Interaction Theory

As an illustration we discuss the electromagnetic interaction. From the principle of minimum substitution, the Lagrangian density is of the form,

$$\mathcal{L} = \mathcal{L}(x) \cdot \mathcal{L}(i \gamma_\mu x^\mu - e A_\mu) \gamma_5 \phi(x) - m \bar{\phi}(x) \gamma_5 \phi(x) - \frac{1}{2} F_{\mu \nu} F^{\mu \nu}$$

Equation of motion

\[
\begin{align*}
\{ (i \gamma_\mu \partial_\mu - m) \phi(x) \} &= e A_\mu \gamma_5 \phi \\
\partial_\nu F^{\nu \mu} &= e \phi \gamma_5 \phi
\end{align*}
\]

non-linear coupled equations

Quantization

Write

\[ \mathcal{L} = \mathcal{L}_0 + \text{Int} \]

\[ \mathcal{L}_0 = \mathcal{L}(i \gamma_\mu \partial_\mu - m) \phi(x) - \frac{1}{2} F_{\mu \nu} F^{\mu \nu} \]

\[ \text{Int} = -e \bar{\phi} \gamma_5 \phi A_\mu \]

Conjugate momenta

\[ \frac{\partial}{\partial x^\nu} \psi(x, t) = \frac{\partial}{\partial x^\nu} \phi(x, t) \]

For em fields, we choose

\[ \psi = 0 \]

\[ \pi = \frac{\partial}{\partial x^\nu} \phi(x, t) = F^{\nu \mu} = -i \]

From equation of motion

\[ F^{\nu \mu} \psi(x, t) \phi(x) \]

Thus \( A^\sigma \) is not zero but it is not an independent dynamical variable,

\[ A^\sigma = \int \frac{d^4 x'}{2 A A^\mu} \psi(x', t) \phi(x', t) = \int \frac{d^4 x'}{12 A A^\mu} \]

Commutation relation

\[ \{ \psi(x, t), \psi^*(x', t) \} = \delta_{\mu \nu} \delta(x-x') \]

\[ \{ \psi(x, t), \psi^*(x', t) \} = \cdots = 0 \]

\[ [A_\mu(x, t), A_\nu(x', t)] = i \delta^{\nu \mu} \delta(x-x') \]

Commutators involving \( A_\mu \) can be worked out as follows

\[ [A_\mu(x', t), \psi(x, t)] = \int \frac{d^4 x'}{A A^\mu} \psi(x', t) \psi(x', t) \]

Hamiltonian

\[ H = \frac{\hbar}{2} \int \frac{d^4 x}{A A^\mu} \psi^* \psi + \frac{\hbar}{2} \int \frac{d^4 x}{A A^\mu} \phi^* \phi + \frac{\hbar}{2} \int \frac{d^4 x}{A A^\mu} F^{\nu \mu} F_{\nu \mu} \]

and

\[ H = \int \frac{d^4 x}{A A^\mu} \left( \frac{\hbar}{2} \left( E + S \right)^2 + \frac{\hbar}{2} \left( E + S \right)^2 \right) \]

In this case, the Hamiltonian does not appear in the interaction.

But if we write

\[ E = E + e A \]

where \( E = -1 \bar{V} A_{\text{ph}} \), \( e = -\frac{e}{g} \)

Then

\[ H = \frac{1}{2} \int \frac{d^4 x}{A A^\mu} \left( E + S \right)^2 \]

The longitudinal part is

\[ \frac{1}{2} \int \frac{d^4 x}{A A^\mu} \left( \int \frac{d^4 x}{A A^\mu} \right) \]

Coulomb interaction

Physical States

In high energy physics, we study the interactions by scattering processes. Suppose that the interactions of interest are all short-range in nature. Then far away from the interaction region, particles behave like free particles.

Choose the physical states to be eigenstates of energy-momentum operators.

Then

\[ p^\mu \psi > = p^\mu \psi > \]

They are required to satisfy following requirements.
They are required to satisfy following requirements

1. the eigenvalues $p_i$ all lie within forward light cone,
   \[ p = p^0 \geq 0. \]

2. there exists a non-degenerate Lorentz invariant ground state $|0\rangle$ with lowest energy
   \[ p|0\rangle = 0 \Rightarrow |0\rangle = 0 \]

3. the exists stable single particle states $|p\rangle$ with $p^2 = m^2$ for each stable particle

4. The vacuum and one particle states form discrete spectrum in $p^0$

We associate a field $\phi(x)$ for each discrete state appearing in the spectrum of $p^0$ and assume that interactions do not violently the spectrum of states.

### In-fields and in-states — asymptotic conditions

For simplicity, consider
\[ \phi = \frac{1}{i\hbar} \psi(x) + \frac{1}{\hbar} \phi(x) \]

Equation of motion
\[ (\Box + m^2) \phi = \frac{1}{i\hbar} \phi \]

Conjugate momenta
\[ \pi(x) = \frac{\partial \phi}{\partial x} = \partial_x \phi \]

Commuation relations
\[ [\pi(x), \phi(y)] = -2\delta(x-y) \]
\[ [\pi(x), \pi(x)] = [\phi(x), \phi(x)] = 0 \]

In scattering problem, at $t \rightarrow \infty$, particles propagate freely. Let $\phi_{in}$ be the operator which creates free particle propagating with physical mass.

\[ (\Box + m^2) \phi_{in}(x) = 0 \]

We will assume that $\phi_{in}(x)$ transforms under coordinate displacements and Lorentz transformation in the same way as $\phi(x)$. In particular,
\[ \left[ P^0 , \phi_{in}(x) \right] = -i \partial_x \phi_{in}(x) \]

$\Rightarrow \phi_{in}(x)$ creates one particle state from vacuum.

**Proof:** consider states with definite momentum, $\mathbf{P}^0 \mid n \rangle = p^0 \mid n \rangle$

\[ \left[ p^0 , \phi_{in}(x) \right] = \langle n | [p^0 , \phi_{in}] | 0 \rangle = p_n \langle n | \phi_{in} \rangle \]

\[ \Rightarrow \left( \Box + m^2 \right) \langle n | \phi_{in} \rangle = \langle n | \phi_{in} \rangle \left( m^2 - p^0 \right) \]

\[ \Rightarrow m^2 = p^0 \]

i.e. $\phi_{in}(x)$ is an one-particle state with mass $m$.

We can expand $\phi_{in}(x)$ as
\[ \phi_{in}(x) = \int d^3k \left[ a_n(k) f(k) + a_n^*(k) f(k)^* \right] \]

\[ f(k) = \frac{1}{(2\pi)^3} \frac{1}{2\pi^2} e^{-ikx} \]

Invert this expansion
\[ a_n(k) = \frac{1}{(2\pi)^3} \int f(k) dk^3 \]

Also we have
\[ \left[ p^0 , a_n(k) \right] = -k^0 a_n(k) \]
\[ \left[ p^0 , a_n^*(k) \right] = k^0 a_n^*(k) \]

States are defined by
\[ \langle k_1, k_2, \ldots, k_n | \rangle = \int \frac{d^3k_1 d^3k_2 \ldots d^3k_n}{(2\pi)^3} \]

normalization
\[ \langle k_n | k_1, k_2, \ldots, k_n \rangle = \sqrt{2^n (2\pi)^3} \]

\[ m^0 \ldots \mathbf{P} \mathbf{W} \ldots m^0 = 0 \] unless $m$ and $(\mathbf{P}, \mathbf{W})$ coincides with $(k, 0 \ldots k)$
\begin{equation*}
\langle \beta, \beta' ; P, m_1 | k_1, \ldots, k_n, m \rangle = 0 \quad \text{unless} \quad m = m_1 \quad \text{and} \quad (\beta, P) \quad \text{coincides with} \quad (k_1, k_2, \ldots, k_n) \end{equation*}

\textbf{Relation between } \phi_m(\omega) \text{ and } \psi(\omega) \:
\begin{equation*}
(\Delta + \mu^2) \psi(\omega) = \hat{j}(\omega) \quad \text{or} \quad (\Delta + \mu^2) \phi_m(\omega) = \hat{g}(\omega) \quad \text{with} \quad \Delta \psi = \hat{g}\psi
\end{equation*}

Solve this equation formally:
\begin{equation*}
\sqrt{x} \ \phi_m(\omega) = \Phi(\omega) - \int \frac{d^4y}{\Delta_{\text{for}}(x-y, \mu^2)} \psi(y)
\end{equation*}

where
\begin{equation*}
(\Delta_{x} + \mu^2) \Delta_{\text{for}}(x-y, \mu^2) = \delta^{(4)}(x-y) \quad \text{and} \quad \Delta_{\text{for}}(x-y, \mu^2) = 0 \quad \text{for} \quad \mu < \mu_0.
\end{equation*}

This suggests that as \( \mu_0 \to +\infty \), \( \Phi(\omega) \to \sqrt{x} \Phi_m \). But this leads to contradiction.

\textbf{Correct asymptotic condition (LSZ):} Let \( |\hat{d}>, |\beta\rangle \) be any two normalizable states. \( \Phi(\omega) \) is defined by smearing \( \Phi(\omega) \) over space-like region:
\begin{equation*}
\Phi(\omega) = i \int d^3x \ f(x, t) \frac{\partial}{\partial t} \Phi(\vec{x}, t)
\end{equation*}

Then the correct asymptotic condition is
\begin{equation*}
\lim_{x_k \to -\infty} \langle \beta | \Phi(\omega) | \beta \rangle = \sqrt{x} \langle \beta | \Phi_0 | \beta \rangle
\end{equation*}

where
\begin{equation*}
\Phi_0 = i \int d^3x \ f(x, t) \frac{\partial}{\partial t} \Phi_0(\vec{x}, t)
\end{equation*}

\textbf{Out fields and out states:} Just like the case of in-field and in states, we can also reduce the dynamics to that of free particles for \( t \to +\infty \) by defining:
\begin{equation*}
(\Delta + \mu^2) \mathcal{P}_{\text{out}}(\omega) = 0
\end{equation*}

\begin{equation*}
\mathcal{P}_{\text{out}}(\omega) \mathcal{K} \left[ \mathcal{A}_{\text{out}}(k) \mathcal{F}(\omega) + \mathcal{A}_{\text{out}}(k) \mathcal{F}_0(\omega) \right] = \mathcal{A}_0(\omega) \mathcal{A}_{\text{out}}(k)
\end{equation*}

\textbf{Asymptotic condition:}
\begin{equation*}
\lim_{t \to 0} \langle \beta | \Phi(\omega) | \beta \rangle = \sqrt{x} \langle \beta | \Phi_0 | \beta \rangle
\end{equation*}

\textbf{S-matrix:}
\begin{equation*}
\text{Description of scattering: Start with state with } n \text{ non-interacting particles. They interact when they are close to each other. After interaction, } n \text{ particles separate.}
\end{equation*}

\begin{itemize}
  \item \text{Initial state: } \langle \beta, \beta' ; P, m_1 | \beta, \beta' ; P, m_1 \rangle = 1
  \item \text{Out state: } \langle \beta, \beta' ; P, m_1 | \beta, \beta' ; P, m_1 \rangle = 1
\end{itemize}

\textbf{S-matrix:}
\begin{equation*}
S_{\beta \beta'} = \langle \beta, \beta' | 1 | m_1, m_1 \rangle
\end{equation*}

\textbf{S-operator:}
\begin{equation*}
\langle \beta | \beta' \rangle = \langle \beta | m_1, m_1 | S \rangle
\end{equation*}

\textbf{Properties of S-matrix:}
\begin{enumerate}
  \item \text{From the stability of vacuum: } |S_{00}| = 1
  \item \text{Stability: The one-particle state requires}
\end{enumerate}
3) $\rho_{\text{in}}(n) = S \rho_{\text{out}} S^{-1}$

Consider $\langle \rho_{\text{out}} | \rho_{\text{out}} (\alpha) \rangle = \langle \rho_{\text{in}} | \rho_{\text{out}} (\alpha) \rangle$

$\rho_{\text{out}}$ is an out state $\Rightarrow \rho_{\text{out}} = \rho_{\text{in}} \rho_{\text{out}}$

Then $\langle \rho_{\text{in}} | S \rho_{\text{out}} (\alpha) \rangle = \langle \rho_{\text{in}} | \rho_{\text{in}} S \rho_{\text{out}} (\alpha) \rangle$.

or $\rho_{\text{out}} = \rho_{\text{in}} S \Rightarrow \rho_{\text{in}} S = \rho_{\text{in}} S \rho_{\text{out}} S^{-1}$

4) Unitarity

Since $\langle \rho_{\text{in}} | S = \langle \rho_{\text{in}} | S^\dagger = 1_{\text{out}}$

$\Rightarrow \langle \rho_{\text{in}} | S S^\dagger | \rho_{\text{out}} \rangle = \langle \rho_{\text{out}} | \rho_{\text{out}} \rangle = S \rho_{\text{out}}$

as operator $SS^\dagger = 1_{\text{out}}$ similar argument $\Rightarrow S S^\dagger = 1$

5) $S$ is translational and Lorentz invariance

Under Lorentz transformation $X^\mu \rightarrow X'^\mu = X^\mu + b^\mu$

then $U(\alpha, b)SU^{-1}(\alpha, b) = S$

(Where $U(\alpha, b) \equiv U_{\alpha, b}$)

Proof $\rho_{\text{in}}(\alpha + b) = U(\alpha, b) \rho_{\text{in}}(\alpha) U^{-1}(\alpha, b) = U S \rho_{\text{out}}(\alpha) S^\dagger U^{-1}$

$= U S \rho_{\text{out}}(\alpha + b) S^\dagger U^{-1}$

But $\rho_{\text{in}}(\alpha + b) = S \rho_{\text{out}}(\alpha + b) S^{-1}$

$\Rightarrow U(\alpha, b)SU^{-1}(\alpha, b) = S$
We now work to set up the framework to compute the transition matrix element. Consider:

\[ \langle \text{out} | \text{d.p} | \text{in} \rangle = \langle \text{out} | \text{d.p} | \text{in} \rangle \]

Using the creation operator, we get:

\[ \langle \text{out} | \text{d.p} | \text{in} \rangle = (\epsilon_{k} \text{d.p}) \langle \text{out} | \text{d.p} | \text{in} \rangle \]

Asymptotic conditions:

\[ \langle \phi_{n} | \phi_{m} \rangle = \frac{1}{\sqrt{V_{n}}} \lim_{p \to \infty} \langle \phi_{n} | \phi_{m} \rangle \]

Identity:

\[ \lim_{k \to \infty} \int d\mathbf{x} \left[ \phi_{n}^{\dagger} \phi_{m} \right] = \int d\mathbf{x} \left[ \phi_{n}^{\dagger} \phi_{m} \right] \]

Put these together, we get the reduction formula:

\[ \langle \text{out} | \text{d.p} | \text{in} \rangle = (\epsilon_{k} \text{d.p}) \langle \text{out} | \text{d.p} | \text{in} \rangle + \frac{1}{V_{n}} \int e^{-i\mathbf{k} \cdot \mathbf{r}} d^{3}r \langle \text{in} | \text{d.p} | \text{in} \rangle \]

To remove a particle from \( p \), write \( p = p' \):

\[ \langle \text{out} | \phi_{n} \rangle \text{d.p} | \text{in} \rangle = \langle \text{out} | \phi_{n} \rangle \text{d.p} | \text{in} \rangle \]

Following the same procedure as before, we can get:

\[ \langle \text{out} | \phi_{n} \rangle \text{d.p} | \text{in} \rangle = \int d^{3}r \langle \text{in} | \phi_{n} \rangle \text{d.p} | \text{in} \rangle \]

It is clear how to remove all particles from an "in" and "out" state.

In and Out fields for Fermions:

\[ u_{\alpha}(\mathbf{x}) = \int d^{3}p \sum_{\sigma} \left[ h_{a}(\mathbf{p}) u_{\alpha}^{+}(\mathbf{p}) + h_{a}^{\dagger}(\mathbf{p}) u_{\alpha}^{\sigma}(\mathbf{p}) \right] \]

where

\[ u_{\alpha}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \left[ \frac{1}{(2\pi)^{3/2}} \right] e^{-i\mathbf{k} \cdot \mathbf{x}} \]

\[ V_{\alpha}(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \right] e^{-i\mathbf{k} \cdot \mathbf{x}} \]

Quantization Page 1
where \( \psi(x) = \frac{1}{(2\pi \sigma^2)^{3/2}} \psi(p,s) e^{-ipx} \)

\( \bar{\psi}(x) = \frac{1}{(2\pi \sigma^2)^{3/2}} \bar{\psi}(p,s) e^{ipx} \)

Inversion

\[ \begin{align*}
  \psi^+ (p,s) &= \int \dx^3 \psi^+ (x, \frac{p}{s}, \sigma) \\
  \bar{\psi}^+ (p,s) &= \int \dx^3 \bar{\psi}^+ (x, \frac{p}{s}, \sigma)
\end{align*} \]

Reduction formulas for fermions

(a) remove electron from the in-state

\[ \langle \text{out} | \psi; \text{in} \rangle = \frac{i}{\sqrt{2 \alpha}} \int \dx^3 \langle \text{out} | \psi^+; \text{in} \rangle e^{i\delta x} \]

(b) remove positron (anti-particle) from the in state

\[ \begin{align*}
  \langle \text{out} | \psi^+; \text{in} \rangle &= \int \dx^3 e^{-i\delta x} \psi^+ (x, \frac{p}{s}) (i\gamma^0 - \gamma^1 m) \psi (x, \frac{p}{s})
\end{align*} \]

(c) remove electron from out state

\[ \begin{align*}
  \langle \beta, p'; \text{out} | \psi; \text{in} \rangle &= \frac{i}{\sqrt{2 \alpha}} \int \dx^3 \bar{\psi} (x, \frac{p'}{s}) e^{i\delta x} (i\gamma^0 - \gamma^1 m) \psi (x, \frac{p}{s})
\end{align*} \]

(d) remove positron from out state

\[ \begin{align*}
  \langle \beta, p'; \text{out} | \psi^+; \text{in} \rangle &= \int \dx^3 \langle \beta, p'; \text{out} | \psi^+; \text{in} \rangle e^{-i\delta x} \bar{\psi}^+ (x, \frac{p'}{s}, \sigma) \psi (x, \frac{p}{s}) e^{i\delta x}
\end{align*} \]
Assume that $\Phi(x)$ and $\pi(x)$ are related to $\Phi_0$ and $\pi_0$ by unitary $U$ matrixs

$$
q(x) = U(x) \Phi_0(x) U^\dagger(x)
$$

$$
\pi(x) = U(x) \pi_0(x) U^\dagger(x)
$$
in fields satisfy the free field equation of motion

$$
\partial_\mu q^\alpha(x) = i [H_0 (\Phi_0, \pi_0), q^\alpha (x)]
$$

$$
\partial_\mu \pi^\alpha(x) = i [H_0 (\Phi_0, \pi_0), \pi^\alpha (x)]
$$

where $H_0 (\Phi_0, \pi_0)$ is the free field Hamiltonian with mass $\mu$.

On the other hand,

$$
\partial_\mu q^\alpha (x) = i [H (\Phi_0, \pi_0), q^\alpha (x)]
$$

$$
\partial_\mu \pi^\alpha (x) = i [H (\Phi_0, \pi_0), \pi^\alpha (x)]
$$

Then from eq. for $q^\alpha$, we get

$$
\partial_\mu q^\alpha = \partial_\mu [q^\alpha U^\dagger] + \partial_\mu U q^\alpha U^\dagger + U \partial_\mu q^\alpha U^\dagger + \partial_\mu U q^\alpha U
$$

$$
= \partial_\mu \pi^\alpha U U^\dagger + U \partial_\mu \pi^\alpha U^\dagger + \partial_\mu U \pi^\alpha U^\dagger + \partial_\mu U \pi^\alpha U
$$

$$
= \partial_\mu \pi^\alpha + i [H (\Phi_0, \pi_0), \pi^\alpha] - \partial_\mu \Phi_0 U U^\dagger
$$

Use

$$
\partial_\mu \Phi_0 = \partial_\mu [H_0 (\Phi_0, \pi_0), \Phi_0]
$$

we get

$$
[\partial_\mu \pi^\alpha + i H_0 (\Phi_0, \pi_0), \pi^\alpha] = 0
$$

where $H_0 (\Phi_0, \pi_0) = H (\Phi_0, \pi_0) - H_0 (\Phi_0, \pi_0)$ contains all the interaction.

Similarly, we can show

$$
[\partial_\mu \pi^\alpha + i H_0 (\Phi_0, \pi_0), \Phi_0] = 0
$$

This means the combination $\partial_\mu \pi^\alpha + i H_0 (\Phi_0, \pi_0)$ commutes with all the operators.

We can take this to be a c-number. For simplicity, we can take this c-number to be zero. Thus

$$
i \frac{\partial q^\alpha}{\partial t} = [H_0 (\Phi_0, \pi_0), q^\alpha]
$$

For convenience, we define

$$
U(t, t') = U(t) U^\dagger (t')
$$

time evolution operator

then

$$
i \frac{\partial U(t, t')}{\partial t} = H_0 (t) U(t, t')
$$

with $U(t, t) = 1$.

We can convert this to integral equation

$$
U(t, t') = 1 - i \int_{t}^{t'} dt_1 H_0 (t_1) U(t_1, t')
$$

Iterate this equation

$$
U(t, t') = 1 - i \int_{t}^{t'} dt_1 H_0 (t_1) + (-i)^2 \int_{t}^{t'} dt_1 H_0 (t_1) \int_{t_1}^{t'} dt_2 H_0 (t_2) + \cdots
$$

$$
+ (-i)^n \int_{t}^{t'} dt_1 \int_{t_1}^{t_2} dt_2 \cdots \int_{t_{n-1}}^{t_n} dt_n H_0 (t_n) H_0 (t_{n-1}) \cdots H_0 (t_2) + \cdots
$$

This can be written as

$$
U(t, t') = 1 - \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t}^{t'} dt_1 \int_{t_1}^{t_2} dt_2 \cdots \int_{t_{n-1}}^{t_n} dt_n T[H_0 (t_n) H_0 (t_{n-1}) \cdots H_0 (t_2)]
$$

$$
T[\exp [-i \int_{t}^{t'} dt x (\Phi_0, \pi_0)]
$$

Perturbation Expansion of Vacuum expectation value
\[ T(x, x, \ldots, x) = \langle \phi| T(\phi(x) \phi(\tilde{x}) \ldots \phi(x)) |0\rangle \]

Using the unitary matrix we can write this in terms of \( \phi_n \):

\[ T = \langle \phi| T(\phi(x) \phi(\tilde{x}) \ldots \phi(x)) |U(0, 0, \ldots, 0)|0\rangle = \langle \phi| T(U^{-1}(t) \phi(\tilde{x}) \phi(\tilde{x}) \ldots \phi(x)) |U(t, 0, \ldots, 0)|0\rangle = \langle \phi| U(t) \phi(\tilde{x}) \phi(\tilde{x}) \ldots \phi(x) |U(t)|0\rangle \]

Let \( t > t_1 \), ..., \( t_n > t \) Then:

\[ T = \langle \phi| U^{-1}(t) T(U(t, 0, \ldots, 0)) |U(t)|0\rangle = \langle \phi| U^{-1}(t) U(t) \phi(\tilde{x}) \phi(\tilde{x}) \ldots \phi(x) |U(0)|0\rangle \]

**Theorem:** \( \langle \phi| \) is an eigensate of \( U(t) \) as \( t \to \infty \)

Consider:

\[ \langle \phi|U(1)\langle0| = \langle \phi|U(1)\langle0| = \langle \phi|U(1)\langle0| \]

\[ = -\frac{i}{\sqrt{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d^2 p}{2\pi} \left( e^{i p} e^{-i \tilde{p} x} \right) \langle \phi| \langle \phi|U(t)\langle0| \]

Carrying out the integral, we get:

\[ \frac{1}{\sqrt{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d^2 p}{2\pi} \left( e^{i p} e^{-i \tilde{p} x} \right) \langle \phi| \langle \phi|U(t)\langle0| \]

Now we take the limit \( t \to \infty \)

Then:

\[ \langle \phi|U(t)\langle0| = \left\{ \langle \phi|U(1)\langle0| = \langle \phi|U(1)\langle0| \right\} \]

In the last form:

\[ U \phi = U \phi = \phi \]

This means:

\[ \langle \phi|U(t)\langle0| = \lambda_+ |0\rangle \]

These phases can be written as:

\[ \lambda_+ = \langle \phi|U(t)\langle0| = \langle \phi|U(t)\langle0| \]

Use this relation:

\[ T(x, x, x, \ldots, x) = \langle \phi| T(\phi(x) \phi(\tilde{x}) \ldots \phi(x)) \exp \left[ \sum_{n=0}^{\infty} \frac{(i \lambda_+)^n}{n!} \right] U(0) |0\rangle \]

or:

\[ T(x, x, x, \ldots, x) = \frac{\langle \phi| T(\phi(x) \phi(\tilde{x}) \ldots \phi(x)) \exp \left[ \sum_{n=0}^{\infty} \frac{(i \lambda_+)^n}{n!} \right] U(0) |0\rangle}{\langle \phi| U(\lambda_+ \pi) \phi(x) \phi(\tilde{x}) \ldots \phi(x) |U(0)|0\rangle} \]

More explicitly:

\[ T(x, x, x, x, x, \ldots, x) = \langle \phi| T(\phi(x) \phi(\tilde{x}) \ldots \phi(x)) \exp \left[ \sum_{n=0}^{\infty} \frac{(i \lambda_+)^n}{n!} \right] U(0) |0\rangle \]

Wish theorem: nearest transverse reduced quadratic normal order
Wick's theorem: Convert time-ordered product to normal ordering

\[ T(\phi_a(x_1) \cdots \phi_a(x_n)) = \langle \phi_a(x_1) \phi_a(x_2) \cdots \phi_a(x_n) \rangle_{\phi_a} + \langle \phi_a(x_1) \phi_a(x_2) \cdots \phi_a(x_n) \rangle_{\text{permutations}} \]

\[ + \{ \langle \phi_a(x_1) \phi_a(x_2) \cdots \phi_a(x_n) \rangle_{\phi_a} \} \cdots \cdots \langle \phi_a(x_1) \phi_a(x_2) \cdots \phi_a(x_n) \rangle_{\text{permutations}} \] in odd

This theorem can be proved by induction.

We will illustrate this for the simple case of \( n=2 \).

It is clear that

\[ T(\phi_a(x) \phi_a(x_2)) = \langle \phi_a(x) \phi_a(x_2) \rangle_{\phi_a} + (c-\text{number}) \]

We can take this equation between vacuum states to compute the c-number term

\[ \langle 0 | T(\phi_a(x) \phi_a(x_2)) | 0 \rangle = (c-\text{number}) \]

Then we get

\[ T(\phi_a(x) \phi_a(x_2)) = \langle \phi_a(x) \phi_a(x_2) \rangle + \langle 0 | T(\phi_a(x) \phi_a(x_2)) | 0 \rangle \]

Most useful application of Wick's theorem

\[ \langle 0 \langle T(\phi_a(x) \phi_a(x_2)) \langle 0 \rangle = \sum \langle \phi_a(x) \phi_a(x_2) \rangle_{\phi_a} \]

**Notation**

\[ \phi_a(x) \phi_a(x) = \langle 0 | T(\phi_a(x) \phi_a(x)) | 0 \rangle \]

**Contraction**

Example:

\[ \langle 0 | T(\phi_a(x) \phi_a(x_2)) | 0 \rangle = \langle 0 | T(\phi_a(x) \phi_a(x_2)) | 0 \rangle \]

**Feynman Propagator**

- Real scalar field

\[ \langle 0 | T(\phi_a(x) \phi_a(x_2)) | 0 \rangle = i \Delta_F(x-y, \mu) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y)}}{k - \mu^2 + i\varepsilon} \]

with

\[ i \Delta_F(k) = \frac{i}{k - \mu^2 + i\varepsilon} \]

- Complex scalar field

\[ \langle 0 | T(\phi_a(x) \phi_a(x_2)) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y)}}{k - \mu^2 + i\varepsilon} \]

- Fermion field

\[ \langle 0 | T(\phi_a(x) \phi_a(x_2)) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{1}{p - \mu^2 + i\varepsilon} \]

- Photon field

\[ \langle 0 | T(\phi_a(x) \phi_a(x_2)) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{1}{p - \mu^2 + i\varepsilon} \]

where \( \mu = (1, 0, 0, 0) \)

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Perturbation Page 3
where \( \varphi_{\mu}(1,0,0,0) \)

It can be shown that only one term contributes is \( f_{\mu\nu} \) as a consequence of the gauge invariance.

Graphic representation

\[
\begin{align*}
&\gamma^\mu \rightarrow \mathcal{A}_\nu(x-y, \mu^2) + \mathcal{S}_\nu(x-y, \mu^2) + \mathcal{D}_\nu^\mu(x-y) \\
&\gamma^\nu \rightarrow x
\end{align*}
\]

Each line (propagator) represents a contraction in Wick's expansion.

E.g.

1. \[ \phi_\mu(x_1) \phi_\nu(x_2) : \phi_\mu(y_1) \phi_\nu(y_2) : \phi_\mu(z_1) \phi_\nu(z_2) : \phi_\mu(w_1) \phi_\nu(w_2) : \]

2. \[ \phi_\mu(x_1) \phi_\nu(x_2) : \phi_\mu(y_1) \phi_\nu(y_2) : \phi_\mu(z_1) \phi_\nu(z_2) : \phi_\mu(w_1) \phi_\nu(w_2) : \]

Vacuum Amplitude

In the denominator of the \( \tau \)-function, there are no external lines

\[
\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int d^n y_1 \cdots d^n y_m <0 \mid T \{ \phi_\mu(y_1), \ldots, \phi_\mu(y_m) \} \mid 0> \]

E.g. and order term for the case \( \mathcal{H}_\mu = \frac{\lambda}{2!} \phi^2 \)

\[
<0 \mid T \{ \mathcal{H}_\mu \phi_\mu(y_1), \mathcal{H}_\mu \phi_\mu(y_2) \} \mid 0> = \left( \frac{\lambda}{2!} \right)^2 <0 \mid T \{ \phi_\mu(y_1), \phi_\mu(y_2) \} \mid 0> \]

\[
= \phi_\mu(y_1) \phi_\mu(y_2) \phi_\mu(y_1) \phi_\mu(y_2) : \phi_\mu(x_1) \phi_\mu(x_2) \\
\]

closed loop diagram: graph with no external lines (lines with open end)

disconnected diagram: a subgraph not connected to any external lines

connected diagram: graph not disconnected

All graphs appearing in the numerator of the \( \tau \)-function can be separated uniquely into connected and disconnected parts

\[
\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int d^n y_1 \cdots d^n y_m <0 \mid T \{ \phi_\mu(y_1), \ldots, \phi_\mu(y_m) \} \mid 0> \]

\[
\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int d^n y_1 \cdots d^n y_m <0 \mid T \{ \phi_\mu(y_1), \ldots, \phi_\mu(y_m) \} \mid 0> \times \frac{(-1)^{m-1}}{(m-3)!} <0 \mid T \{ \phi_\mu(y_1), \ldots, \phi_\mu(y_m) \} \mid 0> \]

\[ \sum_{m=5}^{\infty} \frac{(-1)^m}{m!} \int d^n y_1 \cdots d^n y_m <0 \mid T \{ \phi_\mu(y_1), \ldots, \phi_\mu(y_m) \} \mid 0> \sum_{r=3}^{m-5} \frac{(-1)^r}{r!} <0 \mid T \{ \phi_\mu(y_1), \ldots, \phi_\mu(y_m) \} \mid 0> \]

It is not hard to see that

\[
\tau(x_1, x_2, \ldots, x_\mu) = \int \frac{d^6 x_k}{4!} \frac{\sum \varphi_{\mu}(\mu(x_k - x_\mu))}{\sum \frac{d}{d^6 x_k}} = \int \frac{d^6 x_k}{4!} \frac{\sum \varphi_{\mu}(\mu(x_k - x_\mu))}{\sum \frac{d}{d^6 x_k}} = \sum \varphi_{\mu}(\mu(x_k - x_\mu)) \]

where \( \varphi_{\mu}(\mu(x_k - x_\mu)) \): connected diagrams with \( n \) external lines.
Hence in calculating z-function with n external lines, we can ignore all disconnected graphs.

**Example:** $\lambda_2 = \frac{3}{2} \phi^3$

\[
\phi(x_1) + \phi(x_2) \rightarrow \phi(x_1) + \phi(x_2)
\]

\[
S_{\phi} = \langle \phi(x_1, x_2) | \phi(x_1, x_2) \rangle = \langle \phi(x_1, x_2) | \phi(x_1, x_2) \rangle
\]

\[
= \frac{1}{(2\pi \hbar)^4} \int \frac{d^4 k_1}{k_1^2} \frac{d^4 k_2}{k_2^2} e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \langle \phi(x_1, x_2) | \phi(x_1, x_2) \rangle
\]

\[
= \frac{1}{(2\pi \hbar)^4} \int d^4 k_1 d^4 k_2 \langle \phi(x_1, x_2) | \phi(x_1, x_2) \rangle \frac{1}{k_1^2} \frac{1}{k_2^2} e^{i k_1 \cdot x_1 - i k_2 \cdot x_2}
\]

Perturbative expansion of z-function:

\[
\langle \phi(x_1, x_2) | \phi(x_1, x_2) \rangle = \sum_n \frac{(-i \hbar)^n}{n!} \int \cdots \int d^4 k_n e^{i k_1 \cdot x_1 - i k_n \cdot x_n} \langle \phi(x_1, x_2) | \phi(x_1, x_2) \rangle
\]

Lowest order contribution:

\[
\langle \phi(x_1, x_2) | \phi(x_1, x_2) \rangle = \frac{1}{(2\pi \hbar)^4} \int d^4 k_1 d^4 k_2 \langle \phi(x_1, x_2) | \phi(x_1, x_2) \rangle \frac{1}{k_1^2} \frac{1}{k_2^2} e^{i k_1 \cdot x_1 - i k_2 \cdot x_2}
\]

Using Wick's theorem, we get for the connected diagrams:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{connected_diagram1} \\
\includegraphics[width=0.2\textwidth]{connected_diagram2}
\end{array}
\]

Their contribution to $Z_{\phi}(x_1, x_2)$ is

\[
Z_{\phi}(x_1, x_2) = \frac{1}{(2\pi \hbar)^4} \int d^4 k_1 d^4 k_2 \ e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \text{id} \ e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \text{id} \ e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \text{id} + \ldots
\]

Use the propagator in momentum space:

\[
i \text{id} \left( k \right) = \frac{1}{i \hbar} \frac{1}{-k^2 + \text{i} \varepsilon}
\]

Then

\[
Z_{\phi}(x_1, x_2, k_1, k_2) = \frac{1}{(2\pi \hbar)^4} \int d^4 k_1 d^4 k_2 \text{id} \ e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \text{id} \ e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \text{id} \ e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \text{id} + \ldots
\]

\[
= \frac{1}{(2\pi \hbar)^4} \int d^4 k_1 d^4 k_2 \text{id} \ e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \text{id} \ e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \text{id} + \ldots
\]

\[
= \frac{1}{(2\pi \hbar)^4} \int d^4 k_1 d^4 k_2 \ e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \text{id} \ e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \text{id} + \ldots
\]

\[
S_{\phi} = \frac{1}{(2\pi \hbar)^4} \int d^4 k_1 d^4 k_2 \ e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \text{id} \ e^{i k_1 \cdot x_1 - i k_2 \cdot x_2} \text{id} + \ldots
\]

We see that the external line propagators cancel out.

This is rather simple answer in momentum space.
This is rather simple answer in momentum space.
Consider the vacuum expectation value of the form,
\[ \mathcal{Z}(x_1\ldots x_n) = \langle 0 | U(\phi(x_i)) \ldots U(\phi(x_j)) | 0 \rangle \]
coming from LSZ reduction.

Use $U$-matrix, we can write this as
\[ \mathcal{Z}(x_1\ldots x_n) = \langle 0 | U^\dagger(\phi_1(t_1) U(t_1) \phi_2(t_2) U(t_2) \ldots U(t_{n-1}, t_n) \phi_n(t_n) U(t_n)) | 0 \rangle \]
Write the S-matrix element as

$$S_{fi} = S_{fi}^0 + i (\pi)^{1/2} S^1 \{P_{f} - P_{i}\} T_{fi}$$

For i ≠ f, the transition probability is

$$|S_{fi}|^2 = (\pi)^{1/2} \{P_{f} - P_{i}\} |T_{fi}|^2$$

To interpret S\(\gamma\), we write

$$\left(\pi \gamma\right)^{1/2} \{P_{f} - P_{i}\} = \int d^4x \ e^{-i\left(\mathbf{P}_{f} - \mathbf{P}_{i}\right) \cdot \mathbf{x}}$$

The integration is over some large but finite volume V and time interval T. Then

$$\left(\pi \gamma\right)^{1/2} = VT$$

and

$$|S_{fi}|^2 = V T \left(\pi \gamma\right)^{1/2} \{P_{f} - P_{i}\} |T_{fi}|^2$$

The transition rate (transition probability per unit time) is then

$$\omega_{fi} = \pi \gamma \left(\pi \gamma\right)^{1/2} \{P_{f} - P_{i}\} |T_{fi}|^2$$

Decay rates:

$$\Delta \rightarrow \Delta \Delta + \Delta \Delta + \cdots + \Delta \Delta$$

The number of states in the volume elements \(d^4x\) \(\Delta \Delta\) in momentum space is

$$\frac{d^3k_{\Delta}}{(2\pi)^3 2\omega_{\Delta}}$$

The transition rate, summing over final states, is

$$d\omega = \left(\pi \gamma\right)^{1/2} \left(\pi \gamma\right)^{1/2} \{P_{f} - P_{i}\} \frac{d^3k_{\Delta}}{(2\pi)^3 2\omega_{\Delta}}$$

For the invariant normalization of the physical states we have been using,

$$\langle \Delta | \Psi' \rangle = \left(\pi \gamma\right)^{1/2} \{P_{f} - P_{i}\} \Delta \Delta \rightarrow \langle \Delta | \Psi \rangle = \left(\pi \gamma\right)^{1/2} \{P_{f} - P_{i}\} \Delta \Delta$$

is the # of particle in the initial state.

The decay rate per particle is then

$$d\omega = \frac{d\omega}{\Delta_{\Delta}} = \left(\pi \gamma\right)^{1/2} \{P_{f} - P_{i}\} \frac{d^3k_{\Delta}}{(2\pi)^3 2\omega_{\Delta}}$$

If there are \(m\) identical particles in the final state, divide this by \(m!\)

$$d\omega = \frac{1}{2\omega_{\Delta}} \left(\pi \gamma\right)^{1/2} \{P_{f} - P_{i}\} \frac{d^3k_{\Delta}}{(2\pi)^3 2\omega_{\Delta}}$$

Cross section:

Scattering processes:

$$\Delta_{\Delta} + \Delta_{\Delta} \rightarrow \Delta_{\Delta} + \Delta_{\Delta} + \cdots + \Delta_{\Delta}$$

The transition rate is given by, after summing over final states,

$$d\omega = \left(\pi \gamma\right)^{1/2} \left(\pi \gamma\right)^{1/2} \{P_{f} - P_{i}\} \frac{d^3k_{\Delta}}{(2\pi)^3 2\omega_{\Delta}}$$

We normalize this to one particle in the beam and one particle in the target and divide this by the flux \(\sim\) relative velocity divided by the volume to get differential cross section.

$$d\sigma = \frac{1}{2\omega_{\Delta} \nu} \left(\pi \gamma\right)^{1/2} \left(\pi \gamma\right)^{1/2} \{P_{f} - P_{i}\} \frac{d^3k_{\Delta}}{(2\pi)^3 2\omega_{\Delta}}$$

Velocity factor can be written as
Velocity factor can be written as

\[ I = \left| \frac{p_1 - p_2}{E_1 - E_2} \right| \]

In the c.m. frame \( \vec{p}_1 = -\vec{p}_2 = \vec{p} \) \( \vec{E}_1 = \vec{E}_2, \vec{P}_1 = \vec{E}_2, \vec{P} \)

\[ I = \frac{\vec{p}_1}{E_1 + E_2} \]

\[ (\vec{p}, \vec{p}) = (E_1 + E_2)^2 = E_1^2 + 2E_1E_2 + E_2^2 + \vec{p} \]

\[ (\vec{p}, \vec{p}) = (E_1 + E_2)^2 = E_1^2 + 2E_1E_2 + E_2^2 + \vec{p} \]

\[ (\vec{p}, \vec{P}) = (\vec{p}, \vec{p}) = \frac{m^2 E_1}{(E_1 + E_2)^2} \]

\[ \Rightarrow I = \frac{1}{E_1 + E_2} \left( \frac{m^2 E_1}{(E_1 + E_2)^2} \right) \]

\[ d\sigma = \frac{1}{i} \left( \frac{1}{2 \Delta p} \right)^J \left( \frac{1}{2 \Delta \vec{p}} \right)^J \delta(q, \vec{p} - \vec{p}_1) \left( \delta_{J, J'} \right) \frac{1}{i} \frac{d^2 \lambda}{(2\pi)^2} \frac{d^2 \lambda}{(2\pi)^2} \cdots \frac{d^2 \lambda}{(2\pi)^2} \]
Feynman Rules

Since the final forms for transition matrix elements Tfs are quite simple, we can use simple rules to sidestep all these tedious intermediate steps.

Draw all connected Feynman graphs with appropriate external lines. Label each with momenta and impose momentum conservation for each vertex.

1. For each internal fermion line with momentum $p$, enter the propagator
   \[ iS_f(p) = \frac{i}{p^2 - m^2 + i\epsilon} \]

2. For each internal boson line of spin $0$, with momentum $q$, enter the propagator
   \[ iA_B(q) = \frac{i}{q^2 + i\epsilon} \]

3. For each internal photon line with momentum $k$, enter the propagator
   \[ iD_{\mu\nu}(k) = \frac{-i\gamma_{\mu\nu}}{k^2 + i\epsilon} \]

4. For each internal momentum not fixed by momentum conservation, enter
   \[ \int \frac{d^4q}{(2\pi)^4} \]

5. For each closed fermion loop, enter (-1). Also the should be a factor $g_s$ between graphs which differ only by an interchange of two external identical fermion lines.

At each vertex, the factors depend on the explicit form of interaction.

(a) \[ \frac{\lambda^3}{3!} \]

(b) \[ \frac{\lambda\gamma^\mu}{4!} \]

(c) \[ e^2 \bar{\psi}_1 A^\mu \]

(d) \[ f \bar{\psi}_1 \psi_2 \phi \]

- i\epsilon P_\mu
Example in $\lambda \phi^3$ theory

Consider scattering processes $\phi(k_1) + \phi(k_2) \rightarrow \phi(k_3') + \phi(k_4')$

To second order in $\lambda$, we have following 3 Feynman diagrams for this reaction.

(a) \[ \begin{array}{c} k_1 \leftrightarrow k_2' \rightarrow k_3 \rightarrow k_4' \end{array} \]

(b) \[ \begin{array}{c} k_1 \rightarrow k_2 \leftrightarrow k_3' \rightarrow k_4' \end{array} \]

(c) \[ \begin{array}{c} k_1 \rightarrow k_2 \rightarrow k_3' \leftrightarrow k_4' \end{array} \]

We can write down the matrix element for each graph,

\[ T^{(a)} = \frac{i\lambda^2}{(k_1' - k_3)^2 - \mu^2} \]

\[ T^{(b)} = \frac{i\lambda^2}{(k_1' + k_3')^2 - \mu^2} \]

\[ T^{(c)} = \frac{i\lambda^2}{(k_1' - k_4')^2 - \mu^2} \]

Total amplitude
\[ T = T^{(a)} + T^{(b)} + T^{(c)} \]

Mandelstam variables
\[ S = (k_1 + k_2)^2 \quad \text{total energy in c.m. frame} \]
\[ t = (k_1 - k_3)^2 \quad \text{momentum transfer (scattering angle)} \]
\[ u = (k_1 - k_4)^2 \]

\[ S + t + u = 4\mu^2 \]