Axial Anomaly

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In $\lambda\phi^4$ theory with $U(1)$ symmetry the Noether current is given by

$$J_\mu = i\left[\left(\partial_\mu \phi^\dagger\right)\phi - \left(\partial_\mu \phi\right)\phi^\dagger\right]$$

and it is conserved

$$\partial^\mu J_\mu = 0$$

Using canonical commutation relation,

$$\left[\partial_0 \phi^\dagger\left(\vec{x}, t\right), \phi(\vec{x}', t)\right] = -i\delta^3\left(\vec{x} - \vec{x}'\right)$$

we get

$$\left[J_0\left(\vec{x}, t\right),\ \phi\left(\vec{x}', t\right)\right] = \delta^3\left(\vec{x} - \vec{x}'\right)\phi\left(\vec{x}', t\right) \qquad (1)$$

$$\left[J_0\left(\vec{x}, t\right),\ \phi^\dagger\left(\vec{x}', t\right)\right] = -\delta^3\left(\vec{x} - \vec{x}'\right)\phi^\dagger\left(\vec{x}', t\right) \qquad (2)$$

Consider 3-point function,

$$G_\mu(p, q) = \int d^4x d^4y\ e^{-iq\cdot x - ip\cdot y}\left\langle 0\left| T\left(J_\mu(x)\phi(y)\phi^\dagger(0)\right)\right| 0\right\rangle$$

Contract this with momentum q^μ,

$$\begin{aligned} q^\mu G_\mu(p, q) &= -i\int d^4x d^4y\ e^{-iq\cdot x - ip\cdot y}\ \partial_x^\mu \left\langle 0\left| T\left(J_\mu(x)\phi(y)\phi^\dagger(0)\right)\right| 0\right\rangle \\ &= -i\int d^4x d^4y\ e^{-iq\cdot x - ip\cdot y}\ \begin{array}{l}\{\langle 0 | T\left(\delta(x_0 - y_0)\left[J_0(x), \phi(y)\right]\phi^\dagger(0)\right) | 0\rangle \\ +\langle 0 | T\left(\delta(x_0)\left[J_0(x), \phi^\dagger(0)\right]\phi(y)\right) | 0\rangle\}\end{array} \end{aligned}$$

One important consequence: no renormalization constant is needed for the composite operator J_μ. Recall that the renormalized 3-point function and propagator are

$$G_\mu^R(p,q) = Z_\phi^{-1} Z_J^{-1} G_\mu(p,q), \qquad \Delta^R(p) = Z_\phi^{-1}\Delta(p)$$

where Z_ϕ and Z_J are the renormalization constants for operators ϕ and J_μ respectively. Then Ward identity implies

$$Z_J^{-1} q^\mu G_\mu^R(p,q) = -i\Delta^R(p+q) + i\Delta^R(p)$$

Since RHS is cutoff indep, LHS must be also cutoff indep and we do not need any counter term for J_μ i.e. $Z_j = 1$. Such a non-renormalization result holds for all kind of conserved quantities.

Ward identity at 1-loop

It is instructive to see how Ward identity works in terms of diagrams. Amputated Green's function $\Gamma_\mu(p,q)$ and 1PI self energy $\widetilde{\Sigma}(p)$ are

$$\Gamma_\mu(p,q) = \left[i\Delta^R(p+q)\right]^{-1} G_\mu(p,q)\left[i\Delta(p)\right]^{-1}$$

$$[\Delta(p)]^{-1} = p^2 - \mu^2 - \widetilde{\Sigma}(p)$$

Ward identity takes the form,

$$q^\mu \Gamma_\mu(p,q) = (p+q)^2 - p^2 - \widetilde{\Sigma}(p+q) + \widetilde{\Sigma}(p)$$

In zeroth order we have tree graph contribution and

$$iq^{\mu}\Gamma_{\mu}^{(a)}(p,q) = iq^{\mu}(-i)(2p+q)_{\mu} = [(p+q)^2-\mu^2]-(p^2-\mu^2)$$

This verifies Ward identity to lowest order. This is just an algebric relation between vertex and self energy.
Using dimensional regularization for 1-loop diagram, we get

$$\begin{aligned} iq^{\mu}\Gamma_{\mu}^{(b)}(p,q) &= iq^{\mu}\int\frac{d^nk}{(2\pi)^4}i\lambda\frac{i}{k^2-\mu^2}(-i)(2k+q)_{\mu}\frac{i}{(k+q)^2-\mu^2} \\ &= i\lambda\int\frac{d^nk}{(2\pi)^4}\left[\frac{1}{(k+q)^2-\mu^2}-\frac{1}{k^2-\mu^2}\right] \end{aligned}$$

For $n<2$, first integral is convergent and shift the integration variable $k\to k-q$, to get

$$iq^{\mu}\Gamma_{\mu}^{(b)}(p,q) = i\lambda\int\frac{d^nk}{(2\pi)^4}\left[\frac{1}{k^2-\mu^2}-\frac{1}{k^2-\mu^2}\right]=0$$

This will still be true when we analytically continue to $n>2$. The contribution of self energy graphs are

$$iq^{\mu}\Gamma_{\mu}^{(c)}(p,q) = iq^{\mu}(-i)(2p+q)_{\mu}\frac{i}{(p+q)^2-\mu^2}[\Sigma(p+q)-\Sigma(0)]$$

where

$$-i\Sigma(p+q) = -\frac{i\lambda}{2}\int \frac{d^n k}{(2\pi)^4}\frac{i}{k^2-\mu^2}$$

is independent of external momentum. Thus

$$\widetilde{\Sigma}(p) = \Sigma(p+q) - \Sigma(0) = 0$$

and

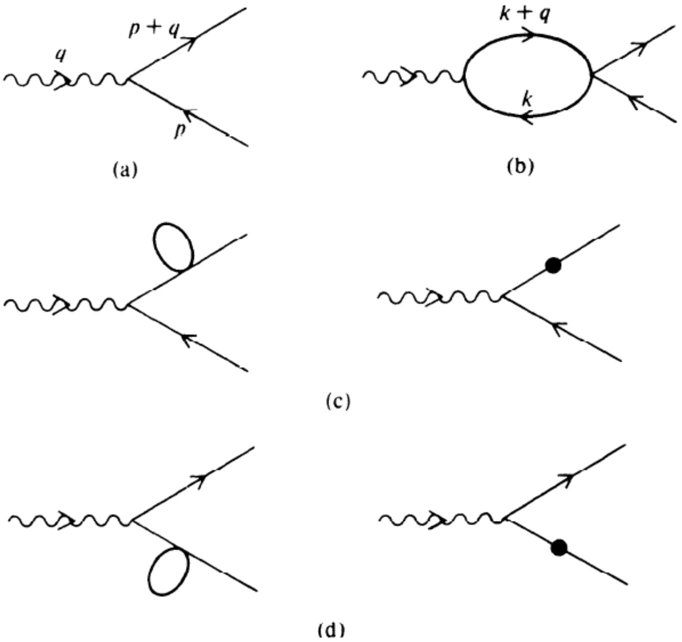
$$iq^{\mu}\Gamma_{\mu}^{(c)}(p,q) = 0$$

Similarly

$$iq^{\mu}\Gamma_{\mu}^{(d)}(p,q) = 0$$

Thus up to 1-loop order the sum of all these contribution gives,

$$iq^{\mu}\Gamma_{\mu}^{(a)}(p,q) = (p+q)^2 - p^2$$


q
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Ward identity for axial vector current

Consider 3-point function in QED

$$T_{\mu\nu\lambda}\left(k_1, k_2, q\right) = i\int d^4x_1 d^4x_2 \left\langle 0\left|T\left(V_\mu\left(x_1\right)V_\nu\left(x_2\right)A_\lambda\left(0\right)\right)\right|0\right\rangle e^{ik_1x_1+ik_2x_2}$$

$$T_{\mu\nu}\left(k_1, k_2, q\right) = i\int d^4x_1 d^4x_2 \left\langle 0\left|T\left(V_\mu\left(x_1\right)V_\nu\left(x_2\right)P\left(0\right)\right)\right|0\right\rangle e^{ik_1x_1+ik_2x_2}$$

where

$$V_\mu\left(x\right) = \bar{\psi}\left(x\right)\gamma_\mu\psi\left(x\right), \quad A_\lambda\left(x\right) = \bar{\psi}\left(x\right)\gamma_\mu\gamma_5\psi\left(x\right), \quad P\left(x\right) = \bar{\psi}\left(x\right)\gamma_5\psi\left(x\right), \quad q = k_1 + k_2$$

From equations of motion, divergence of V_μ, A_λ are

$$\partial^\mu V_\mu\left(x\right) = 0, \qquad \partial^\lambda A_\lambda\left(x\right) = 2imP\left(x\right)$$

From

$$\partial^\mu T\left(J_\mu\left(x\right)O\left(y\right)\right) = T\left(\partial^\mu J_\mu\left(x\right)O\left(y\right)\right) + \delta\left(x_0 - y_0\right)\left[J_0\left(x\right), O\left(y\right)\right]$$

and

$$\delta\left(x_0 - y_0\right)\left[V_0\left(x\right), A_0\left(y\right)\right] = 0$$

where $O\left(y\right)$ arbitrary local operator. We get the Ward identities

$$k_1^\mu T_{\mu\nu\lambda} = k_2^\nu T_{\mu\nu\lambda} = 0 \qquad (4)$$

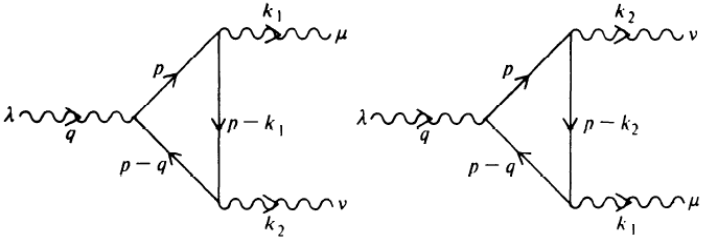
$$q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} \tag{5}$$

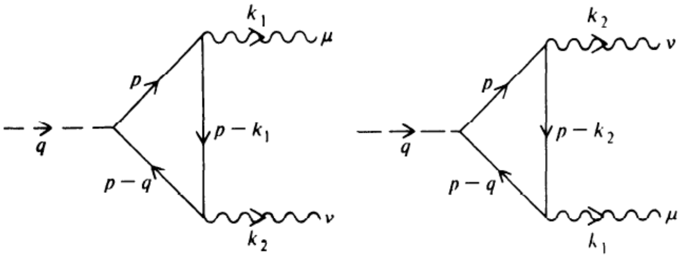
But in lowest order contribution to $T_{\mu\nu\lambda}$ and $T_{\mu\nu}$, Ward identities are not satisfied,

$$T_{\mu\nu\lambda} = i\int \frac{d^4p}{(2\pi)^4}(-1)\left\{Tr\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{\not p - \not q - m}\gamma_\nu\frac{i}{\not p - \not k_1 - m}\gamma_\mu\right] + \left(\begin{array}{c} k_1 \leftrightarrow k_2 \\ \mu \longleftrightarrow \nu \end{array}\right)\right\} \tag{6}$$

and

$$T_{\mu\nu} = i\int \frac{d^4p}{(2\pi)^4}(-1)\left\{Tr\left[\frac{i}{\not p - m}\gamma_5\frac{i}{\not p - \not q - m}\gamma_\nu\frac{i}{\not p - \not k_1 - m}\gamma_\mu\right] + \left(\begin{array}{c} k_1 \leftrightarrow k_2 \\ \mu \longleftrightarrow \nu \end{array}\right)\right\}$$





Use the relation

$$\not{q}\gamma_5 = \gamma_5 \left(\not{p} - \not{q} - m\right) + \left(\not{p} - m\right)\gamma_5 + 2m\gamma_5$$

we get

$$q^{\lambda} T_{\mu\nu\lambda} = 2m T_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \tag{7}$$

with

$$\Delta^{(1)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} Tr\left\{ \frac{1}{\not{p} - m}\gamma_5\gamma_\nu \frac{i}{\not{p} - \not{k}_1 - m}\gamma_\mu - \frac{i}{\not{p} - \not{k}_2 - m}\gamma_5\gamma_\nu \frac{i}{\not{p} - \not{q} - m}\gamma_\mu \right\}$$

$$\Delta^{(2)}_{\mu\nu} = \int \frac{d^4 p}{(2\pi)^4} Tr \left\{ \frac{1}{\not p - m} \gamma_5 \gamma_\mu \frac{i}{\not p - \not k_2 - m} \gamma_\nu - \frac{i}{\not p - \not k_1 - m} \gamma_5 \gamma_\mu \frac{i}{\not p - \not q - m} \gamma_\nu \right\}$$

If $\Delta^{(i)}_{\mu\nu} = 0$, we get the Ward identity. Superficially this appears to be the case. Two integrals in $\Delta^{(1)}_{\mu\nu}$ cancel each other if we can shift the integration variable p to $p + k_2$ in the second term. But integrals are linearly divergent and a translation of integration variable will produce extra terms with $\Delta^{(1)}_{\mu\nu} \neq 0$, $\Delta^{(2)}_{\mu\nu} \neq 0$.

Linearly divergent integral

Consider the integral,

$$\Delta(a) = \int_{-\infty}^{\infty} dx \left[f(x+a) - f(x) \right]$$

If each integral is convergent, a shift $x \to x - a$ in the first integral will give $\Delta(a) = 0$. However if integrals are divergent we need to be more careful. Expand by Taylor expansion to get

$$\begin{aligned} \Delta(a) &= \int_{-\infty}^{\infty} dx \left[a f'(x) + \frac{a^2}{2} f''(x) + \cdots \right] \\ &= a \left[f(\infty) - f(-\infty) \right] + \frac{a^2}{2} \left[f'(\infty) - f'(-\infty) \right] + \cdots \end{aligned}$$

If $\int_{-\infty}^{\infty} dx f(x)$ is convergent then $f(\pm\infty), f'(\pm\infty), \cdots$ all vanish and $\Delta(a) = 0$. But for a linearly divergent integral $f(\pm\infty) \neq 0$, and $f'(\pm\infty) = 0$, $\cdots$ and

$$\Delta(a) = a \left[f(\infty) - f(-\infty) \right]$$

This is a "surface" term in one dimension. Note that eventhough $\int_{-\infty}^{\infty} dx f(x)$ is divergent but $\Delta(a)$ is finite because divergences cancel out between the first and second term in $\Delta(a)$.
The generalization to $n-$dimension is straightforward,

$$\begin{aligned} \Delta(a) &= \int_{-\infty}^{\infty} d^n r \left[f(r+a) - f(r) \right] \\ &= \int_{-\infty}^{\infty} d^n r \left[a^\lambda \frac{\partial}{\partial r^\lambda} f(r) + a^\lambda \frac{\partial}{\partial r^\lambda} a^\sigma \frac{\partial}{\partial r^\sigma} f(r) f(r) + \cdots \right] \end{aligned}$$

After applying Gauss's theorem, all but the first term vanish upon integration to $r = R \longrightarrow \infty$

$$\Delta(a) = a^{\lambda} \frac{R_{\lambda}}{R} f(R) S_n(R)$$

where $S_n(R)$ is the surface area of the hypersphere of radius R. For the case of 4-dim Minkowski space, we have

$$\Delta(a) = a^{\lambda} \int d^4 x \partial_{\lambda} f(x) = 2i\pi^2 a^{\lambda} \lim_{R \to \infty} R^2 R_{\lambda} f(R) \qquad (8)$$

Ambiquities in $T_{\mu\nu\lambda}$

The 1-loop $T_{\mu\nu\lambda}$ given in Eq(6) is linearly divergent and not uniquely defined. Suppose we make a shift of integration variable so that the propagator $\not p - m$ is replacd by $\not p + \not a - m$ with

$$a = \alpha k_1 + (\alpha - \beta)\, k_2$$

Then

$$\begin{aligned}
\Delta_{\mu\nu\lambda}(a) &= T_{\mu\nu\lambda} = i\int \frac{d^4p}{(2\pi)^4}(-1) \begin{array}{c} \{Tr\left[\frac{i}{\not p+\not a-m}\gamma_\lambda\gamma_5\frac{i}{\not p-\not q-m}\gamma_\nu\frac{i}{\not p-\not k_1-m}\gamma_\mu\right] \\ -\left[\frac{i}{\not p-m}\gamma_\lambda\gamma_5\frac{i}{\not p-\not q-m}\gamma_\nu\frac{i}{\not p-\not k_1-m}\gamma_\mu\right]+\left(\begin{array}{c} k_1\leftrightarrow k_2 \\ \mu\longleftrightarrow\nu\end{array}\right)\}\end{array} \\
&= \Delta^{(1)}_{\mu\nu\lambda}(a) + \Delta^{(2)}_{\mu\nu\lambda}(a)
\end{aligned}$$

Apply the result in Eq(8) , we have

$$\begin{aligned}
\Delta^{(1)}_{\mu\nu\lambda}(a) &= -\int \frac{d^4p}{(2\pi)^4} a^\lambda \frac{\partial}{\partial p^\lambda} Tr\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{\not p - \not q - m}\gamma_\nu\frac{i}{\not p - \not k_1 - m}\gamma_\mu\right] \\
&= \frac{-i2\pi^2 a^\lambda}{(2\pi)^4}\lim_{p\to\infty} p^2 p_\lambda Tr\left(\gamma_\alpha\gamma_\lambda\gamma_5\gamma_\beta\gamma_\nu\gamma_\delta\gamma_\mu\right) p^\alpha p^\beta p^\delta / p^6 \\
&= \frac{-i2\pi^2 a^\lambda}{(2\pi)^4}\lim \frac{p_\lambda p^\rho}{p^2} 4i\varepsilon_{\mu\nu\lambda\rho}
\end{aligned}$$

After the replacement $\dfrac{p^\lambda p^\rho}{p^2}$ by $\dfrac{g^{\lambda\rho}}{4}$, we have

$$\Delta^{(1)}_{\mu\nu\lambda}(a) = \frac{a^\lambda \varepsilon_{\rho\mu\nu\lambda}}{8\pi^2}$$

Since $\Delta^{(2)}_{\mu\nu\lambda}(a)$ is related to $\Delta^{(1)}_{\mu\nu\lambda}(a)$ by $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$, we have

$$\Delta_{\mu\nu\lambda}(a) = \Delta^{(2)}_{\mu\nu\lambda}(a) + \Delta^{(2)}_{\mu\nu\lambda}(a) = \frac{\beta}{8\pi^2}\varepsilon_{\rho\mu\nu\lambda}(k_1 - k_2)^\rho$$

Thus the amplitude $T_{\mu\nu\lambda}$ has an ambiguity in arbitrary parameter β

$$T_{\mu\nu\lambda}(\beta) = T_{\mu\nu\lambda}(0) - T_{\mu\nu\lambda} - \frac{\beta}{8\pi^2}\varepsilon_{\rho\mu\nu\lambda}(k_1 - k_2)^\rho \tag{9}$$

Try to determine this arbitrariness in β by imposing vector and axial vector Ward identities Eqs (4,5). In Eq (7), two surface terms can be evaluated by using the relation in Eq (8)

$$\begin{aligned}
\Delta^{(1)}_{\mu\nu} &= -\int \frac{d^4p}{(2\pi)^4} k_2^\lambda \frac{\partial}{\partial p^\lambda} Tr\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\gamma_\nu \frac{i}{\not p - \not k_1 - m}\gamma_\mu\right] \\
&= -\frac{k_2^\lambda}{(2\pi)^4} 2i\pi^2 \lim_{p\to\infty}\frac{p_\lambda}{p^2} Tr\left(\gamma_\alpha\gamma_5\gamma_\nu\gamma_\beta\gamma_\mu\right) p^\alpha k_1^\beta = -\frac{1}{8\pi^2}\varepsilon_{\mu\nu\sigma\rho}k_1^\sigma k_2^\rho
\end{aligned}$$

and

$$\Delta_{\mu\nu}^{(1)} = \Delta_{\mu\nu}^{(2)}$$

The axial Ward identity is then

$$q^{\lambda} T_{\mu\nu\lambda}(\beta) = 2m T_{\mu\nu}(0) - \frac{1-\beta}{4\pi^2} \varepsilon_{\mu\nu\sigma\rho} k_1^{\sigma} k_2^{\rho}$$

For the vector Ward identity,

$$k_1^{\mu} T_{\mu\nu\lambda}(0) = \int \frac{d^4p}{(2\pi)^4}(-1) \begin{array}{l} \{Tr\left[\frac{1}{\not p-m}\gamma_{\lambda}\gamma_5\frac{1}{\not p-\not q-m}\gamma_{\nu}\frac{1}{\not p-\not k_1-m}\not k_1\right]+ \\ +Tr\left[\frac{1}{\not p-m}\gamma_{\lambda}\gamma_5\frac{1}{\not p-\not q-m}\not k_1\frac{1}{\not p-\not k_2-m}\gamma_{\nu}\right]\} \end{array}$$

Using

$$\not k_1 = (\not p - m) - [\not p - \not k_1 - m] = [\not p - \not k_2 - m] - [\not p - \not q - m]$$

we get

$$k_1^{\mu} T_{\mu\nu\lambda}(0) = \int \frac{d^4p}{(2\pi)^4}(-1)\, Tr\left[\gamma_{\lambda}\gamma_5\frac{1}{\not p-\not q-m}\gamma_{\nu}\frac{1}{\not p-\not k_1-m}\not k_1 - \gamma_{\lambda}\gamma_5\frac{1}{\not p-\not k_2-m}\gamma_{\nu}\frac{1}{\not p-m}\right]$$

Again RHS is a surface term and can be evaluated by using Eq (8) ,

$$\begin{aligned} k_1^{\mu} T_{\mu\nu\lambda}(0) &= \frac{k_1^{\sigma}}{(2\pi)^4} \int d^4p \frac{\partial}{\partial p^{\sigma}} Tr\left[\gamma_\lambda \gamma_5 \frac{1}{\not p - \not q - m} \gamma_\nu \frac{1}{\not p - \not k_1 - m} \not k_1\right] \\ &= \frac{k_1^{\sigma}}{(2\pi)^4} 2i\pi^2 \lim_{p\to\infty} Tr\left(\gamma_\alpha \gamma_5 \gamma_\nu \gamma_\beta \gamma_\mu\right) k_2^{\alpha} p^{\beta} = \frac{-1}{8\pi^2} \varepsilon_{\lambda\sigma\nu\rho} k_1^{\rho} k_2^{\sigma} \end{aligned}$$

Then with Eq(9) we get

$$k_1^{\mu} T_{\mu\nu\lambda}(\beta) = \frac{(1+\beta)}{8\pi^2} \varepsilon_{\lambda\sigma\nu\rho} k_1^{\rho} k_2^{\sigma}$$

For arbitrary β we can not satisfy both vector and axial Ward identities. If we choose to satisfy the vector Ward identity, i. e. $\beta = -1$, then there will be an extra term in axial Ward identity

$$q^{\lambda} T_{\mu\nu\lambda}(\beta) = 2m T_{\mu\nu}(0) - \frac{1}{2\pi^2} \varepsilon_{\mu\nu\sigma\rho} k_1^{\sigma} k_2^{\rho}$$

and the axial current is not conserved any more

$$\partial^{\lambda} A_{\lambda}(x) = 2imP(x) + \frac{1}{4\pi^2} \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

Remarks

1. The anomaly is indep of fermion masses and should be present in the massless theory.

2. Adler and Bardeen have shown that the coefficient in the anormaly term is not affected by the higher order radiative corrections
3. It seems that we have choice to put the anomalous term either in vector or axial vector Ward identy. But it is not hard to see that the Ward identiy for 3-point function with all axial current $\langle T\left(AAA\right)\rangle$ also has anomaly and there is no choice but to put the anomaly in the axial current.

ABJ anomaly for non-Abelian symmetries

The 3-point function for non-Abelian currents of interest is of the form,

$$T^{abc}_{\mu\nu\lambda}\left(k_1,k_2,q\right)=i\int d^4x_1d^4x_2\left\langle 0\left|T\left(V^a_\mu\left(x_1\right)V^b_\nu\left(x_2\right)A^c_\lambda\left(0\right)\right)\right|0\right\rangle e^{ik_1x_1+ik_2x_2}$$

where

$$V^a_\mu\left(x\right)=\bar{\psi}\left(x\right)\gamma_\mu T^a\psi\left(x\right),\qquad A^b_\lambda\left(x\right)=\bar{\psi}\left(x\right)\gamma_\mu\gamma_5T^b\psi\left(x\right)$$

where T^a is the internal symmetry matrix. It is not hard to see that the anomaly in the axial Ward identity is

$$q^\lambda T^{abc}_{\mu\nu\lambda}=\text{(commutator terms) - }\frac{1}{2\pi^2}\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta D^{abc}$$

where

$$D^{abc}=\frac{1}{2}tr\left(\left\{T^a,T^b\right\}T^c\right)$$

$\pi^0 \to \gamma\gamma$

Important application of axial anomaly is $\pi^0 \to \gamma\gamma$ decay. This amplitude is defined as

$$\left\langle \gamma\left(k_1,\varepsilon_1\right)\gamma\left(k_2,\varepsilon_2\right)\left|\pi^0\left(q\right)\right.\right\rangle = i\left(2\pi\right)^4\delta^4\left(q-k_1-k_2\right)\varepsilon_1^{\mu}\left(k_1\right)\varepsilon_2^{\nu}\left(k_2\right)\Gamma_{\mu\nu}\left(k_1,k_2,q\right)$$

with

$$\Gamma_{\mu\nu}\left(k_1,k_2,q\right) = e^2\int d^4yd^4ze^{ik_1z+ik_2y}\left\langle 0\left|T\left(J_{\mu}^{em}\left(z\right)J_{\nu}^{em}\left(y\right)\right)\right|\pi^0\left(q\right)\right\rangle$$

which has structure,

$$\Gamma_{\mu\nu}\left(k_1,k_2,q\right) = i\varepsilon_{\mu\nu\alpha\beta}k_1^{\alpha}k_2^{\beta}\Gamma\left(q^2\right) \qquad (10)$$

Consider

$$\Gamma_{\mu\nu\lambda}\left(k_1,k_2,q\right) = \int d^4xd^4ye^{ik_1z+ik_2y}\left\langle 0\left|T\left(A_{\lambda}^3\left(x\right)J_{\mu}^{em}\left(0\right)J_{\nu}^{em}\left(y\right)\right)\right|0\right\rangle$$

which satisfies the Ward identity,

$$q^{\lambda}\Gamma_{\mu\nu\lambda}\left(k_1,k_2,q\right) \quad = \quad -i\int d^4xd^4ye^{i}$$

$$+T\delta\left(x_0-y_0\right)\left[A_0^3\left(x\right),J_{\nu}^{em}\left(y\right)\right]J_{\mu}^{em}\left(0\right)+T\delta\left(x_0\right)\left[A_0^3\left(x\right),J_{\mu}^{em}\left(0\right)\right]J_{\nu}^{em}\left(y\right)\}|0 \quad >$$

It is easy to see that the commutators here all vanish,

$$q^{\lambda}\Gamma_{\mu\nu\lambda}\left(k_1,k_2,q\right) = -i\int d^4xd^4ye^{iqx+ik_2y} < 0|T\{\left(\partial^{\lambda}A_{\lambda}^3\left(x\right)J_{\mu}^{em}\left(0\right)J_{\nu}^{em}\left(y\right)\right)$$

Naively we would identify the right side as $\pi^0 \rightarrow \gamma\gamma$ amplitude,

$$\Gamma_{\mu\nu}(k_1, k_2, q) = \frac{-ie^2(-q^2 + m_\pi^2)}{f_\pi m_\pi^2} \int d^4x d^4y e^{iqx+ik_2y} < 0|T\{\left(\partial^\lambda A_\lambda^3(x) J_\mu^{em}(0) J_\nu^{em}(y)\right)$$

so that

$$q^\lambda \Gamma_{\mu\nu\lambda}(k_1, k_2, q) = \frac{f_\pi m_\pi^2}{e^2(-q^2 + m_\pi^2)} \Gamma_{\mu\nu}(k_1, k_2, q) = \frac{f_\pi m_\pi^2}{e^2(-q^2 + m_\pi^2)} i\varepsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \Gamma(q^2)$$

Then as we let $q \rightarrow 0$, the result is

$$\Gamma(0) = 0$$

i.e. amplitude for $\pi^0 \rightarrow \gamma\gamma$ vanishes as $q \rightarrow 0$. However, one must include the anomaly in the Ward identity,

$$q^\lambda \Gamma_{\mu\nu\lambda}(k_1, k_2, q) = \frac{f_\pi m_\pi^2}{e^2(-q^2 + m_\pi^2)} \Gamma_{\mu\nu}(k_1, k_2, q) - i\frac{D}{2\pi^2} \varepsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta$$

where D is the coefficient of anomaly. The low energy theorem is

$$\lim_{q\rightarrow 0} \Gamma_{\mu\nu}(k_1, k_2, q) = i\frac{e^2 D}{2\pi^2 f_\pi} \varepsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta$$

or

$$\Gamma(0) = \frac{e^2 D}{2\pi^2 f_\pi}$$

Thus in low energy limit amplitude for $\pi^0 \to \gamma\gamma$ comes entirely from anomaly. To compute D , we write the currents in terms of quark fields u, d, s,

$$J_\mu^{em}(x) = \bar{q}(x)\gamma_\mu Q q(x), \qquad A_\mu^3(x) = \bar{q}(x)\gamma_\mu\gamma_5\frac{\lambda_3}{2}q(x)$$

with

$$Q = \frac{1}{3}\begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

The coefficient D takes the value,

$$D = \frac{1}{2}Tr\left(\{Q, Q\}\frac{\lambda_3}{2}\right) = \frac{1}{6}$$

yielding

$$\Gamma(0) = 0.0123\ m_\pi^{-1}$$

which is about factor 3 smaller than the experimental value $\Gamma\left(m_\pi^2\right) = 0.0375\ m_\pi^{-1}$. This lends support to the idea that quarks carry color degree of freedom and gives an additional factor of 3 coming from summing over colors

$$\Gamma(0) = 0.037\ m_\pi^{-1}.$$

Axial vector current in 2-dim

The Lagrangian in 2-dim QED is

$$\mathcal{L} = \bar{\psi}(x)\, iD^{\mu}\gamma_{\mu}\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad \text{with} \qquad D_{\mu} = \partial_{\mu} + ieA_{\mu}$$

Choose the γ matrices to be

$$\gamma^{0} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^{1} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

The axial vector current is

$$J_{\mu}^{5} = \bar{\psi}(x)\,\gamma_{\mu}\gamma_{5}\psi(x)\,, \qquad \text{with} \quad \gamma_{5} = \gamma^{0}\gamma^{1}$$

The equations of motion

$$\partial\!\!\!/\psi = -ieA\!\!\!/\psi, \qquad \partial_{\mu}\bar{\psi}\gamma^{\mu} = \bar{\psi}ieA\!\!\!/$$

J_{μ}^{5} is a composite operator constructed from the fermion fields. The product of local operators are often singular so define the product by point spliting,

$$J_{\mu}^{5}(x) = \lim_{\varepsilon\to 0}\left\{\bar{\psi}\left(x+\frac{\varepsilon}{2}\right)\gamma_{\mu}\gamma_{5}\exp\left[-ie\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} dz\cdot A(z)\right]\psi\left(x-\frac{\varepsilon}{2}\right)\right\}$$

The phase factor will make the axial current gauge invariant. Now we can compute the divergence

$$\begin{aligned}\partial^{\mu} J_{\mu}^{5}(x) &= \lim_{\varepsilon\to 0}\{\partial_{\mu}\bar{\psi}\left(x+\frac{\varepsilon}{2}\right)\gamma_{\mu}\gamma_{5}\exp\left[-ie\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} dz\cdot A(z)\right]\psi\left(x-\frac{\varepsilon}{2}\right) \\ &\quad +\bar{\psi}\left(x+\frac{\varepsilon}{2}\right)\gamma_{\mu}\gamma_{5}\exp\left[-ie\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} dz\cdot A(z)\right]\partial^{\mu}\psi\left(x-\frac{\varepsilon}{2}\right) \\ &\quad +\bar{\psi}\left(x+\frac{\varepsilon}{2}\right)\gamma_{\mu}\gamma_{5}\left[-ie\varepsilon^{\nu}\partial_{\mu}A_{\nu}(x)\right]\psi\left(x-\frac{\varepsilon}{2}\right)\}\end{aligned}$$

Using equations of motion,

$$\begin{aligned}\partial^{\mu} J_{\mu}^{5}(x) &= \lim_{\varepsilon\to 0}\{\bar{\psi}\left(x+\frac{\varepsilon}{2}\right)\left[ie\not{A}\left(x+\frac{\varepsilon}{2}\right)-ie\not{A}\left(x-\frac{\varepsilon}{2}\right)-ie\varepsilon^{\nu}\partial_{\mu}A_{\nu}(x)\right]\gamma_{5}\psi\left(x-\frac{\varepsilon}{2}\right) \\ &= \lim_{\varepsilon\to 0}\{\bar{\psi}\left(x+\frac{\varepsilon}{2}\right)\left[-ie\gamma^{\mu}\varepsilon^{\nu}\left(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}\right)\right]\gamma_{5}\psi\left(x-\frac{\varepsilon}{2}\right)\end{aligned}$$

This expression looks like it is going to vanish as $\varepsilon\longrightarrow 0$. However we need take into account the singularity in the product of fermion field. Fermion propagator in 2-dim is

$$\left\langle 0\left|T\left(\psi(y)\bar{\psi}(z)\right)\right|0\right\rangle = \int\frac{d^{2}k}{(2\pi)^{2}}e^{-ik(y-z)}\frac{i\not{k}}{k^{2}} = -\not{\partial}\left[\frac{i}{4\pi}\ln(y-z)^{2}\right] = \frac{-i}{2\pi}\frac{\gamma^{\alpha}(y-z)_{\alpha}}{(y-z)^{2}}$$

which is singular as $y \to z$. Then we have

$$\left\langle 0 \left| T\left(\bar{\psi}\left(x+\frac{\varepsilon}{2}\right)\Gamma\psi\left(x-\frac{\varepsilon}{2}\right)\right)\right| 0\right\rangle = \frac{-i}{2\pi} Tr\left[\frac{\gamma^{\alpha}\varepsilon_{\alpha}}{\varepsilon^{2}}\Gamma\right]$$

and

$$\partial^{\mu} J_{\mu}^{5}(x) = \lim_{\varepsilon\to 0}\left\{\frac{-i}{2\pi} Tr\left[\frac{\gamma^{\alpha}\varepsilon_{\alpha}}{\varepsilon^{2}}\gamma^{\mu}\gamma_{5}\right]\left(-ie\varepsilon^{\nu}F_{\mu\nu}\right)\right\}$$

Using $Tr\left(\gamma^{\mu}\gamma^{\alpha}\gamma_{5}\right) = 2\varepsilon^{\alpha\mu}$, we get

$$\partial^{\mu} J_{\mu}^{5}(x) = \frac{e}{2\pi}\lim_{\varepsilon\to 0}\left(2\frac{\varepsilon^{\mu}\varepsilon^{\nu}}{\varepsilon^{2}}\right)\varepsilon^{\mu\alpha}F_{\nu\alpha}$$

or

$$\partial^{\mu} J_{\mu}^{5}(x) = \frac{e}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu}$$

Here we have taken the symmetric limit

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\mu}\varepsilon^{\nu}}{\varepsilon^{2}} = \frac{1}{2}g^{\mu\nu}$$

In the free fermion theory, the integral of axial vector current conservation law gives,

$$\int d^{2}x\partial^{\mu}A_{\mu}(x) = N_{R} - N_{L} = 0$$

Combine this with vector current conservation

$$N_R + N_L = 0$$

we get conservation for N_R, N_L separately.
In 2-dim QED, the anomalous term is also a total derivative,

$$\varepsilon^{\mu\nu} F_{\mu\nu} = 2\partial_\mu \left(\varepsilon^{\mu\nu} A_\mu\right)$$

we still maintain the global conservation law if the quantity $\varepsilon^{\mu\nu} A_\mu$ falls off sufficiently at infinity. Let us analyze this problem by studying the fermions in one dim in a backgroun A^1 field that is constant in space and has a very slow time dependence. We will assume that the system has a finite length L, with periodic boundary condition. The Hamiltonian is

$$H = \int d^2 x \psi^\dagger \left(-i\alpha^1 D_1\right) \psi = \int d^2 x \{-i\psi_+^\dagger \left(\partial_1 - ieA^1\right) \psi_+ + i\psi_-^\dagger \left(\partial_1 - ieA^1\right) \psi_-\}$$

where

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

Note that Wilson line

$$\exp\left[-ie\int_0^L dx A_1\right]$$

forms a gauge invariant loop due to the periodic boundary condition. For constant A^1 field we can diagonalize H . The eigenstates of the covariant derivatives are wavefunctions

$$e^{ik_n x}, \qquad \text{with} \quad k_n = \frac{2\pi n}{L}, \qquad n = -\infty, \cdots \infty$$

Thus the single particle eigenstates of H have energies

$$\psi_+ : \qquad E_n = \left(k_n - eA^1\right)$$

$$\psi_- : \qquad E_n = -\left(k_n - eA^1\right)$$

Each type of fermions has infinite towers of equally spaced levels. The ground state will have all negative energy levels filled and we interpret holes as antiparticles. If the field A^1 changes by

$$\Delta A^1 = \frac{2\pi}{eL}$$

which brings the Wilson loop to its original value, the spectrum of H returns to its original form. In this process, each level of ψ_+ moves down to next position and ψ_- move up to the next position as shown.

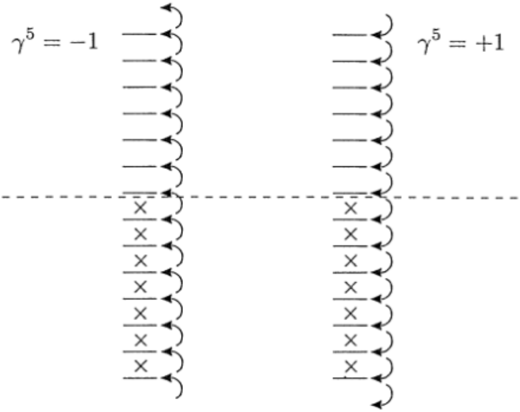


Figure 19.2. Effect on the vacuum state of the Hamiltonian H of one-dimensional QED due to an adiabatic change in the background A^1 field.

Thus one right moving fermion disappears from the vacuum and one extra left moving fermion appears. At the same time

$$\int d^2x \left(\frac{e}{\pi} \varepsilon^{\mu\nu} F_{\mu\nu} \right) = \int dxdt \frac{e}{\pi} \partial_0 A_1 = \frac{e}{\pi} L \left(-\Delta A^1 \right) = 2$$

Thus the integrated form of the anomalous non-conservation law is indeef satisfied,

$$N_R - N_L = \int d^2x \left(\frac{e}{\pi} \varepsilon^{\mu\nu} F_{\mu\nu} \right)$$

Anomaly in 4-dimension

We now want to derive the axial anomaly in the operator form for the 4-dimensional case. The steps leading to the final result remain the same and most of the equations remain valid except for the singularity in the fermion propagator which is of the form,

$$\left\langle 0 \left| T\left(\psi(y)\bar{\psi}(z)\right)\right| 0\right\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(y-z)} \frac{i\not{k}}{k^2} = -\not{\partial}\left[\frac{i}{2\pi^2}\frac{1}{(y-z)^2}\right] = \frac{-i}{\pi^2}\frac{\gamma^\alpha (y-z)_\alpha}{(y-z)^4}$$

This is highly singular as $y - z \to 0$, but it gives zero when traced it with $\gamma^\mu\gamma_5$. We need to consider the higher order term

$$\int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} e^{-i(k+p)y} e^{ikz} \frac{i(\not{k}+\not{p})}{(k+p)^2}\left(-ie\not{A}\right)\frac{i\not{k}}{k^2}$$

This contribution leads to

$$\begin{aligned}\left\langle 0 \left| T\left(\bar{\psi}\left(x+\frac{\varepsilon}{2}\right)\gamma^\mu\gamma_5\psi\left(x-\frac{\varepsilon}{2}\right)\right)\right| 0\right\rangle &= \int \frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4} e^{-ik\varepsilon} e^{ipx}\, Tr\left[\gamma^\mu\gamma_5 \frac{i(\not{k}+\not{p})}{(k+p)^2}\left(-ie\not{A}\right.\right. \\ &= \int \frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4} e^{-ik\varepsilon} e^{ipx} \frac{4e\varepsilon^{\mu\alpha\beta\gamma}(k+p)_\alpha A_\beta(p)\, k_\gamma}{k^2 (k+p)^2}\end{aligned}$$

To evaluate the limit $\varepsilon \longrightarrow 0$, we can expand the integrand for large k. Then

$$\begin{aligned}
\left\langle 0 \left| T\left(\bar{\psi}\left(x+\frac{\varepsilon}{2}\right)\gamma^{\mu}\gamma_{5}\psi\left(x-\frac{\varepsilon}{2}\right)\right)\right| 0\right\rangle &\sim 4e\varepsilon^{\mu\alpha\beta\gamma}\int\frac{d^{4}p}{(2\pi)^{4}}e^{ipx}p_{\alpha}A_{\beta}(p)\,e^{-ik\varepsilon}\int\frac{k_{\gamma}}{k^{4}}\frac{d^{4}k}{(2\pi)^{4}}e^{-ik\varepsilon} \\
&= 4e\varepsilon^{\mu\alpha\beta\gamma}\left(\partial_{\alpha}A_{\beta}(x)\frac{\partial}{\partial\varepsilon^{\gamma}}\left(\frac{i}{16\pi^{2}}\ln\frac{1}{\varepsilon^{2}}\right)\right) \\
&= 2e\varepsilon^{\mu\alpha\beta\gamma}F_{\alpha\beta}\left(\frac{-i}{8\pi^{2}}\frac{\varepsilon^{\gamma}}{\varepsilon^{2}}\right)
\end{aligned}$$

We then get for the divergence of axial current

$$\partial^{\mu}J_{\mu}^{5}(x)=\frac{e}{4\pi^{2}}\lim_{\varepsilon\to 0}\{\varepsilon^{\alpha\beta\mu\gamma}F_{\alpha\beta}\left(\frac{-i\varepsilon^{\gamma}}{\varepsilon^{2}}\right)\left(-ie\varepsilon^{\nu}F_{\mu\nu}\right)$$

Again we will take the symmetric limit to get

$$\partial^{\mu}J_{\mu}^{5}(x)=-\frac{e^{2}}{16\pi^{2}}\varepsilon^{\alpha\beta\mu\gamma}F_{\alpha\beta}F_{\mu\nu}$$

Path integral derivation of axial anomaly

For the fermions, the generating functional can be written as a path integral of the form

$$Z\left[\eta,\bar{\eta}\right]=\int\left[d\psi\right]\left[d\bar{\psi}\right]\exp\left[i\int\left(\mathcal{L}+\bar{\eta}\psi+\bar{\psi}\eta\right)\right] \tag{11}$$

For simplicity, we will take the Lagrangian to have the form $\mathcal{L}=\bar{\psi}i\not{D}\psi$ with $D_\mu=\partial_\mu-igA_\mu$ being the covariant derivative, and A_μ the $U\left(1\right)$ gauge field. One way to define the integration measure of the path integral is to expand ψ and $\bar{\psi}$ in terms of a complete set of orthonormal functions, $\phi_n\left(x\right)$,

$$\psi\left(x\right)=\sum_n a_n\phi_n, \qquad \bar{\psi}\left(x\right)=\sum_n\phi_n^*\left(x\right)\bar{a}_n \tag{12}$$

where

$$\int\phi_n^*\left(x\right)\phi_m\left(x\right)d^4x=\delta_{nm} \tag{13}$$

and take

$$\left[d\psi\right]\left[d\bar{\psi}\right]=\prod_n da_n\prod_m d\bar{a}_m \tag{14}$$

We now perform an axial transformation.

$$\psi\rightarrow\psi'=e^{i\alpha\gamma_5}\psi. \tag{15}$$

To compute the Jacobian for this transformation we expand the transformed field $\psi' = e^{i\alpha\gamma_5}\psi$, in a complete set of basis functions,

$$\psi' = \sum_n b_n \phi_n(x) . \tag{16}$$

The coefficients of expansion can be projected out by using the orthogonality relation,

$$\begin{aligned} b_n &= \int d^4x \phi_n^*(x) \psi'(x) = \int d^4x \phi_n^*(x) e^{i\gamma_5\alpha} \psi(x) \\ &= \int d^4x \phi_n^*(x) e^{i\gamma_5\alpha} \sum_m a_m \phi_m(x) = \sum_m C_{nm} a_m \end{aligned} \tag{17}$$

where

$$C_{nm} = \int d^4x \phi_n^*(x) e^{i\gamma_5\alpha} \phi_m(x) . \tag{18}$$

Similarly,

$$\overline{\psi}' = \sum_n \overline{b}_n \phi_n^*(x) , \qquad \overline{b}_n = \sum_m C_{nm} \overline{a}_m . \tag{19}$$

Thus the Jacobian of the transformation $(a_n, \overline{a}_n) \to \left(b_n, \overline{b}_n\right)$ is

$$J = (\det C)^2 \tag{20}$$

For infinitesimal α, we have

$$C_{nm} \approx \delta_{nm} + i\alpha \int d^4x \phi_n^* (x) \gamma_5 \phi_m (x) \tag{21}$$

or in matrix form

$$C \approx 1 + i\alpha D \qquad \text{with} \qquad D_{nm} = \int d^4x \phi_n^* (x) \gamma_5 \phi_m (x) . \tag{22}$$

Thus we get for the determinant:

$$\det C \approx \det (1 + i\alpha D) \approx 1 + i\alpha \, TrD \approx \exp (i\alpha \, TrD) \tag{23}$$

where

$$TrD = \sum_n \int d^4x \phi_n^* (x) \gamma_5 \phi_n (x) . \tag{24}$$

This is just the identity $\det \left(e^A\right) = e^{TrA}$. Thus we can write the Jacobian as an exponential:

$$J = (\det C)^2 \approx e^{2i\alpha \, TrD} = \exp\{2i\alpha \sum_n \int d^4x \phi_n^* (x) \gamma_5 \phi_n (x)\}. \tag{25}$$

This means that the effect of an axial transformation can be included as an extra term in the Lagrangian,

$$\delta\mathcal{L}_\alpha = 2\alpha \sum_n \phi_n^* (x) \gamma_5 \phi_n (x) . \tag{26}$$

TrD is quite singular: If we take $\phi_n(x)$ to be the plane wave $\phi_n(x) = u(p,s)\, e^{-ipx}$, we get

$$TrD = \int d^4x e^{ipx} u^\dagger(p,s)\gamma_5 u(p,s)\, e^{-ipx} = \delta^4(0)\, u^\dagger(p,s)\gamma_5 u(p,s) \qquad (27)$$

which is not well-define because $\delta^4(0) \to \infty$, while $u^\dagger(p,s)\gamma_5 u(p,s) \to 0$. It has been suggested by Fugikawa that we can regulate $Tr(D)$ by Gaussian cutoff,

$$Tr(D) = \lim_{M\to\infty} \sum_n \int d^4x \left(\phi_n^* \gamma_5 \exp\left(-\frac{\lambda_n^2}{M^2}\right)\phi_n \right) \qquad (28)$$

where λ_n is the eigenvalue of the operator $i\not{D}$,

$$i\not{D}\chi_n = \lambda_n \chi_n. \qquad D_\mu = \partial_\mu - igA_\mu \qquad (29)$$

Calculate $Tr(D)$ in the limit $M \to \infty$.

$$i\not{D}\chi_n = \lambda_n \chi_n, \qquad . \qquad (30)$$

For the special case of $g = 0$, we have $\lambda_n = \not{k}$, and

$$\exp\left(-\frac{\lambda_n^2}{M^2}\right) = \exp\left(-\frac{k^2}{M^2}\right) \qquad (31)$$

and the integral over k is convergent. For the general case we choose $\phi_n(x)$ to be the eigenfunctions of the operator $i\not{D}$ and write TrD as

$$TrD = \sum_n \int d^4x \phi_n^*(x) \gamma_5 \exp\left(\frac{\not{D}^2}{M^2}\right) \phi_n(x). \tag{32}$$

Since the trace is invariant under the change of basis (unitary transformation), we can now use the plane wave state

$$\phi_n(x) = e^{-ikx}, \qquad \left(\sum_n \to \int \frac{d^4k}{(2\pi)^4}\right) \tag{33}$$

to compute the trace. Simple algebra gives the result

$$\begin{aligned} \not{D}\not{D} &= \gamma_\mu\gamma_\nu D^\mu D^\nu = \gamma_\mu\gamma_\nu(\frac{1}{2}[D^\mu, D^\nu] + \frac{1}{2}\{D^\mu, D^\nu\}) \\ &= \frac{1}{4}\{\gamma_\mu, \gamma_\nu\}\{D^\mu, D^\nu\} + \frac{1}{2}\gamma_\mu\gamma_\nu(-igF^{\mu\nu}) \\ &= \frac{1}{2}g_{\mu\nu}\{D^\mu, D^\nu\} - \frac{ig}{4}\left[\gamma_\mu, \gamma_\nu\right]F^{\mu\nu} = D^2 - \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{34}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \qquad \sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right] \tag{35}$$

Also,

$$D^2 = (\partial_\mu - igA_\mu)(\partial^\mu - igA^\mu) = \partial^2 - 2igA^\mu\partial_\mu - ig\partial_\mu A^\mu + g^2A^\mu A_\mu$$

$$D^2 e^{-ikx} = \left[-\left(k_\mu + gA_\mu\right)^2 - ig\partial_\mu A^\mu\right] e^{-ikx} \tag{36}$$

Thus we have

$$\exp\left(-\frac{D^2}{M^2}\right) e^{-ikx} = \exp\left[-\frac{\left(k_\mu + gA_\mu\right)^2}{M^2} - \frac{ig\partial_\mu A^\mu}{M^2}\right] e^{-ikx} \tag{37}$$

Put all these together we get,

$$\begin{aligned} TrD &= \int \frac{d^4k}{(2\pi)^4} \int d^4x Tr\left(\gamma_5 \exp\left(\frac{\not{D}^2}{M^2}\right)\right) = \int d^4x \int \frac{d^4k}{(2\pi)^4} \\ &\quad \times Tr\left(\gamma_5 \exp\left[-\frac{\left(k_\mu + gA_\mu\right)^2}{M^2} - \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}\frac{1}{M^2} - \frac{ig}{M^2}\partial_\mu A^\mu\right]\right) \end{aligned}$$

Change the integration variable, $k_\mu - gA_\mu = k'_\mu M$,

$$TrD = \int d^4x M^4 \int \frac{d^4k'}{(2\pi)^4} e^{-k'^2} Tr\left(\gamma_5 \exp\left[-\frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}\frac{1}{M^2} - \frac{ig}{M^2}\partial_\mu A^\mu\right]\right)$$

It is clear that last term in exponential, not containing any γ-matrices, will not contribute as $Tr\gamma_5 = 0$. We can expand the exponential ,

$$\begin{aligned} \exp\left[-\frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}\frac{1}{M^2}\right] &= \exp\left[-\frac{ig}{2}\gamma_\mu\gamma_\nu F^{\mu\nu}\frac{1}{M^2}\right] \\ &= 1 - \frac{ig}{2}\gamma_\mu\gamma_\nu F^{\mu\nu}\frac{1}{M^2} + \frac{1}{2}\left(\frac{ig}{2}\right)^2 \gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta F^{\mu\nu}F^{\alpha\beta}\frac{1}{M^4} + \ldots \end{aligned} \tag{38}$$

Only the first and the M^{-4} terms will survive as the M^{-2} term will vanish after taking the trace, while the higher order terms vanish in the limit $M \to \infty$. Using the relation,

$$Tr\left(\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\right) = 4i\varepsilon_{\mu\nu\alpha\beta} \tag{39}$$

we get

$$TrD = -\frac{g^2}{8}\int d^4x \int \frac{d^4k}{(2\pi)^4} 4i\varepsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}e^{-k^2}. \tag{40}$$

From

$$\int \frac{d^4k}{(2\pi)^4}e^{-k^2} = \frac{i}{16\pi^2} \tag{41}$$

we get

$$TrD = \frac{g^2}{32\pi^2}\int d^4x\varepsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}. \tag{42}$$

Thus the effective term in the Lagrangian is of the form,

$$\delta\mathcal{L} = 2\alpha\frac{g^2}{32\pi^2}\varepsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}. \tag{43}$$

Since the divergence of the axial vector current is just the coefficient of $\alpha(x)$ in $\delta\mathcal{L}$ under the axial transformation, we see that the Jacobian here will contribute to $\partial_\mu A^\mu$ as

$$\partial_\mu A^\mu = \frac{g^2}{16\pi^2}\varepsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}. \tag{44}$$

Or, if we define

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}, \tag{45}$$

this can be written as

$$\partial_\mu A^\mu = \frac{g^2}{8\pi^2}F^{\mu\nu}\tilde{F}_{\mu\nu}, \tag{46}$$

which is just the axial anomaly equation.

Axial anomaly and $\eta \to \gamma\gamma$

The decay $\eta \to \gamma\gamma$ is very similar to $\pi^0 \to \gamma\gamma$. Suppose that the process also proceeds, just like the case for π^0, through the axial anomaly. Parametrize the matrix elements for the decays, as in CL-eqns (6.61) and (6.63),

$$\mathcal{A}\left[P\left(q\right) \to \gamma\left(k_1,\varepsilon_1\right)\gamma\left(k_2,\varepsilon_2\right)\right] = \varepsilon_1^\mu\left(k_1\right)\varepsilon_2^\nu\left(k_2\right)i\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta\Gamma_P\left(q^2\right) \tag{47}$$

where P stands for either of the pseudoscalar mesons η or π^0.

If we assume η is a pure octet, $\eta = \phi^8$, we can show that

$$\frac{\Gamma_\pi\left(0\right)}{\Gamma_\eta\left(0\right)} = \sqrt{3} \tag{48}$$

from the theory of anomaly. From CL-eqns (6.69) and (6.72), we see that

$$\Gamma_\pi\left(0\right) = \frac{e^2}{4\pi^2 f_\pi}Tr\left(Q^2\lambda_3\right), \qquad \text{and} \qquad \Gamma_\eta\left(0\right) = \frac{e^2}{4\pi^2 f_\pi}Tr\left(Q^2\lambda_8\right)$$

Using

$$Q = \frac{1}{3}\begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

we get

$$\frac{\Gamma_\pi(0)}{\Gamma_\eta(0)} = \frac{Tr\left(\lambda_3 Q^2\right)}{Tr\left(\lambda_8 Q^2\right)} = \sqrt{3} \tag{49}$$

We can also show that the ratio of decay rates is given by

$$\frac{\Gamma\left(\pi^0 \to \gamma\gamma\right)}{\Gamma\left(\eta \to \gamma\gamma\right)} = \left(\frac{m_\pi}{m_\eta}\right)^3 \left[\frac{\Gamma_\pi\left(m_\pi^2\right)}{\Gamma_\eta\left(m_\eta^2\right)}\right]^2 \tag{50}$$

Assume that

$$\frac{\Gamma_\pi\left(m_\pi^2\right)}{\Gamma_\eta\left(m_\eta^2\right)} \approx \frac{\Gamma_\pi(0)}{\Gamma_\eta(0)} \tag{51}$$

compute the decay ratio, and compare it with the experimental results.Since the amplitude is proportional to f_π^{-1}, and the decay rate $\propto f_\pi^{-2}$. This means that we need m_P^3 in the decay rates to get the right dimension,

$$\Gamma\left(P \to \gamma\gamma\right) \propto m_P^3 \Gamma_P\left(m_P^2\right) \tag{52}$$

Then we have for the ratio

$$\frac{\Gamma\left(\pi^0 \to \gamma\gamma\right)}{\Gamma\left(\eta \to \gamma\gamma\right)} = \left(\frac{m_\pi}{m_\eta}\right)^3 \left[\frac{\Gamma_\pi\left(m_\pi^2\right)}{\Gamma_\eta\left(m_\eta^2\right)}\right]^2 \tag{53}$$

If we assume

$$\frac{\Gamma_\pi\left(m_\pi^2\right)}{\Gamma_\eta\left(m_\eta^2\right)} \approx \frac{\Gamma_\pi\left(0\right)}{\Gamma_\eta\left(0\right)} = \sqrt{3}, \tag{54}$$

we get

$$\frac{\Gamma\left(\pi^0 \to \gamma\gamma\right)}{\Gamma\left(\eta \to \gamma\gamma\right)} = \left(\frac{m_\pi}{m_\eta}\right)^3 \times 3 = 0.045 \tag{55}$$

Experimentally, this ratio is about 0.0165. The discrepancy probably is due to the assumption (51). That $m_\pi^2 \approx 0.02\, GeV^2$ is quite close to 0, the approximation $\Gamma_\pi\left(m_\pi^2\right) \approx \Gamma_\pi\left(0\right)$ should be fairly good, while $m_\eta^2 \approx 0.3\, GeV^2$ and $\Gamma_\eta\left(m_\eta^2\right) \approx \Gamma_\eta\left(0\right)$ is probably not a reliable approximation. Another possibility is that the η meson does not transform as a pure member of the $SU\left(3\right)$ octet.

The Ward Identity and Unitarity

In non-Abelian gauge theory, we need to choose a gauge to carry out the quantization. In the renormalizable R_ξ gauge, the gauge boson propagator is of the form,

$$i\Delta^{ab}_{\mu\nu}(k) \rightarrow -i\delta^{ab}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]\frac{1}{k^2+i\varepsilon}$$

and has the asymptotic behavior as $k \rightarrow \infty$,

$$i\Delta_{\mu\nu}(k) \sim -i\left[\frac{g_{\mu\nu}}{k^2}\right]$$

Then the theory is renormalizable by power counting. But in this gauge the ghost fields are needed and the propator is of the form,

$$i\Delta^{ab}(k) = -i\delta^{ab}\frac{1}{k^2+i\varepsilon}$$

These ghost fields will give rise to unphysical states which might cause problem in the physical interpretation of the theory. It turns out that we can use Ward identity to eliminate the ghost field contribution so that the unitarity is preserved.
Consider a simple $SU(2)$ gauge theory with fermion (f) in a doublet representation. The requirement that $S-$matrix must be unitary,

$$SS^\dagger = S^\dagger S = 1 \qquad \text{or} \qquad \sum_c S_{ac}S^*_{bc} = \delta_{ab}$$

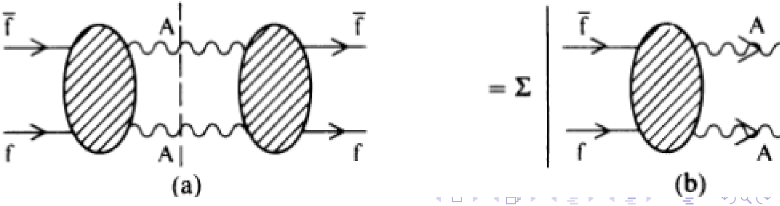
implies that the scattering amplitude T_{ab} which is related to S_{ab} by

$$S_{ab} = \delta_{ab} + i\,(2\pi)^4\,\delta^4\,(p_a - p_b)\;T_{ab}$$

which satisfy the relation

$$\mathrm{Im}\;T_{ab} = \frac{1}{2}\sum_c T_{ac}\,T^*_{bc}\,(2\pi)^4\,\delta^4\,(p_a - p_b)$$

In other words, the requirement that the S-matrix must be unitary implies that the imaginary part of the scattering amplitude T_{ab} os directly related to a sum over products of matrix elements connecting the initial and final states to all physical states with the same energy-momentum as the initial and final states. For our caluculation we shall consider the fermion and anti-fermion scattering amplitude $T\left(\bar{f}f \rightarrow \bar{f}f\right)$ with the intermediate states being the two gauge boson states. This is represented schematically in following graph,



The imaginary part of the scattering amplitude on the left-hand side of Eq() can be calculated by replacing the propagators in the intermediate states by their imaginary parts and multiplying them by the on-shell scattering amplitudes $T\left(\bar{f}f \to AA\right)$ and $T^{*}\left(AA \to \bar{f}f\right)$.
For the vector boson propagator we take the 't Hooft-Feynam gauge withe gauge parameter $\xi = 1$,

$$\Delta_{\mu\nu}^{ab} = \delta^{ab}\left(-g_{\mu\nu}\right)\frac{1}{k^2 + i\varepsilon}$$

It has the imaginary part

$$\pi\delta^{ab}g_{\mu\nu}\delta\left(k^2\right)\theta\left(\omega\right), \qquad \text{with} \qquad \omega = \left|\vec{k}\right| \tag{56}$$

Similarly, the imaginary part of the ghost field propagator is

$$\pi\delta^{ab}\delta\left(k^2\right)\theta\left(\omega\right) \tag{57}$$

The step function in Eqs (56,57) have the effect of constraining the intermediate gauge particle states and ghost states to the same physical region. The unitarity condition for the 4-th order amplitude then reads

$$\int d\rho_2\left[\frac{1}{2}T_{\mu\nu}^{ab}T_{\mu'\nu'}^{ab*}g^{\mu\mu'}g^{\nu\nu'} - S^{ab}S^{ab*}\right] = \frac{1}{2}\int d\rho_2\left[\frac{1}{2}T_{\mu\nu}^{ab}T_{\mu'\nu'}^{ab*}P^{\mu\mu'}\left(k_1\right)P^{\nu\nu'}\left(k^2\right)\right] \tag{58}$$

where $T^{ab}_{\mu\nu}$ and S^{ab} are the $\bar{f}f \to A^a_\mu A^b_\nu$ and $\bar{f}f \to c^{a\dagger}c^b$ amplitudes where A^a_μ and c^a are gauge and ghost fields respectively. The ρ_2 integration is over 2 (massless)-particle phase space. The $P^{\mu\nu}$ are the polarization sum of the gauge particles

$$P^{\mu\mu'}(k_1) = \sum_{\sigma=1,2} \varepsilon_1^{\mu}(k_1,\sigma)\,\varepsilon_1^{\mu'}(k_1,\sigma)$$

$$P^{\nu\nu'}(k_1) = \sum_{\sigma=1,2} \varepsilon_2^{\nu}(k_2,\sigma)\,\varepsilon_2^{\nu'}(k_2,\sigma)$$

where $\varepsilon_1^{\mu}(k_1,\sigma)$ and $\varepsilon_2^{\nu}(k_2,\sigma)$ are polarization 4-vector of the gauge bosons.
We note that in this case LHS of Eq (58) receives a contribution coming from ghost fields while RHS does not because ghosts are not physical states. This is the feature which makes the demonstration of unitarity relation non-trivial. As we shall see, what ultimately allows the unitarity relation to hold is that the polarization sum $P^{\mu\mu'}(k_1)$ is not just $g^{\mu\mu'}$ *and* the effect of the ghost fields is jut to make up the difference.
We shall carry out the lowest non-trivail order calculation as in Eq (58). The imaginary part of the amplitude $\bar{f}f \to \bar{f}f$

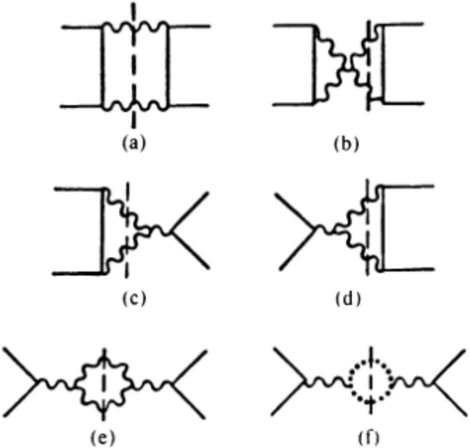
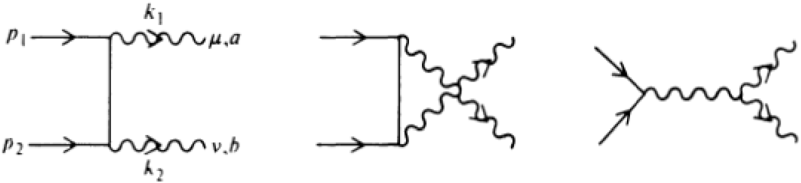
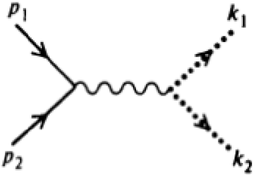


FIG. 9.5. Fourth-order cut-diagrams for $f\bar{f} \to f\bar{f}$ where the intermediate-state particles are ga
particles and FP ghosts.

has been written as squares of the $\bar{f}f \to AA$ amplitude



and of $\bar{f} f \to c^{\dagger} c$ amplitudes



The factor of $\frac{1}{2}$ on the LHS of Eq (58) on the LHS in Eq(58) arises because there are 9 diagrams when one square the $\bar{f}f \rightarrow AA$ amplitude in the Figure, eight of them are just the twice those of Fig $(a)-(d)$ and the 9-th one corresponds to (e) with closed gauge boson loop having a symmetry factor of $\frac{1}{2}$. The FP ghost field c behaves like a fermion with $c \neq c^{\dagger}$; hence there is a minus sign and no symmetry factor of the SS^{*} term.
The lowest-order diagrams for $T^{ab}_{\mu\nu}$ and S^{ab} are shown below

$$\begin{aligned} T^{ab}_{\mu\nu} \quad = \quad & -ig^2\bar{v}(p_2)\frac{\tau^b}{2}\gamma_\nu\frac{1}{(\not{p}_1-\not{k}_1)-m}\frac{\tau^a}{2}\gamma_\mu u(p_1)-ig^2\bar{v}(p_2)\frac{\tau^a}{2}\gamma_\nu\frac{1}{(\not{k}_1-\not{p}_2)-m}\frac{\tau^b}{2}\gamma_\mu u(p_1) \\ & -g^2\varepsilon^{abc}\left[(k_1-k_2)_\lambda g_{\mu\nu}+(k_1+2k_2)_\mu g_{\lambda\nu}+(2k_1+k_2)_\nu g_{\mu\lambda}\right]\frac{1}{(k_1+k_2)^2}\bar{v}(p_2)\frac{\tau^c}{2}\gamma^\lambda u(p_1) \end{aligned}$$

$$S^{ab} = -ig^2\varepsilon^{abc}\frac{1}{(k_1+k_2)^2}\bar{v}(p_2)\frac{\tau^b}{2}\not{k}_1 u(p_1)$$

Gauge-particle polarization

The gauge particle being massless has only 2 physical polarization states, $\varepsilon^{\mu}(k,\sigma)$, $\sigma = 1,2$. Thus the 3 4-vectors, $k_{\mu}, \varepsilon^{\mu}(k,1), \varepsilon^{\mu}(k,2)$ do not completely span the 4-dim space. We can furnish another vector η_{μ} such that

$$\eta \cdot \varepsilon(k,\sigma) = 0, \qquad \sigma = 1,2$$

where $\varepsilon^{\mu}(k,\sigma)$ satisfy

$$\varepsilon(k,1) \cdot \varepsilon(k,2) = 0, \qquad \& \qquad k \cdot \varepsilon(k,\sigma) = 0$$

Since $k^2 = 0$ and η_{μ} can not be portional to k_{μ} we must have $k \cdot \eta = 0$. By the usual procedure of establishing completeness relations, these orthogonality condition and normalization $\varepsilon^2(k,\sigma) = -1$, yields for the polarization sum,

$$P_{\mu\nu} = -g_{\mu\nu} + Q_{\mu\nu}$$

with

$$Q_{\mu\nu} = \left[(k \cdot \eta)\left(k_{\mu}\eta_{\nu} + k_{\nu}\eta_{\mu}\right) - \eta^2 k_{\mu}k_{\nu}\right]\frac{1}{(k \cdot \eta)^2}$$

Clearly, the extra terms $Q_{\mu\nu}$ substracts out the non-transverse polarization states. The task of checking the unitarity relation of Eq (58) involves verifying that the FP ghost term percisely compensates for the extra projection terms in the polarization sum. To simply the calculation, we will adopt the convenient choics $\eta^2 = 0$. Then we get

$$Q^{\mu\mu'}(k_1, \eta_1) = \left[\left(k_1^{\mu}\eta_1^{\mu'} + k_1^{\mu'}\eta_1^{\mu}\right)\right]\frac{1}{(k_1 \cdot \eta_1)}$$

$$Q^{\nu\nu'}(k_2, \eta_2) = \left[\left(k_2^{\nu}\eta_2^{\nu'} + k_2^{\nu'}\eta_2^{\nu}\right)\right]\frac{1}{(k_2 \cdot \eta_2)}$$

Ward identities from lowest-order diagram

We first compute $k_1 T^{ab}_{\mu\nu}$. The first two terms

$$\begin{aligned} & -ig^2 \bar{v}(p_2) \frac{\tau^b}{2}\frac{\tau^a}{2}\gamma_\nu \frac{(\not p_1 - \not k_1) + m}{(p_1 - k_1)^2 - m^2} \not k_1 u(p_1) - ig^2 \bar{v}(p_2) \frac{\tau^a}{2}\frac{\tau^b}{2} \not k_1 \frac{(\not k_1 - \not p_2) + m}{(k_1 - p_2)^2 - m^2} \gamma_\mu u(p_1) \\ = \quad & -ig^2 \bar{v}(p_2) \left[\frac{\tau^b}{2}, \frac{\tau^a}{2}\right] \gamma_\nu u(p_1) = g^2 \varepsilon^{abc} \bar{v}(p_2) \frac{\tau^c}{2} \gamma_\nu u(p_1) \end{aligned}$$

The last term yields

$$\begin{aligned} & -g^2 \varepsilon^{abc} \left[(k_1 - k_2)_\lambda k_{1\nu} + 2k_2 \cdot k_1 g_{\lambda\nu} + (2k_1 + k_2)_\nu k_{1\lambda}\right] \frac{1}{(k_1 + k_2)^2} \bar{v}(p_2) \frac{\tau^c}{2} \gamma^\lambda u(p_1) \\ = \quad & -g^2 \varepsilon^{abc} \bar{v}(p_2) \frac{\tau^c}{2} \gamma^\lambda u(p_1) - g^2 \varepsilon^{abc} \frac{k_{1\nu}}{(k_1 + k_2)^2} \bar{v}(p_2) (\not k_1 + \not k_2) \frac{\tau^c}{2} u(p_1) - g^2 \varepsilon^{abc} \frac{k_{2\nu}}{(k_1 + k_2)^2} \bar{v} \end{aligned}$$

Thus we have

$$k_1^\mu T^{ab}_{\mu\nu} = -iS^{ab} k_{2\nu}$$

Similarly,

$$k_2^\nu T^{ab}_{\mu\nu} = -iS^{ab} k_{1\mu}$$

These are example of non-Abelian theories.

It is then easy to check that the unitary condition in Eq (58) is indeed satisfied as the RHS reads,

$$\frac{1}{2}\int d\rho_2 \left[\frac{1}{2} T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} \left[-g^{\mu\mu'} + \left(k_1^{\mu}\eta_1^{\mu'} + k_1^{\mu'}\eta_1^{\mu}\right)(k_1\cdot\eta_1)^{-1}\right] \left[-g^{\nu\nu'} + \left(k_2^{\nu}\eta_2^{\nu'} + k_2^{\nu'}\eta_2^{\nu}\right)(k_2\cdot\eta\right.\right.$$

$$\begin{aligned}
&= \frac{1}{2}\int d\rho_2 \{TT^* gg + [(k_1 T\eta_2)(\eta_2 T^* k_2) + (\eta_1 T k_2)(k_2 T^* \eta_2)](k_1\eta_1)^{-1}(k_2\eta_2)^{-1} \\
&\quad -[(k_1 T)\cdot(\eta_1 T^*) + (\eta_1 T)\cdot(k_1 T^*)](k_1\eta_1)^{-1} - [(Tk_2)\cdot(T^*\eta_2) + (T\eta_2)\cdot(T^* k_2)](k_2\eta_2)^{-1}\} \\
&= \frac{1}{2}\int d\rho_2 \{TT^* gg + 2SS^* - 2SS^* - 2SS^*\} = \frac{1}{2}\int d\rho_2 \{TT^* gg - 2SS^*\}
\end{aligned}$$

To summarize, the unitary condition (58) relateds the LHS where we have used the covariant guage Feynman rule with their spurious states of longitudinal polarization and FP ghosts, to the RHS where only the physical transvere polarization states appear because of the (axial) gauge conditions. The spurious states of covariant gauge on LHS do cancel among themselves and in the axial gauge on the RHS there are only physical states. In short, the FP ghosts fields are needed in order to maintain the unitarity conditon.

The issue of unitarity is particularly relevent for the spontaneous broken gauge theories. In such theory one encounters further unphysical particles , the would be Goldstone-bosons. One needs to check their decoupling by using Ward identies. Thus we must be sure that Ward identities are satisfied to all orders in perturbation theory. Since there are reflection of theory's symmetries, it is important that we adopt regularization procedure that respect these symmetries. One of the virtures of the dimensional regularization is that it clearly preserve the generalized Ward identities.

Anomaly cancellation in Standard Model

Since the ABJ anomaly spoils the renormalizability of the gauge theory, the fermionic gauge coupling must not introduce anomalous Ward identity. Thus for the fermion representation R with representation matrix $T^a(R)$, the trace $Tr(\{T^a(R), T^b(R)\} T^a(R))$ should be zero to ensure that the Ward identities are anomaly free.

In the Standard model the fermions are either doublets or singlets under $SU(2)$. The matrix T^a will be either the Pauli matrix τ^a or the $U(1)$ hypercharge Y. Since the group $SU(2)$ is anomly free

$$Tr\left(\{\tau^i, \tau^j\}\tau^k\right) = 2\delta^{ij} Tr\left(\tau^k\right) = 0$$

we will consider cases where at least on of the $T's$ is the hypercharges Y. Because every member of a given $SU(2)$ multiplet has the same hypercharge, for the case ot two $T's$ being a Y we have,

$$Tr\left(\tau^i YY\right) \sim Tr\left(\tau^i\right) = 0$$

and for the case of one T being a Y we have

$$Tr\left(\{\tau^i, \tau^j\} Y\right) = 2\delta^{ij} Tr(Y)$$

and

$$Tr(Y) = \sum_i Y_i = \sum_{lepton} Y + \sum_{quark} Y$$

Explict calculation gives

$$\sum_{lepton} Y = -1 \times 2 - 2 = -4$$

$$\sum_{quark} Y = 3\left(\frac{1}{3}\times 2+\frac{4}{3}-\frac{2}{3}\right) = 4$$

For the case when all $T's$ are the hypercharge, we have

$$Tr\,(YYY) = 8\,Tr\left(Q^3 - 3Q^2T_3 + 3QT_3^2 - T_3^2\right) \sim Tr\left(Q^2T_3 - QT_3^2\right)$$

where $Tr\left(T_3^3\right) = 0$. Explicit calculation gives

$$\sum_{lepton}\left(Q^2T_3 - QT_3^2\right) = -\frac{1}{2}+\frac{1}{4} = -\frac{1}{4}$$

$$\sum_{quark}\left(Q^2T_3 - QT_3^2\right) = \left(\frac{2}{3}-\frac{1}{6}-\frac{1}{2}+\frac{1}{4}\right) = \frac{1}{4}$$

Thus the anomaly cancels among the fermions.
A simplier condition for anomaly condition is to note that

$$TrY \sim TrQ$$

and

$$Tr\left(Q^2T_3 - QT_3^2\right) \sim Tr\,(T_3QY) \sim Tr\left(T_3^2Y\right) \sim TrQ$$

Thus the ABJ anomaly is proportional to

$$TrQ = \sum_i Q_i = 0$$

Lepton and quark charges cancel when 3 colors are taken into account.