

In the standard model with one generation, the fermion content with respect to $SU(3)_C \times SU(2)_L$ are given by,

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix} \sim (\mathbf{1}, \mathbf{2}), \qquad e_L^+ \sim (\mathbf{1}, \mathbf{1}),$$

$$\begin{pmatrix} u_\alpha \\ d_\alpha \end{pmatrix} \sim (\mathbf{3}, \mathbf{2}), \qquad u_L^{c\alpha} \sim (\mathbf{3}^*, \mathbf{1}), \qquad d_L^{c\alpha} \sim (\mathbf{3}^*, \mathbf{1})$$

where we have used the relaitons

$$\psi^c = C\gamma^0\psi^*, \qquad (\psi_R)^c = (\psi^c)_L \equiv \psi_L^c$$

The $SU(3)_C \times SU(2)_L$ contents of simple $SU(5)$ representations are ;

| | | |
|---|---|---|
| lowest rep | $\psi_i$ | $\mathbf{5} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$ |
| lowest conjugate | $\psi^i$ | $\mathbf{5}^* = (\mathbf{3}^*, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$ |
| anti-symm | $\psi_{ij}$ | $\mathbf{10} = (\mathbf{3}^*, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{1}, \mathbf{1})$ |

Comparing with first generation fermions, we see that they can be accomodated as $\mathbf{5}^*+\mathbf{10}$ representation of $SU(5)$,

$$\mathbf{5}^* : \left(\psi^i\right)_L = \left(d^{c1}, d^{c2}, d^{c3}, e^-, \nu\right)_L$$

and

$$10 : \left(\chi_{ij}\right)_L = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u^{c3} & -u^{c2} & u_1 & d_1 \\ -u^{c3} & 0 & u^{c1} & u_2 & d_2 \\ u^{c2} & -u^{c1} & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & -0 \end{pmatrix}$$

**Charge Quantization**

One consequence of $SU(5)$ unification is a explanation for charge quantization. In general, if unification group is simple $\Longrightarrow$ charge quantization, because eigenvalues of a non-Abelian group are discrete while Abelian $U(1)$ group are continuous.In $SU(3)_C \times SU(2)_L$ model electric charge operator is

$$Q = T_3 + \frac{Y}{2}$$

Express this relation in terms of generators of $SU(5)$. Write

$$Q = T_3 + cT_0$$

It is straightforward to see that

$$c = -\sqrt{\frac{5}{3}}$$

## Gauge bosons

The $SU(5)$ adjoint rep $A_i^j$ has dimension 24 and the $SU(3)_C \times SU(2)_L$ content is

$$\mathbf{24} = (\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{2}) + (\mathbf{3}^*,\mathbf{2})$$

| | | |
|---|---|---|
| $A_\beta^\alpha$ | $(\mathbf{8},\mathbf{1})$ | $SU(3)_C$ color gluons |
| $A_s^r$ | $(\mathbf{1},\mathbf{3})$ | $SU(2)_L$ weak bosons |
| $A_\alpha^\alpha - A_r^r$ | $(\mathbf{1},\mathbf{1})$ | $U(1)$ weak boson |
| $A_\alpha^r$ | $(\mathbf{3},\mathbf{2})$ | Lepto-quark |
| $A_r^\alpha$ | $(\mathbf{3}^*,\mathbf{2})$ | Lepto-quark |

The lepto-quark

$$A_\alpha^r = (X_\alpha, Y_\alpha), \qquad A_r^\alpha = \begin{pmatrix} X^\alpha \\ Y^\alpha \end{pmatrix}$$

have fractional charges

$$Q(X) = -\frac{4}{3}, \qquad Q(Y) = -\frac{1}{3}$$

and are important in the Baryon number violation. Put all $SU(5)$ gauge bosons in $5 \times 5$ matrix

$$A_\mu = \sum_{a=0}^{23} A_\mu^a \frac{\lambda^a}{2}$$

we get

$$A = \begin{pmatrix} G_1^1 & G_2^1 & G_3^1 & X_1 & Y_1 \\ G_1^2 & G_2^2 & G_3^2 & X_2 & Y_2 \\ G_1^3 & G_2^3 & G_3^3 & X_3 & Y_3 \\ X^1 & X^2 & X^3 & W^3 + B & W^+ \\ Y^1 & Y^2 & Y^3 & W^- & -W^3 + B \end{pmatrix} \tag{2}$$

## Spontaneous symmetry breaking

Spontaneous symmetry breaking in two stages, characterized by $v_1$ and $v_2$

$$SU(5) \xrightarrow{v_1} SU(3)_C \times SU(2)_L \times U(1)_Y \xrightarrow{v_2} SU(3)_C \times U(1)_{EM}$$

where $v_1 \gg v_2$. In 1st stage, $X, Y$ masses being superheavy, $M_{X,Y} \gg M_{W,Z}$. This can be achieved with scalars in adjoint $\left(H^i_j\right)$ for $v_1$ and vector $(\phi_i)$representations for $v_2$. The general $SU(5)$ −invariant 4-th order potential is,

$$V(H,\phi) = V_1(H) + V_2(\phi) + \lambda_4 tr\left(H^2\right)\left(\phi^\dagger\phi\right) + \lambda_5\left(\phi^\dagger H^2\phi\right)$$

with

$$V_1(H) = -m_1^2 tr\left(H^2\right) + \lambda_1\left[tr\left(H^2\right)\right]^2 + \lambda_2 tr\left(H^4\right)$$

$$V_2(\phi) = -m_2^2\left(\phi^\dagger\phi\right) + \lambda_3\left(\phi^\dagger\phi\right)^2$$

$H$ is $5\times 5$ traceless hermitian matrix. Use discrete symmetry $H \to -H$ and $\phi \to -\phi$ to remove cubic terms. We first minimize the potential $V_1(H)$. It turns out that for $\lambda_2 > 0$ and $\lambda_1 > -\frac{7}{30}$, $V_1(H)$ has an extremum at $H = \langle H\rangle$ with

$$\langle H\rangle = v_1\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

where

$$v_1^2 = \frac{m_1^2}{[60\lambda_1 + 14\lambda_2]}$$

and symmetry breaking is

$$SU(5) \longrightarrow SU(3) \times SU(2) \times U(1)$$

and gauge bosons $X$, and $Y$ obtain masses $\propto v_1$. As we will see later $v_1$ could be of order of $10^{15}$ $Gev$ or so.
The fact $H$ develops $VEV$ also effects the $\phi$ system. The color triplet $\phi_t : (\mathbf{3}, \mathbf{1})$ and flavor doublet $\phi_d : (\mathbf{1}, \mathbf{2})$ components of $\phi = (\phi_t, \phi_d)$ acquires mass terms,

$$m_t^2 = -m_2^2 + (30\lambda_4 + 4\lambda_5)\, v_1^2 \tag{3}$$

$$m_d^2 = -m_2^2 + (30\lambda_4 + 9\lambda_5)\, v_1^2 \tag{4}$$

After first stage of symmetry breaking all masses are of order of $v_1$ , superheavy.
For the second stage of symmetry breaking need a $SU(2)$ doublet to get $v_2 \sim 250$ $Gev$. Here we assume "somehow" the $m_d^2$ in Eq (4) is much smaller than $v_1^2$ and will survive to low energy($\sim 250$ $Gev$) as the superheavy particle (with masses of order $v_1$) decouple. The relevant physics is described by the effective potential,

$$V_{eff}(\phi_d) = -m_d^2 \left(\phi_d^\dagger \phi_d\right) + \lambda_3 \left(\phi_d^\dagger \phi_d\right)^2$$

which produces symmetry breaking

$$SU(2) \times U(1) \longrightarrow U(1)$$

$$\phi_d = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \qquad\qquad v_2 = \sqrt{\frac{m_d^2}{\lambda_3}} \sim 250 \ Gev$$

This feature where $v_1 \gg v_2$ is usually called the **gauge hierachy**.

## Coupling constant unification

In standard model,for energies $\leq 10^2 Gev$ there are 3 coupling constants: $g_s, g$, and $g'$ for , $SU(3)_C, SU(2)_L$ and $U(1)_Y$ respectively. In grand unified theory unifies into one coupling constant.

After $v_1$ $X, Y$ gauge bosons of $SU(5)$ acquire masses and decouple from the coupling constant renormalization.This leads to 3 different coupling constants. Since the energy dependence of coupling constants is only logarithmic, unification scale $M_X$ is expected to be many orders of magnitude larger than $10^2 Gev$.

The covariant derivative for $SU(5)$ is

$$D_\mu = \partial_\mu + ig_5 \sum_{a=0}^{23} A_\mu^a \frac{\lambda^a}{2}$$

and for $SU(3)_C \times SU(2)_L \times U(1)_Y$

$$D_\mu = \partial_\mu + ig_s \sum_{a=1}^{8} G_\mu^a \frac{\lambda^a}{2} + ig \sum_{r=1}^{3} W_\mu^r \frac{\lambda^r}{2} + ig' B_\mu \frac{Y}{2}$$

All non-Abelian groups here are normalized as $Tr(\lambda^a \lambda^b) = 2\delta^{ab}$ and we have

$$g_5 = g_3 = g_2 = g_1 \tag{5}$$

with

$$g_3 = g_s, \qquad g_2 = g$$

The coupling $g_1$ is that of the Abelian $U(1)$ subgroup. Thus

$$ig_1 \lambda^0 A^0_\mu = ig' Y B_\mu$$

and $A^0_\mu$ is identified with $B_\mu$ gauge field. Note that

$$Y = \begin{pmatrix} -\frac{2}{3} & & & & \\ & -\frac{2}{3} & & & \\ & & -\frac{2}{3} & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$$

From this we get

$$Y = -\sqrt{\frac{5}{3}}\lambda^0, \qquad g' = -\sqrt{\frac{3}{5}}g_1$$

The weak mixing angle is then

$$\sin^2\theta_W = \frac{g'^2}{g^2 + g'^2} = \frac{3}{8} \tag{6}$$

The relations given Eqs (5,6) are valid at unification scale. To compare them with experimental data we need to evolve them down to low energy $\sim 10^2 Gev$. The evolution of the $SU(n)$ coupling constant is of the form,

$$\frac{dg_n}{d(\ln \mu)} = -b_n g_n^3$$

where

$$b_n = \frac{1}{48\pi^2}\left(11n - 2N_F\right) \qquad \text{for} \quad n \geq 2$$

$$b_1 = -\frac{N_F}{24\pi^2}$$

Then we get

$$b_n - b_1 = \frac{11n}{48\pi^2}$$

In our case, the solution for the effective coupling constants are

$$\frac{1}{g_1^2(\mu)} = \frac{1}{g_1^2(\mu_0)} + 2b_1 \ln\left(\frac{\mu}{\mu_0}\right)$$

$$\frac{1}{g_2^2(\mu)} = \frac{1}{g_2^2(\mu_0)} + 2b_2 \ln\left(\frac{\mu}{\mu_0}\right)$$

$$\frac{1}{g_3^2(\mu)} = \frac{1}{g_3^2(\mu_0)} + 2b_3 \ln\left(\frac{\mu}{\mu_0}\right)$$

In terms of more familiar parameters,

$$\frac{g_1^2(\mu)}{4\pi} = \left(\frac{5}{3}\right)\frac{\alpha(\mu)}{\cos^2\theta_W}, \qquad \frac{g_2^2(\mu)}{4\pi} = \frac{\alpha(\mu)}{\sin^2\theta_W}, \qquad \frac{g_3^2(\mu)}{4\pi} = \alpha_s(\mu)$$

we get

$$\frac{1}{\alpha_s(\mu)} = \frac{1}{\alpha_5} + 8\pi b_3 \ln\left(\frac{\mu}{M_X}\right)$$

$$\frac{\sin^2\theta_W}{\alpha(\mu)} = \frac{1}{\alpha_5} + 8\pi b_2 \ln\left(\frac{\mu}{M_X}\right)$$

$$\frac{3}{5}\frac{\cos^2\theta_W}{\alpha(\mu)} = \frac{1}{\alpha_5} + 8\pi b_1 \ln\left(\frac{\mu}{M_X}\right)$$

where we have used

$$g_1(M_X) = g_2(M_X) = g_2(M_X) = g_5, \qquad \text{and} \quad \frac{g_5^2}{4\pi} = \alpha_5$$

Taking a linear combination to eliminate $\ln\left(\frac{\mu}{M_X}\right)$ we get

$$\frac{2}{\alpha_s} - \frac{3}{\alpha}\sin^2\theta_W + \frac{3}{5\alpha}\cos^2\theta_W = 8\pi\left[2(b_3 - b_1) - 3(b_2 - b_1)\right]\ln\left(\frac{\mu}{M_X}\right) = 0$$

This implies

$$\sin^2\theta_W = \frac{1}{6} + \frac{5\alpha(\mu)}{9\alpha_s(\mu)}$$

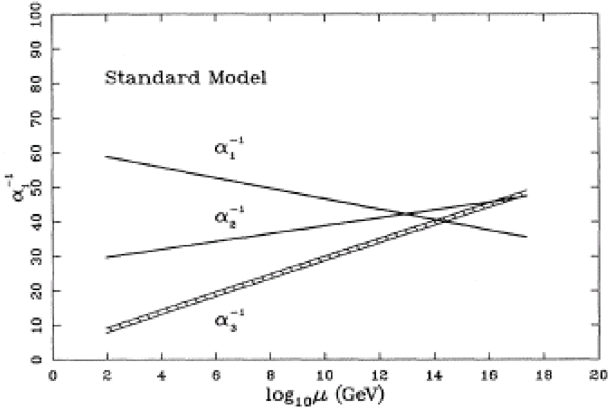
Using the measured values of $\alpha(\mu)$ and $\alpha_s(\mu)$ we get

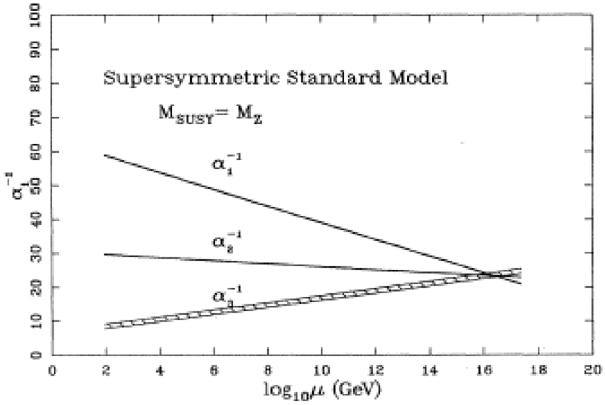
$$\sin^2\theta_W \simeq .21$$

and

$$M_X \simeq 4 \times 10^{14}\,Gev$$

Or using the measured values of $\sin^2\theta_W$, we get






This shows that these 3 coupling constants do not unify very well. It turns out that if we use supersymmetric version of unification we getThis has become one of the motivation for supersymmetry.

**Baryon number violation**

The gauge couplings of $\mathbf{5}^{*}+\mathbf{10}$ fermions $\left(\psi^{i}, \chi_{ij}\right)$, are of the form, using the gauge boson matrix in Eq (2)

$$\begin{aligned} g\bar{\psi}\gamma^{\mu}A_{\mu}^{T}\psi + trg\bar{\chi}\gamma^{\mu}\{A_{\mu}, \chi\} \quad = \quad & -\sqrt{\frac{1}{2}}gW_{\mu}^{\dagger}\left(\bar{\nu}\gamma^{\mu}e + \bar{u}_{\alpha}\gamma^{\mu}e_{\alpha}\right) \\ & +\sqrt{\frac{1}{2}}gX_{\mu\alpha}^{a}[\varepsilon^{\alpha\beta\gamma}\bar{u}_{\alpha}\gamma^{\mu}q_{\beta a} + \varepsilon^{ab}(\bar{q}_{\alpha b}\gamma^{\mu}e^{+} - \bar{l}_{b}\gamma^{\mu}d_{\alpha}^{c})] \end{aligned}$$

Note that $X$ bosons couple to two-fermion channels with different baryon numbers:

1) quarks and leptons ($B = \dfrac{1}{3}$)

2) quarks and antiquark ($B = \dfrac{2}{3}$)

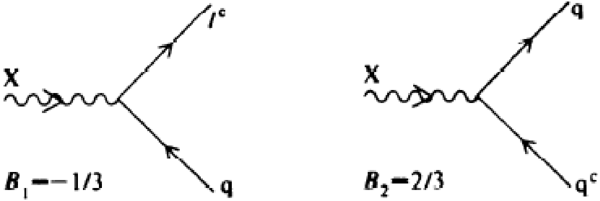



FIG. 14.4. X bosons as leptoquarks and diquarks.

Consequently , mediation of $X - boson$, a $B = -\frac{1}{3}$ channel can be converted into a $B = \frac{2}{3}$ one, a baryon number violation process. Since $M_X$ is very heavy, effective 4-fermion local interaction for baryon violating processes is,

$$\mathcal{L}_{\Delta B=1} = \frac{g^2}{2M_X^2}\varepsilon^{\alpha\beta\gamma}\varepsilon^{ab}(\bar{u}^c_\alpha\gamma^\mu q_{\beta a})(\bar{d}^c_\alpha\gamma_\mu l_b + \bar{e}^+\gamma_\mu q_{\alpha b})$$

This will give following decays,

$$p \longrightarrow e^+\pi^0, \qquad e^+\omega, \qquad \cdots$$

To calculate these decay rates needs to renormalize these effective Lagrangian from $M_X$ down to 1 *Gev* and use some hadronic model to compute the hadronic matrix elements. Note the factor $M_X^2$ in the denominator which makes the decay rates very small because $M_X \sim 10^{15}\,Gev$ or more. In the 80's the proton decay experiments have been actively pursued and none has been found. The best limit is

$$\tau(p \longrightarrow e^+\pi^0) \geq 1.6 \times 10^{33}\ years$$

**Baryon number asymmetry in the universe**

Universe seems to be made out of mostly matter and no anti-matter.
In hot Big Bang Model, the matter and anti-matter are produced in equal amount.
How this matter- antimatter asymmetric universe can happen? One quantitative measure is ratio of baryon number density $n_B$ to the Cosmic Background Radiation (CMB) photon density $n_\gamma$,

$$\eta = \frac{n_B}{n_\gamma} \simeq (6.1 \pm 0.2) \times 10^{-10}$$

Sakharov has came up with 3 conditions needed to generate this asymmetry,

1. Baryon number violation
   If baryon number were conserved, then initial $n_B = 0$ of hot Big Bang model can not change as the universe evolves.
2. *C* and *CP* violations
   For a baryon violating reaction, $X \longrightarrow qq$, there will also be a mirror processes $\bar{X} \longrightarrow \bar{q}\bar{q}$ creating exactly negative amount of $n_B$, no net baryon number if these two processes can occur with same rate. Thus $C$ and $CP$ violations are needed to ge i.e.,
   $\Gamma(X \longrightarrow qq) \neq \Gamma\left(\bar{X} \longrightarrow \bar{q}\bar{q}\right)$ .

3. Out of thermal equilibrium
   $CPT$ invariance requires particle and anti-particle to have the same mass, hence to be equally weighted in the Boltzmann distribution if in thermal equilibrium.

GUTs, such as $SU(5)$, togather with expansion of the universe can satisfy all these conditions. Unfortunately the $CP$ violation in the Standard Model is not large enough to account for observed aymmetry $\eta \sim 10^{10}$.